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Global existence and asymptotic behavior of solutions to the mixed problems for the discrete Boltzmann equation

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1. Introduction

The discrete Boltzmann equation is the fundamental equation describing the time-evolution of a discrete velocity gas which consists of particles with a finite number of velocities ([6]). The aim of this note is to survey the author's recent works [10,11,12] concerning the global existence and asymptotic behavior of solutions to the mixed problems for the discrete Boltzmann equation on a bounded region $0<x<d$.

The general form of the discrete Boltzmann equation in one space dimension is written as

$$(1) \qquad c_i\left(\frac{\partial F_i}{\partial t} + v_i \frac{\partial F_i}{\partial x}\right) = Q_i(F), \qquad i\in\Lambda,$$

where Λ is a finite set $\{1,...,m\}$, c_i are positive constants, and each $F_i=F_i(x,t)$ represents the mass density of gas particles with the i-th velocity at time t and position x; each v_i denotes the x-component of the i-th velocity and hence v_i, $i\in\Lambda$, are not necessarily distinct. For the derivation of (1) from the original discrete Boltzmann equation in $\mathbb{R}^n$ (n=2 or 3), we refer the reader to

[9]. Collision terms $Q_i(F)$ on the right hand side of (1) are given as

$$Q_i(F) = \sum_{jk\ell}(A^{ij}_{k\ell}F_kF_\ell - A^{k\ell}_{ij}F_iF_j), \qquad i\in\Lambda,$$

where the summation is taken over all $j,k,\ell\in\Lambda$, and where the coefficients $A^{ij}_{k\ell}$ are nonnegative constants satisfying

(A1) $$A^{ji}_{k\ell} = A^{ij}_{k\ell} = A^{ij}_{\ell k},$$

(A2) $$A^{ij}_{k\ell}(v_i+v_j-v_k-v_\ell) = 0,$$

(A3) $$A^{ij}_{k\ell} = A^{k\ell}_{ij},$$

for any $i,j,k,\ell\in\Lambda$. (A2) implies the conservation of momentum (in the x-direction) in the microscopic collision process and (A3) is called the micro-reversibility condition.

We prescribe the initial data:

(2) $$F_i(x,0) = F_{i0}(x), \qquad i\in\Lambda.$$

Let $\Lambda_+=\{i\in\Lambda; v_i>0\}$ and $\Lambda_-=\{i\in\Lambda; v_i<0\}$, and we impose the boundary conditions as follows: on the left boundary x=0, either

(3) $$F_i(0,t) = B^0_i, \qquad i\in\Lambda_+, \quad \text{or}$$

(3)' $\quad c_iF_i(0,t) = \Sigma_j^- B_{ij}^0 F_j(0,t), \qquad i\in\Lambda_+,$

and on the right boundary x=d, either

(4) $\quad F_i(d,t) = B_i^1, \qquad i\in\Lambda_-,$ or

(4)' $\quad c_iF_i(d,t) = \Sigma_j^+ B_{ij}^1 F_j(d,t), \qquad i\in\Lambda_-.$

Here the boundary data B_i^0 and B_i^1 are positive constants, the coefficients B_{ij}^0 are nonnegative constatns, and $\Sigma_j^\pm$ mean the summations taken over all $j\in\Lambda_\pm$, respectively. For the boundary condition (3)', we require as in [7] that

$(B1)_0 \quad \Sigma_i^+ v_iB_{ij}^0 + c_jv_j = 0, \qquad j\in\Lambda_-,$

$(B2)_0 \quad c_iM_i^0 = \Sigma_j^- B_{ij}^0 M_j^0, \qquad i\in\Lambda_+,$

where $M^0=(M_i^0)_{i\in\Lambda}$ is some constant Maxwellian; a vector $M=(M_i)_{i\in\Lambda}$ is called Maxwellian if M_i are all positive and satisfy $A_{k\ell}^{ij}(M_iM_j-M_kM_\ell)=0$ for any $i,j,k,\ell\in\Lambda$. Analogous conditions are required also for (4)':

$(B1)_1 \quad \Sigma_i^- v_iB_{ij}^1 + c_jv_j = 0, \qquad j\in\Lambda_+,$

$(B2)_1 \quad c_iM_i^1 = \Sigma_j^+ B_{ij}^1 M_j^1, \qquad i\in\Lambda_-,$

where $M^1=(M_i^1)_{i\in\Lambda}$ is a constant Maxwellian not necessarily equal to the M^0 in $(B2)_0$. The conditions $(B1)_0$ and $(B1)_1$ imply that the total momentum density $\Sigma_i c_i v_i F_i$ vanishes on the boundaries x=0 and x=d, respectively. For the meaning of $(B2)_{0,1}$, we refer the reader to [7].

2. Global solutions

For convenience of later references, we shall list up our mixed problems formulated in Section 1.

Problem (I) : {(1),(2),(3),(4)},

Problem (II) : {(1),(2),(3),(4)'},

Problem (III) : {(1),(2),(3)',(4)'}.

Note that the problem consisting of (1),(2),(3)' and (4) is essentially the same as the above problem (II).

For a nonnegative integer k, we denote by $C^k(\Omega)$ the space of k-times continuously differentiable functions on a set Ω, and by $C_+^k(\Omega)$ the totality of strictly positive function in $C^k(\Omega)$. Then our global existence result can be stated as follows.

Theorem 1. *Suppose that* $F_0=(F_{i0})_{i\in\Lambda}$ *is in* $C_+^1([0,d])$. *Then the mixed problem* (I), (II) *or* (III) *has a unique global solution* $F=(F_i)_{i\in\Lambda}$ *in* $C_+^1([0,d]\times[0,\infty))$, *provided that* F_0 *satisfies the corresponding compatibility conditions up to order one.*

Remark. A similar global existence result holds true also for the mixed problem (1),(2),(3) (or (3)') on the half-space $0<x<\infty$ ([11]). For global existence results to the pure initial value problem (1),(2) on the whole space $-\infty<x<\infty$, see [1,2].

To give an outline of the proof of Theorem 1, we introduce

(5)
$$E_0 = \max_i \sup_{0\le x\le d} F_{i0}(x), \qquad E(t) = \max_i \sup_{\substack{0\le x\le d \\ 0\le \tau\le t}} F_i(x,\tau),$$

$$\Phi(t,r) = \sup_{|I|\le r} \int_I \sum_i c_i F_i(x,t)dx,$$

where the supremum in the last expression is taken over all the intervals I contained in [0,d], with the length $|I|\le r$. The standard method based on the contraction mapping priciple shows that each problem has a unique local solution in $C^1_+([0,d]\times[0,T_0])$ for some $T_0>0$ depending only on the sup-norm E_0 of the initial data. Therefore, key to the proof of Theorem 1 is to derive a suitable a priori estimate for the sup-norm E(t) of solutions in $C^1_+([0,d]\times[0,T])$ for any fixed T>0. The desired a priori estiamte is given in the following

Proposition 1. *Let* $T>0$ *and let* $F\in C^1_+([0,d]\times[0,T])$ *be a solution to the mixed problem* (I), (II) *or* (III). *Then there exists a constant* $K(E_0,T)$ *such that*

(6) $$E(t) \le K(E_0,T), \qquad t\in[0,T].$$

Here $K(E_0,T)$ *depends only on* E_0 *and* T, *and increases monotonously as* $E_0\longrightarrow\infty$ *or* $T\longrightarrow\infty$.

The a priori estimate (6) can be derived easily from the difference inequality (7) for $E(t)$, combined with the estimate (8) for $\Phi(t,r)$:

Lemma 1. *There is a positive constant* C *such that for any* $t\geq 0$ *and* $h>0$ *satisfying* $t+h\leq T$ *and* $2vh\leq d$ *(where* $v=\max_i|v_i|$*), we have*

(7) $$E(t+h) \leq CE(t) + C(E_0h + \Phi(t,2vh))E(t+h).$$

Moreover there is another positive constant C *such that for any* $0\leq t\leq T$ *and* $0<r\leq d$, *we have*

(8) $$\Phi(t,r) \leq CE_0(1+T)\delta(r),$$

where $\delta(r)$ *is a continuous and increasing function of* r *such that* $\delta(r)\longrightarrow 0$ *as* $r\longrightarrow 0$.

Remark. The term E_0h in (7) comes from the boundary data in (3) and (4), and hence this term is unnecessary for the problem (Ⅲ).

Lemma 1 can be proved by a method similar to the one employed in [2] for the pure initial value problem. (Of course we need suitable modifications to deal with the boundary effects.) More specifically, in the proof of (7), we use the characteristic method and the identities obtained by integrating the conservation equations of mass and momentum, given in (9) below, over various regions in the rectangle $[0,d]\times[0,T]$.

$$\frac{\partial}{\partial t}\sum_i c_iF_i + \frac{\partial}{\partial x}\sum_i c_iv_iF_i = 0,$$
(9)
$$\frac{\partial}{\partial t}\sum_i c_iv_iF_i + \frac{\partial}{\partial x}\sum_i c_iv_i^2F_i = 0.$$

On the other hand, the estimate (8) is essentially based on the identity obtained by integrating the equation of modified H-function, given in (10) below, over the rectangle $[0,d]\times[0,t]$ with $0<t\leq T$, and on the argument of Crandall-Tartar [15].

(10)
$$\frac{\partial}{\partial t}\sum_i c_iF_i\log(F_i/M_i) + \frac{\partial}{\partial x}\sum_i c_iv_iF_i\log(F_i/M_i)$$
$$= -\frac{1}{4}\sum_{ijk\ell} A_{k\ell}^{ij}(F_iF_j - F_kF_\ell)\log(F_iF_j/F_kF_\ell)$$
$$-\sum_i c_iv_iF_i\frac{\partial}{\partial x}\log M_i,$$

where $M=(M_i)_{i\in\Lambda}$ is a Maxwellian depending smoothly in x and is chosen according to the boundary conditions of each mixed problem. For the details, see [10,11].

3. Stationary solutions

We consider the corresponding stationary problems (pure boundary value problems) for the discrete Boltzmann equation:

(11)
$$c_iv_i\frac{dF_i}{dx} = Q_i(F), \qquad i\in\Lambda,$$

(12) $\quad F_i(0) = B_i^0, \quad i\in\Lambda_+,$ or

(12)' $\quad c_iF_i(0) = \sum_j^- B_{ij}^0F_j(0), \qquad i\in\Lambda_+,$

(13) $\quad F_i(d) = B_i^1, \quad i\in\Lambda_-,$ or

(13)' $\quad c_iF_i(d) = \sum_j^+ B_{ij}^1F_j(d), \qquad i\in\Lambda_-.$

There are essentially the following three stationary problems.

Problem (i) : {(11),(12),(13)},

Problem (ii) : {(11),(12),(13)'},

Problem (iii): {(11),(12)',(13)'}.

Let $\Lambda_0=\{i\in\Lambda; v_i=0\}$. We observe that if $\Lambda_0\neq\phi$, the ordinary differential equations (11) partially degenerate into the algebraic equations

(14) $\quad Q_i(F) = 0, \qquad i\in\Lambda_0,$

so that in order to solve the stationary problems, we need to require the solvability of (14) with respect to $(F_i)_{i\in\Lambda_0}$. Here we introduce

(15) $\quad \hat{W} = \{\hat{F}=(F_i)_{i\in\Lambda_+\cup\Lambda_-}\ ;\ F_i>0 \text{ for } i\in\Lambda_+\cup\Lambda_-\},$

and propose the following condition.

Condition 1. (Solvability of (14)). There exists a set of mappings $(\pi_i)_{i\in\Lambda_0}$ with the following conditions:

(i) Each π_i is a C^1-mapping of $\hat{W}$ into $\mathbb{R}_+$ (the totality of positive real numbers) and is nondecreasing.

(ii) For any given $\hat{F}=(F_i)_{i\in\Lambda_+\cup\Lambda_-}\in\hat{W}$, we put $F_i=\pi_i(\hat{F})$ for $i\in\Lambda_0$. Then $F=(F_i)_{i\in\Lambda}$ solves (14).

Under the above condition we can solve the stationary problems as follows.

Theorem 2. *Assume Condition 1 if* $\Lambda_0\neq\phi$. *Then the stationary problem (i) or (ii) has a solution* $F=(F_i)_{i\in\Lambda}$ *in* $C^1_+([0,d])$.

Remark. This result is an improvement of the earlier work [4] where $\Lambda_0=\phi$ is assumed in technical reason. Uniqueness of the solutions is unknown in general. (In this respect, see Theorem 3 below.) The stationary problem (iii) is unsolved as yet. For an existence result of the stationary problem (11),(12) on the half-space $0<x<\infty$, see [3].

As in [4], we can prove the above theorem by applying the following fixed point theorem of Leray-Schauder type.

Fixed point theorem by Browder-Potter ([14]). *Let* S *be a closed convex subset of a Banach space* X. *Let* $\Phi^\lambda(F)$ *be a continuous mapping of* $(F,\lambda)\in S\times[0,1]$ *into a compact sutset of* X *such that*

1) $\Phi^0(\partial S) \subset S$,

2) *for* $0 \le \lambda \le 1$, $\Phi^\lambda(\cdot)$ *has no fixed point on* ∂S.

Then $\Phi^1(\cdot)$ *has a fixed point in* S.

In application to our stationary problem (i) or (ii), we take $X = C^0([0,d])$ and

$$(16) \qquad S = \{F = (F_i)_{i \in \Lambda} \in X \ ; \ 0 \le F_i(x) \le R \text{ for } x \in [0,d],\ i \in \Lambda\},$$

where R>0 is some large constant. We shall define the mapping $\Phi^\lambda = (\Phi_i^\lambda)_{i \in \Lambda}$ as follows: Let $(G,\lambda) \in S \times [0,1]$. Then $F_i = \Phi_i^\lambda(G)$, $i \in \Lambda_+ \cup \Lambda_-$, are defiend by solving the linearized problem

$$(17)_1 \qquad c_i v_i \frac{dF_i}{dx} = \lambda(q_i(G) - r_i(G)F_i), \qquad i \in \Lambda_+ \cup \Lambda_-,$$

with the boundary conditions (12),(13) (or (12),(13)'), where

$$q_i(G) = \sum_{jk\ell} A_{k\ell}^{ij} G_k G_\ell, \qquad r_i(G) = \sum_{jk\ell} A_{ij}^{k\ell} G_j.$$

Furthermore, since the resulting $F_i = \Phi_i^\lambda(G)$, $i \in \Lambda_+ \cup \Lambda_-$, are strictly positive, we can define

$$(17)_2 \qquad F_i = \Phi_i^\lambda(G) \equiv \pi_i((\Phi_j^\lambda(G))_{j \in \Lambda_+ \cup \Lambda_-}), \qquad i \in \Lambda_0,$$

where π_i are the mappings in Condition 1. Note that $F = \Phi^\lambda(G)$ thus defined is strictly positive and satisfies (14). Also, we see easily that our $F=\Phi^{\lambda}(G)$ is continuous mapping of $(G,\lambda)\in S\times[0,1]$ into a compact subset of X and satisfies $\Phi^{0}(S)\subset S$ (a stronger version of $\Phi^{0}(\partial S)\subset S$). Therefore, for the proof of Theorem 2, it suffices to check the condition 2) of the fixed point theorem. To this end, we suppose that F is a fixed point, namely, $F=\Phi^{\lambda}(F)$. Then we have

$$c_i v_i \frac{dF_i}{dx} = \lambda Q_i(F), \qquad i\in\Lambda,$$

with the boundary conditions under consideration. In particular, we have the nonstationary version of the conservation equations of mass and momentum

$$\text{(18)} \qquad \frac{d}{dx}\sum_i c_i v_i F_i = 0, \qquad \frac{d}{dx}\sum_i c_i v_i^2 F_i = 0.$$

The strict positivity of F and the identities obtained by integrating (18) over $[0,d]$ or $[0,x]$ with $0<x\le d$ yield the a priori estimate $0<F_i(x)\le C$ for $x\in[0,d]$ and $i\in\Lambda$, where C is a positive constant independent of both R and λ. This implies that $F\notin\partial S$. Thus we have verified the condition 2) and the proof of Theorem 2 is complete. See [12] for details.

4. Stationary solutions near Maxwellian

We wish to show the uniqueness of solutions to the stationary problems (i) and (ii) in a neighborhood of a constant Maxwellian. To this end, after introducing several notations, we formulate a condition which we call the stability condition in stationary case.

Let

(19) $$V = \mathrm{diag}(v_i)_{i\in\Lambda},$$

and let $\mathcal{M}_0$ be the totality of vectors $\phi=(\phi_i)_{i\in\Lambda}\in\mathbb{R}^m$ satisfying

(20) $$A^{ij}_{k\ell}(\phi_i+\phi_j-\phi_k-\phi_\ell) = 0,$$

for any $i,j,k,\ell\in\Lambda$. A vector $\psi=(c_i\phi_i)_{i\in\Lambda}$ is called collision invariant of the discrete Boltzmann equation (1) if $\phi=(\phi_i)_{i\in\Lambda}\in\mathcal{M}_0$. The totality of collision invariants is denoted by $\mathcal{M}$. Notice that $\mathcal{M}$ contains the vectors $(c_i)_{i\in\Lambda}$ and $(c_iv_i)_{i\in\Lambda}$. Our stability condition is then formulated as follows.

Condition 2. (Stability condition in stationary case). Let $\psi\in\mathcal{M}$ and let $V\psi=0$. Then $\psi=0$.

Note that Condition 2 with $\mathcal{M}$ replaced by $\mathcal{M}_0$ leads to an equivalent condition. This stability condition enables us to prove the existence and uniqueness of solutions to the stationary problems (i) and (ii).

Theorem 3. *We consider the stationary problem (i) under Condition 2. Let* $M=(M_i)_{i\in\Lambda}$ *be any fixed constant Maxwellian and put*

(21) $$\delta = \Sigma_i^+ \ |B_i^0 - M_i| + \Sigma_i^- \ |B_i^1 - M_i|.$$

If δ is small enough, then there exists a unique solution $F=(F_i)_{i\in\Lambda}$ *which belongs to* $C^1([0,d])$ *and satisfies the estimate* $|F-M|_1\le C\delta$ *for some positive constant* C, *where* $|\cdot|_1$ *denotes the norm of* $C^1([0,d])$.

Remark. A similar existence and uniqueness result holds true also for the stationary problem (ii); in this case we should take $M=M^1$ (in $(B2)_1$) and $\delta=\Sigma_i^+|B_i^0-M_i^1|$.

For the proof of Theorem 3, we first consider a linearization of the collision term. Let us denote by $Q_i(F,G)$ the bilinear form corresponding to the collision term $Q_i(F)$ and put $Q(F,G)=(Q_i(F,G))_{i\in\Lambda}$. Then we get the expression

$$(22)\qquad Q(M+I_Mf,M+I_Mf) = -L_Mf + \Gamma_M(f,f)$$

for $f\in\mathbb{R}^m$, where $M=(M_i)_{i\in\Lambda}$ is a Maxwellian, $I_M=\mathrm{diag}(M_i)_{i\in\Lambda}$ and

$$(23)\qquad L_Mf = -2Q(M,I_Mf),\qquad \Gamma_M(f,g) = Q(I_Mf,I_Mg).$$

The following property of the linearlized collision operator L_M is well known. See, for example, [6,8].

Lemma 2. L_M *is real symmetric and nonnegative definite. The null space of* L_M *coincides with the subspace* $\mathcal{M}_0$ *defined by (20), and hence Condition 2 is equivalent to the following condition.*

(24) Let $L_M\phi = 0$ and let $V\phi = 0$. Then $\phi = 0$.

Now we briefly sketch the proof of Theorem 3. We define the constant vector $\bar{F}=(\bar{F}_i)_{i\in\Lambda}$ by

(25) $$\bar{F}_i = \begin{cases} B_i^0, & i\in\Lambda_+, \\ M_i, & i\in\Lambda_0, \\ B_i^1, & i\in\Lambda_-. \end{cases}$$

Also, we introduce another constant vector $\bar{f}=(\bar{f}_i)_{i\in\Lambda}$ and a new unknown $f=(f_i)_{i\in\Lambda}$ by

(26) $$\bar{F} = M + I_M\bar{f}, \quad F = \bar{F} + I_Mf = M + I_M(\bar{f}+f),$$

respectively. Then the stationary problem (i) can be transformed into

(27) $$\tilde{I}_M V\frac{df}{dx} + L_Mf = - L_M\bar{f} + \Gamma_M(\bar{f}+f,\bar{f}+f),$$

(28) $$f_i(0) = 0,\ i\in\Lambda_+, \qquad f_i(d) = 0,\ i\in\Lambda_-,$$

where $\tilde{I}_M=\mathrm{diag}(c_iM_i)_{i\in\Lambda}$. We wish to solve the problem (27),(28), which is equivalent to the original problem (i), by applying the contraction mapping principle. To this end, we consider the linearized equation

(29) $$\tilde{I}_M V\frac{df}{dx} + L_Mf = h,$$

with the boundary conditions (28). Let $L^2(0,d)$ be the space of square integrable functions on $(0,d)$, with the norm $\|\cdot\|$. For a positive integer ℓ, we denote by $H^\ell(0,d)$ the standard L^2-sense Sobolev space of order ℓ, on $(0,d)$, equipped with the norm $\|\cdot\|_\ell$. With these notations, the following existence and regularity result for the linearized problem (29),(28) can be stated.

Proposition 2. *Suppose that* h *is in* $H^1(0,d)$. *Then the linearized stationary problem (29),(28) has a unique solution* f *in* $H^1(0,d)$. *Moreover we have the estimates*

$$(30)_1 \qquad \|f\|_1 \le C\|h\|_1,$$

$$(30)_2 \qquad \|f\| + \Sigma_i^+ \left\|\frac{df_i}{dx}\right\| + \Sigma_i^- \left\|\frac{df_i}{dx}\right\| \le C\|h\|.$$

where C *is a positive constant.*

Once Proposition 2 is established, the proof of Theorem 3 is immediate. In fact, we put

$$(31) \qquad S = \{f \in H^1(0,d) \ ; \ \|f\|_1 \le R\delta\},$$

for a suitably large constant $R>0$, and define the mapping $g \longrightarrow f$ by solving the linearized problem (29),(28) with $h=-L_M\bar{f}+\Gamma_M(\bar{f}+g,\bar{f}+g)$. Then it follows from the estimates $(30)_1$ and $|\bar{f}|\le C\delta$ (for a constant

c) that $g \longrightarrow f$ is a contraction mapping of S into itself if δ is small enough. This completes the proof of Theorem 3.

It remains to prove Proposition 2. We introduce the boundary subspaces $\mathscr{B}_0$ and $\mathscr{B}_1$ associated with the boundary conditions in (28):

$$(32)\qquad \begin{aligned} \mathscr{B}_0 &= \{f=(f_i)_{i\in\Lambda}\in\mathbb{R}^m \ ;\ f_i=0 \quad \text{for} \quad i\in\Lambda_+\}, \\ \mathscr{B}_1 &= \{f=(f_i)_{i\in\Lambda}\in\mathbb{R}^m \ ;\ f_i=0 \quad \text{for} \quad i\in\Lambda_-\}. \end{aligned}$$

Then a simple calculation shows the following

Lemma 3. *The boundary subspace $\mathscr{B}_0$ is maximal nonnegative with respect to the boudnary matrix $-\tilde{I}_M V$ such that*

$$(33)\qquad -\langle \tilde{I}_M Vf, f\rangle \geq c\textstyle\sum_i^- f_i^2, \qquad f=(f_i)_{i\in\Lambda}\in\mathscr{B}_0,$$

where c *is a positive constant and* $\langle\ ,\ \rangle$ *is the inner product of* $\mathbb{R}^m$. *A similar statement holds true for the subspace $\mathscr{B}_1$ and the corresponding matrix* $\tilde{I}_M V$.

By virtue of Lemmas 2 and 3, the standard energy method can be applied to our problem (29),(28) and we obtain the estimates $(30)_{1,2}$ (as a priori estimates) with the aid of the Poincarè inequality; the condition (24) which is equivalent to Condition 2 is used to derive the estimates of $(f_i)_{i\in\Lambda_0}$. On the other hand, the existence of solutions to the problem (29),(28) follows from the arguments essentially based on Friedrichs' theory [5] for symmetric positive systems. We omit the details and refer the reader to [12].

5. Large-time behavior of solutions

It is expected that the solution to the mixed problem (I) or (II) converges to the corresponding unique stationary solution constructed in Theorem 3 as $t \longrightarrow \infty$. This is indeed true for the problem (I) if we assume the following stability condition previously formulated in [13].

Condition 3. (Stability condition in nonstationary case). Let $\psi \in \mathcal{K}$ and let $V\psi = \lambda\psi$ for $\lambda \in \mathbb{R}$. Then $\psi = 0$.

Note that this condition is equivalent to

(34) Let $L_M\phi = 0$ and let $V\phi = \lambda\phi$ for $\lambda \in \mathbb{R}$. Then $\phi = 0$.

Theorem 4. *We consider the mixed problem* (I) *under Condition 3. Suppose that* F_0 *is in* $H^1(0,d)$ *and satisfies the compatibility conditions of order zero. Let* M *be any fixed constant Maxwellian and suppose that* $\|F_0 - M\|_1$ *is small enough. Then there exists a unique global solution* F *in* $C^0([0,\infty);H^1(0,d)) \cap C^1([0,\infty);L^2(0,d))$. *Moreover, this solution* $F(x,t)$ *converges, uniformly in* $x \in [0,d]$, *to the solution* $F^\infty(x)$ *of the corresponding stationary problem (i), which is constructed in Theorem 3, at an exponential rate* $e^{-\alpha t}$, $\alpha > 0$, *as* $t \longrightarrow \infty$.

Remark. We have proved in Section 4 the existence and uniqueness of solution also for the stationary problem (ii). However, we do not know in general whether the solution to the mixed problem (II) converges to that stationary solution as $t\longrightarrow\infty$. In this respect, we refer the reader to [12]. The problem concerning the asymptotic behavior of solutions to the mixed problem (III) is completely open.

We give an outline of the proof of Theorem 4. Let F^{∞} be the stationary solution of the problem (i) constructed in Theorem 3. We introduce the known function f^{∞} and and a new unknown f by

$$(35) \qquad F^{\infty} = M + I_M f^{\infty}, \qquad F = F^{\infty} + I_M f = M + I_M(f^{\infty}+f),$$

respectively. Then, as in the preceding section, we can transform the mixed problem (I) into

$$(36) \qquad \tilde{I}_M\left(\frac{\partial f}{\partial t} + V\frac{\partial f}{\partial x}\right) + L_M f = \Gamma_M(2f^{\infty}+f,f),$$

$$(37) \qquad f(x,0) = f_0(x),$$

$$(38) \qquad f_i(0,t) = 0, \quad i\in\Lambda_+, \qquad f_i(d,t) = 0, \quad i\in\Lambda_-,$$

where $f_0=I_M^{-1}(F_0-F^{\infty})=I_M^{-1}(F_0-M)-f^{\infty}$. The corresponding linearized problem is

$$(39) \qquad \tilde{I}_M\left(\frac{\partial f}{\partial t} + V\frac{\partial f}{\partial x}\right) + L_M f = h,$$

with the initial and the boundary conditions, (37) and (38). For $T>0$, we put

$$X_T^1 = C^0([0,T];H^1(0,d)) \cap C^1([0,T];L^2(0,d)),$$

(40)

$$|||f|||_{1,T}^2 = \sup_{0\le t\le T} |||f(t)|||_1^2 + \int_0^T |||f(t)|||_1^2 dt,$$

where $|||f|||_1^2 = \|f\|_1^2 + \|\partial f/\partial t\|^2$. Then we can prove the following result for the linearized problem (39),(37),(38).

Proposition 3. *Consider the linearized mixed problem (39), (37),(38). Let* $T>0$ *and suppose that* h *is in* X_T^1. *Suppose furthermore that* f_0 *is in* $H^1(0,d)$ *and satisfies the compatibility conditions of order zero. Then there exists a unique solution* f *in* X_T^1. *Moreover, there are positive constants* α_0 *and* C *(independent of* T*) such that for any* $\alpha\in[0,\alpha_0]$, *we have*

(41)
$$e^{\alpha t}|||f(t)|||_1^2 + \int_0^t e^{\alpha\tau}|||f(\tau)|||_1^2 d\tau \le C|||f(0)|||_1^2 + $$
$$+ Ce^{\alpha t}\|h(t)\|^2 + C\int_0^t e^{\alpha\tau}|||h(\tau)|||_1^2 d\tau, \qquad t\in[0,T].$$

Once Proposition 3 is established, Theorem 4 can be proved by applying the contraction mapping principle. In fact, we define a closed convex subset S by

(42)
$$S = \{f\in X_T^1 \ ; \ |||f|||_{1,T} \le R\|F_0 - M\|_1\},$$

where $R>0$ is a suitably large constant, and consider the mapping $g \longrightarrow f$ defined by solving the linearized problem (39),(37),(38) with $h=\Gamma_M(2f^\infty+g,g)$. Since $|f^\infty|_1 \le C\delta$ by Theorem 3, we have $\|f_0\|_1 \le C\|F_0-M\|_1$ and hence $|||f(0)|||_1 \le C\|F_0-M\|_1$, where C is some constant. Therefore, applying the estimate (41) with $\alpha=0$, we find that $g \longrightarrow f$ is a contraction mappping (with respect to the norm $|||\cdot|||_{1,T}$) of S into itself, if $\|F_0-M\|_1$ is small enough. Thus we have a unique fixed point f in S. This fixed point f is the desired solution of the problem (36),(37),(38), which is equivalent to the original mixed problem (I), and satisfies the estimate

$$(43) \qquad e^{\alpha t}|||f(t)|||_1^2 + \int_0^t e^{\alpha\tau}|||f(\tau)|||_1^2 d\tau \le C\|F_0-M\|_1^2, \qquad t\in[0,T],$$

for any $\alpha\in[0,\alpha_0]$, where C is a constant independent of T. This proves Theorem 4 since $T>0$ is arbitrary.

Proposition 3 is proved by a rather technical energy method which is similar to the one employed in [8] for the pure initial value problem. More specifically, our energy method is essentially based on the properties stated in Lemmas 2 and 3, and on the existence of a skew-symmetric matrix K given in Lemma 4 below; we also make use of the elliptic estimate $(30)_2$ in Proposition 2 and the Poincarè inequality. For a complete proof of Proposition 3, we refer the reader to [12].

Lemma 4. *We assume the condition (34) which is equivalent to Condition 3. Then there exists a skew-symmetric matrix* K *with the* *following properties.*

(i) $Kf=0$ *holds for any* $f=(f_i)_{i\in\Lambda}\in\mathbb{R}^m$ *with* $f_i=0$, $i\in\Lambda_+\cup\Lambda_-$.

(ii) *There is a positive constant* c *such that for any* $f=(f_i)_{i\in\Lambda}\in\mathbb{R}^m$, *we have*

$$\langle(KV-VK)f,f\rangle + \langle VL_M Vf,f\rangle \geq c\left(\sum_i^+ f_i^2 + \sum_i^- f_i^2\right).$$

We remark that Lemma 4 is a simple corollary of Theorem 1.1 in [13] or Theorem 3.2 in [9].

References

[1] J.M. Bony, Solutions globales bornées pour les modèles discrets de l'équation de Boltzmann, en dimension 1 d'espace, Journées "Equations aux Dérivées Partielles", Ecole Polytechnique, 1987.

[2] H. Cabannes and S. Kawashima, Le problème aux valeurs initiales en théorie cinetique discrète, C.R. Acad. Sci. Paris, 307 (1988), 507-511.

[3] C. Cercignani, R. Illner, M. Pulvirenti and M. Shinbrot, On non-linear stationary half-space problems in discrete kinetic theory, J. Stat. Phys., 52 (3/4) (1988), 885-896.

[4] C. Cercignani, R. Illner and M. Shinbrot, A boundary value problem for discrete-velocity models, Duke Math. J., 55 (1987), 889-900.

[5] K.O. Friedrichs, Symmetric positive linear differential equations, Comm. Pure Appl. Math., 11 (1958), 333-418.

[6] R. Gatignol, Théorie cinétique de gaz à répartition discrète de

vitesses, Lecture Notes in Physics 36, Springer-Verlag, New York, 1975.

[7] R. Gatignol, Kinetic theory boundary conditions for discrete velocity gases, Phys. Fluids, 20 (1977), 2022-2030.

[8] S. Kawashima, Global existence and stability of solutions for discrete velocity models of the Boltzmann equation, "Recent Topics in Nonlinear PDE", Lecture Notes in Num. Appl. Anal. 6, Kinokuniya, 1983, 59-85.

[9] S. Kawashima, Large-time behavior of solutions of the discrete Boltzmann equation, Comm. Math. Phys., 109 (1987), 563-589.

[10] S. Kawashima, Initial-boundary value problem for the discrete Boltzmann equation, Journées "Equations aux Dérivées Partielles", Ecole Polytechnique, 1988.

[11] S. Kawashima, Global solutions to the initial-boundary value problems for the discrete Boltzmann equation, submitted to Nonlinear Analysis.

[12] S. Kawashima, Existence and stability of stationary solutions to the discrete Boltzmann equation, in preparation.

[13] Y. Shizuta and S. Kawashima, Systems of equations of hyperbolic-parabolic type with applications to the discrete Boltzmann equation, Hokkaido Math. J., 14 (1985), 249-275.

[14] D.R. Smart, Fixed Point Theorems, Cambridge University Press, New York, 1974.

[15] L. Tartar, Some existence theorems for semilinear hyperbolic systems in one space variable, MRC Technical Summary Report, University of Wisconsin, 1980.