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ON $L^2(\mathbb{R}^n)$ WELL-POSEDNESS FOR SOME SINGULAR OR DEGENERATE HYPERBOLIC EQUATIONS

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INTRODUCTION

In this note, we consider the $L^2(\mathbb{R}^n)$ well-posedness of the Cauchy problem for the following singular or degenerate partial differential equation

$$\left.\begin{aligned} &(0.1) \quad \partial_t^2 u - \mathcal{A}(t)u = f(t,x) \quad \text{in} \quad [-1,1]\setminus\{0\} \times \mathbb{R}^n, \\ &(0.2) \quad u(t_0) = u_0, \quad \partial_t u(t_0) = u_1 \quad \text{in} \quad \mathbb{R}^n, \end{aligned}\right\} \text{(CP)}$$

where $\partial_t = \frac{\partial}{\partial t}$, $\mathcal{A}(t)$ is a $2m$-th order elliptic pseudo-differential operator which is singular or degenerates at $t = 0$, and t_0 is an arbitrary point of $[-1,1]$.

The C^∞ or H^∞ well-posedness of (CP) is studied by various authors. The sufficient conditions are obtained by Protter [8], Oleinik [7], Ivrii [4], Imai [3], Segala [9], Tahara [10], Tarama [11] and Yamazaki [13], (see also the references in [12], [13] and [14]). But their estimates include some loss of spacial regularities at the singular or degenerate point $t = 0$. Thus in order to obtain a solution $u(t,\cdot)$ in the Sobolev space $H^s(\mathbb{R}^n)$, one has to take the initial data in a smaller Sobolev space $H^{s'}(\mathbb{R}^n)$ with $s' > s$.

On the other hand, in [12] and [14] we proved the $L^2(\mathbb{R}^n)$ well-posedness without loss of regularities, when

the principal part is a product of a function depending only on t and a non-negative self-adjoint operator independent of t. The purpose of this note is to give a sufficient condition under which (CP) has a unique solution which conserves spacial regularity over singular or degenerate point $t = 0$. That is, the solution belongs to the same Sobolev space $H^s(\mathbb{R}^n)$ before and after the singular or degenerate point. This generalizes the results of [12] and [14] to the case where the principal part cannot be written as a product as in these papers. At the same time, we employ general functions measuring the singularity or degeneracy of the principal part, and relax the assumptions on the lower order terms imposed in [14]. The proof is based on the energy estimate, which has the same spacial regularity before and after the singular or degenerate point.

Detail will appear in [15].

§1 Preliminary

In this section, we provide notations, definitions and some lemmas on pseudo-differential operators.

Let $\mathbb{R}^+$ denote the interval $(0,\infty)$. Let $\mathfrak{X}$ and $\mathfrak{Y}$ be Banach spaces. For an operator $\mathcal{A}$ from $\mathfrak{Y}$ to $\mathfrak{X}$, the operator norm of $\mathcal{A}$ is defined by $\|\mathcal{A}\|_{\mathfrak{Y},\mathfrak{X}} = \sup\{\|\mathcal{A}\|_{\mathfrak{X}} : y \in \mathfrak{Y}, \|y\|_{\mathfrak{Y}} = 1\}$, which may be ∞. Let $B(\mathfrak{Y},\mathfrak{X})$ denote the space of all bounded operators from $\mathfrak{Y}$ to $\mathfrak{X}$. Let $AC([-1,1];\mathfrak{X})$ denote the set of $\mathfrak{X}$-valued absolutely continuous functions on the interval $[-1,1]$.

For a multi-index $\alpha = (\alpha_1,\cdots,\alpha_n) \in \mathbb{N}^n$ and

$x = (x_1, \cdots, x_n)$, $\xi = (\xi_1, \cdots, \xi_n) \in \mathbb{R}^n$, we put $|\alpha| = \alpha_1 + \cdots + \alpha_n$, $\alpha! = \alpha_1! \cdots \alpha_n!$, $x \cdot \xi = x_1\xi_1 + \cdots + x_n\xi_n$ and $\langle\xi\rangle = (|\xi|^2+1)^{1/2}$.

For the coordinate variable $x = (x_1, \cdots, x_n) \in \mathbb{R}^n$, let dx denote the Lebesgue measure on $\mathbb{R}^n$, and put $\bar{d}x = (2\pi)^{-n}dx$. We omit the domain of integration if it is the whole space $\mathbb{R}^n$. We put $\partial_{x_j} = \partial/\partial x_j$, $\partial_x^\alpha = (\partial_{x_1})^{\alpha_1} \cdots (\partial_{x_n})^{\alpha_n}$, $i = \sqrt{-1}$, $D_{x_j} = -i\ \partial/\partial x_j$ and $D_x^\alpha = (D_{x_1})^{\alpha_1} \cdots (D_{x_n})^{\alpha_n}$.

Let $\mathcal{S}$ (resp. $\mathcal{S}'$) denote the space of rapidly decreasing functions (resp. tempered distributions) on $\mathbb{R}^n$. For $u(x)$, $v(\xi) \in \mathcal{S}'$, put $\hat{u}(\xi) = \int e^{-ix\cdot\xi}u(x)dx$, the Fourier transform of $u(x)$, and $\mathcal{F}^{-1}[v] = \int e^{ix\cdot\xi}v(\xi)\bar{d}\xi$, the inverse Fourier transform of $v(\xi)$.

The norm of the Sobolev space $H^s(\mathbb{R}^n)$ $(s \in \mathbb{R})$ is denoted by $\|\cdot\|_s$, and the inner product of $L^2(\mathbb{R}^n)$ is denoted by $(\cdot,\cdot)$. We abbreviate $H^s(\mathbb{R}^n)$ to H^s, and $L^2(\mathbb{R}^n)$ to L^2, respectively.

DEFINITION 1.1. Let $\lambda(\xi)$ be a positive real-valued C^∞-function on $\mathbb{R}^n_\xi$. We say that a C^∞-function $\varphi(x,\xi)$ on $\mathbb{R}^{2n}_{x,\xi}$ is *a symbol of class* $S^k(\lambda)$, if for every non-negative integer ℓ,

$$|\varphi|^{(k)}_{\lambda,\ell} = \max_{|\alpha+\beta|\le\ell} \sup_{(x,\xi)} |\partial_\xi^\alpha D_x^\beta \varphi(x,\xi)|\lambda(\xi)^{-1}\langle\xi\rangle^{-k+|\alpha|}$$

is finite. In particular if $\lambda(\xi) \equiv 1$, we write

$$S^k(\lambda) = S^k \quad \text{and} \quad |\varphi|^{(k)}_{\lambda,\ell} = |\varphi|^{(k)}_{\ell}.$$

If $k = 0$, then we write $S^0(\lambda) = S(\lambda)$.

DEFINITION 1.2. Let $\lambda(t,\xi)$ be a symbol for each $t \in [-1,1]$. Let $\varphi(t,x,\xi)$ be a symbol of class $S^k(\lambda(t))$ for each $t \in [-1,1]$. Then we say that φ belongs to $BS^k(\lambda)$ if $\varphi(t,x,\xi)$ is a symbol of class $S^k(\lambda(t))$ for each t and $\sup_{t\in[-1,1]} |\varphi(t)|^{(k)}_{\lambda(t),\ell} < \infty$. If $k = 0$, then we write $BS^0(\lambda) = BS(\lambda)$.

DEFINITION 1.3. Let $\varphi(x,\xi)$ be a symbol. *The pseudo-differential operator with symbol* $\varphi(x,\xi)$ (resp. *with Weyl symbol* $\varphi(x,\xi)$) is defined by the formula

$$\varphi(x,D_x)u(x) = \int e^{ix\cdot\xi}\varphi(x,\xi)\hat{u}(\xi)\bar{d}\xi,$$

$$(\text{resp. } \varphi^w(x,D_x)u(x) = \int e^{i(x-y)\cdot\xi}\varphi((x+y)/2,\xi)u(y)dy\bar{d}\xi,)$$

for $u \in \mathcal{S}$.

See Hörmander [2], for the Weyl symbol calculus of pseudo-differential operators.

We can show the next two lemmas by using Gårding's inequality.

LEMMA 1.1. *For an arbitrary non-negative integers* k *and* k' *with* $k \le k'$, *there exist positive constants* C *and* ℓ *for which the following holds. Let* $\lambda(\xi)$ *be a positive real-valued symbol such that*

$$\lambda(\xi) \in S(\lambda) \quad \textit{and} \quad \lambda(\xi),\ \lambda(\xi)^{-1} \in S^{k'}.$$

Let $\varphi(x,\xi) = \varphi_1(x,\xi) + \varphi_2(x,\xi)$ *be a symbol in* $S^{k'}$ *such that*

$$\varphi_1(x,\xi) \in S^{2k}(\lambda^2), \quad \mathrm{Re}\ \varphi_1(x,\xi) \ge 0, \quad \varphi_2(x,\xi) \in S^{2k-1}(\lambda^2).$$

Let $\Phi = \varphi(x,D_x)$ *or* $\varphi^w(x,D_x)$. *Then*

$$\mathrm{Re}(\Phi u,u) \geq -C\{|\varphi_1\lambda^{-2}|_\ell^{(2k)} + |\varphi_2\lambda^{-2}|_\ell^{(2k-1)} + |\varphi|_\ell^{(k')}(|\lambda^{-1}|_\ell^{(k')})^2\}\|\lambda(D_x)u\|_{k-(1/2)}^2,$$

for every $u \in \mathcal{S}$.

LEMMA 1.2. *For arbitrary non-negative integers* k *and* k' *with* $k \leq k'$, *there exist positive constants* C *and* ℓ *for which the following holds. Let* $\lambda_1(\xi)$ *and* $\lambda_2(\xi)$ *be positive real-valued symbols such that*

$$\lambda_2(\xi) \in S(\lambda_2) \quad \text{and} \quad \lambda_1(\xi),\ \lambda_2(\xi),\ \lambda_2(\xi)^{-1} \in S^{k'}.$$

Let $\psi(x,\xi)$ *be a symbol such that*

$$\psi(x,\xi) \in S^{2k}(\lambda_1^2), \quad |\psi(x,\xi)| \leq \lambda_1(\xi)^2\langle\xi\rangle^{2k},$$

and let $\Psi = \psi(x,D_x)$ *or* $\psi^w(x,D_x)$. *Then we have*

$$|(\Psi u,v)| \leq \frac{1}{2}(\|\lambda_1(D_x)\lambda_2(D_x)u\|_k^2 + \|\lambda_1(D_x)\lambda_2^{-1}(D_x)v\|_k^2)$$

$$+ C\{|\psi\lambda_1^{-2}|_\ell^{(2k)} + |\psi|_\ell^{(k')}(|\lambda_1^{-1}|_\ell^{(k')})^2|\lambda_2|_\ell^{(k')}|\lambda_2^{-1}|_\ell^{(k')}\}$$

$$\times\ (\|\lambda_1(D_x)\lambda_2(D_x)u\|_{k-1/2}^2 + \|\lambda_1(D_x)\lambda_2^{-1}(D_x)v\|_{k-1/2}^2),$$

for every $u,\ v \in \mathcal{S}$.

§2. Result

Let $\mathcal{A}(t)$ be a $2m$-th order pseudo-differential operator in $\mathbb{R}^n$ represented by the Weyl symbol

$$\mathcal{A}(t) = a^w(t,x,D_x) = a_{2m}^w(t,x,D_x) + \sum_{j=0}^{m} a_j^w(t,x,D_x),$$

where

$$a_j \in S^j \quad (j = 0,1,\cdots,m,2m),\ D_x = (-i\partial_{x_1},\cdots,-i\partial_{x_n}).$$

We put the following assumption on a_{2m}:

(A2m) For every $(t,x,\xi) \in [-1,1]\setminus\{0\} \times \mathbb{R}^n_x \times \mathbb{R}^n_\xi$, $a_{2m}(t,x,\xi)$ is real-valued, and the function: $t \to a_{2m}(t,x,\xi)$ is an S^{2m}-valued C^2-class function on $[-1,1]\setminus\{0\}$ satisfying the following:

$$\partial_{x_j} a_{2m}(t,x,\xi) \in S^{m+1},$$

$$a_{2m}(t,x,\xi) \geq \eta(t)\langle\xi\rangle^{2m},$$

$$|\partial_t^k a_{2m}(t)|_\ell^{(2m)} + |\partial_t^k \partial_{x_j} a_{2m}(t)|_\ell^{(m+1)} \leq M_{a,\ell} t^{-k}\eta(t),$$

for every $t \in [-1,1]\setminus\{0\}$, $\ell = 0,1,\cdots$, $j = 1,2,\cdots,n$, and $k = 0,1,2$, where $M_{a,\ell}$ is a positive constant, and η is an integrable even function on $(-1,1)$ such that

(η) $\quad M_\eta^{-1}\eta(t) \leq t^{-1}\int_0^t \eta(s)ds \leq M_\eta \eta(t)$ for $0 < t \leq 1$,

with some positive constant M_η.

We easily see that if η is absolutely continuous on $(0,1)$ and if there are constants C_1, C_2 and δ such that $C_1 < 1$ and

(η)' $\quad -C_1 \leq \eta(t)^{-1}\eta'(t)t \leq C_2$ for a.e. t on $(0,\delta)$,

then η is integrable and satisfies (η).

We put $H^\kappa(\lambda) = \{u;\ \lambda(D_x)u \in H^\kappa(\mathbb{R}^n)\}$, for a real number κ and a C^∞-function $\lambda(\xi)$. Next, we define positive C^∞-function $\zeta(\tau(\xi))$ by (4.1) and (4.5) in section 4, and put

$$\pi_t^\kappa = \begin{cases} H^{m/2+\kappa}(\mathbb{R}^n) \times H^{-m/2+\kappa}(\mathbb{R}^n) & \text{if } t \neq 0, \\ H^{m/2+\kappa}(\zeta(\tau(\xi))^{-1/2}) \times H^{-m/2+\kappa}(\zeta(\tau(\xi))^{1/2}) & \text{if } t = 0. \end{cases}$$

In particular, if $\eta(t) = |t|^{\alpha}$ $(\alpha > -1)$, then π_0^{κ} equals $H^{m/2-\gamma+\kappa}(\mathbb{R}^n) \times H^{-m/2+\gamma+\kappa}(\mathbb{R}^n)$, where $\gamma = m\alpha/(2\alpha+4)$.

We put the assumptions on the lower order terms:

(Aj) For each $j = 0,1,\cdots,m$, the function $t \to a_j(t,x,\xi)$ is an S^j-valued continuous function on $[-1,1]\setminus\{0\}$ and for every $\ell = 0,1,\cdots$, it satisfies

$$|a_j(t)|_{\ell}^{(j)} \le M_{aj,\ell}\eta_j(t),$$

where $M_{aj,\ell}$ is a positive constant, and η_j is a nonnegative integrable function on $(-1,1)$ such that

(ηj) $$\operatorname*{ess\,sup}_{-1<t<1} |t|^{1-(j/m)}\eta(t)^{-j/(2m)}\int_0^t \eta_j(s)ds < \infty$$

for $j = 0,1,\cdots,m-1,m$, and furthermore,

(ηm*) $$\eta(t)^{-1/2}\eta_m(t) \in L^1(-1,1),$$

Now we state our result.

THEOREM. *Assume the above conditions* (Aj) *for* $j = 0,1,\cdots,m,2m$. *Let* κ *be an arbitrary real number. Let* $f \in L^1(-1,1;H^{m+\kappa}(\mathbb{R}^n))$ *be a function satisfying* (f) *or* (f)$'$;

(f) $$f \in C([-1,1]\setminus\{0\};H^{m/2+\kappa}(\mathbb{R}^n)),$$

(f)$'$ $$f \in C^1([-1,1]\setminus\{0\};H^{-m/2+\kappa}(\mathbb{R}^n)).$$

Then for every $(u_0,u_1) \in \pi_{t_0}^{\kappa+m}$, (CP) *has a unique solution* u *in the following sense,*

(i) $(u(t), \partial_t u(t)) \in \pi_t^{\kappa+m}$ *for every* $-1 \le t \le 1$,

(ii) $u \in AC([-1,1];H^{\kappa}(\mathbb{R}^n))$

$$\cap \bigcap_{j=0}^{2} C^j([-1,1]\setminus\{0\};\ H^{(3/2-j)m+\kappa}(\mathbb{R}^n)),$$

(iii) $\partial_t u \in AC([-1,1];H^{-3m/2+\kappa}(\mathbb{R}^n))$,

(iv) (0.1) *holds in* $H^{\kappa-m/2}(\mathbb{R}^n)$ *for every* $t \in [-1,1]\setminus\{0\}$,

(v) (0.2) *holds.*

REMARK 2.1. In the proof, we essentially use the estimates of only a finite number of seminorms of symbols. Hence it suffices to assume (Aj) ($j = 0,1,\cdots,m,2m$) for some integer ℓ depending only on n and m.

The above theorem asserts that the solution u and its time derivative $\partial_t u$, which start before the singular or degenerate point $t = 0$, belong to the same Sobolev spaces $H^{m+\kappa}(\mathbb{R}^n)$ and $H^{\kappa}(\mathbb{R}^n)$, respectively, after $t = 0$. Hence the spacial regularity of solution is conserved over the degenerate point. Furthermore note that the sum of the regularity of each space of π_t^{κ} is the same, since $H^{-m/2+\kappa}(\mathbb{R}^n)$ and $H^{-m/2+\kappa}(\xi(\tau(\xi))^{1/2})$ are the dual space of $H^{m/2+\kappa}(\mathbb{R}^n)$ and $H^{m/2+\kappa}(\xi(\tau(\xi))^{-1/2})$, respectively in H^{κ}. Thus the sum of the spacial regularity of the solution and that of its time derivative is conserved for all t.

Next we mention the necessity of (Am), a condition on the lower order terms. Assume that $m = 1$, and $\eta(t) = t^2$. $\eta(t) = t^2$. Then (η1) and (η1*) are satisfied if and only if $t^{-1}\eta_1(t)$ is integrable. We cannot relax this condition to $t^{-1}\eta_1(t) \le Ct^{-1}$ ($C > 0$), that is, $\eta_1(t) \le C$. In fact even for a simple case $n = 1$,

$a_2(t,x,D_x) = -t^2\partial_x^2$ and $a_1(t,x,D_x)$ is a positive constant (and thus η_1 is a positive constant), we see from Chi Min-You [1] or Ivrii and Petkov [5] that we cannot determine π_0^k, the space to which the solution belongs, only from the principal part a_{2m}. [1] gave the expression of the unique solution of (CP) with $a_0 \equiv f \equiv u_1 \equiv 0$ and $a_1 = 4N + 1$, where N is a non-negative integer), and this expression tells that the regularity loss at $t = 0$ tends to infinity as N becomes infinity. [5] obtained the necessary condition of the well-posedness of (CP) in a general form, and it follows from their result that the regularity loss of the mapping from f to u, a solution of (CP) with $u_0 \equiv u_1 \equiv 0$, tends to infinity as a_1 ($\equiv$ a constant) becomes infinity.

EXAMPLE. We give some examples of the function η which satisfies the condition of the theorem;

(2.1) $\eta(t) = |t|^\alpha(1-\log|t|)^\beta \qquad (\alpha > -1, \quad \beta \in \mathbb{R}),$

(2.2) $\eta(t) = |t|^\alpha\{|\sin(\log(2-\log|t|))| + \frac{\beta}{1-\log|t|}\}$

$$(\alpha > -1, \quad (\alpha+1)\beta > 1),$$

(2.3) $\eta(t) = \{(1-\log|t|)^\alpha|\sin(\log(2-\log|t|))|$

$$+ \gamma(1-\log|t|)^{\alpha-1}\}^\beta \qquad (\alpha, \beta \in \mathbb{R}, \gamma > |\beta|).$$

Here we note that if η is a function defined by (2.2) with $0 < \alpha < 1$ or by (2.3), then no monotone function $\psi(t)$ satisfies $\psi(t) \le \eta(t) \le C\psi(t)$ on $(0,\delta)$ for any positive constants δ and C. The function η defined by (2.3) oscillates between $(1-\log|t|)^{(\alpha-1)\beta}$ and $(1-\log|t|)^{\alpha\beta}$. When $0 < \alpha < 1$, this gives an example of

$\eta(t)$ with no positive upper or lower bound in any neighborhood of the origin.

REMARK 2.2. In [12] and [14], the above problem is treated when $a_{2m}(t,x,D_x)$ has a special form $\varphi(t)\Lambda$, where Λ is a non-negative self-adjoint operator independent of t, and $\eta(t) = |t|^{\alpha}$ $(\alpha > -1)$. In that case, we obtained the energy estimate by using the spectral decomposition of Λ together with an abstract theorem shown in these papers. But here we cannot use the same method, because $a_{2m}(t,x,D_x)$ has different spectral decomposition at each t, and we use the pseudo-differential calculus and Gårding's inequality instead, together with the abstract theorem. Besides, we give a general method to construct $\xi(\tau(\xi))$ in π_t^{κ}, in order to treat more general $\eta(t)$.

§3 Abstract linear evolution equations

In this section, we state a slight modification of a theorem in [14], adapted to the Cauchy problem (CP). The statement here is simpler than that in [14].

Let X, Y and Z be Banach spaces with norm $\|\cdot\|_X$, $\|\cdot\|_Y$ and $\|\cdot\|_Z$ respectively. Let $\{A(t);-1\le t\le 1\}$ be a family of linear operators in Z, and $F(t)$ be a Z-valued function on $[-1,1]$ and belongs to Y a.e. t on $(-1,1)$. We consider a linear evolution equation in Z :

$$\left.\begin{aligned} &\frac{d}{dt}U(t) + A(t)U(t) = F(t) \quad \text{for } -1 \le t \le 1, \\ &U(t_0) = U_0, \end{aligned}\right\} \text{(ACP)}$$

where t_0 is an arbitrary number with $-1 \le t_0 \le 1$, and $A(t)$ is singular or degenerate at $t = 0$.

DEFINITION 2.1. Let $\{W_t; -1 \le t \le 1\}$ be a family of Banach spaces in a Banach space Z with norms $\{\|\cdot\|_{W_t}\}$. We say that $\|\cdot\|_{W_s}$ is *differentiable with respect to* s *at* t if the following holds: W_{t+h} equals W_t as a linear space for sufficiently small $|h|$ with $t+h \in [-1,1]$, and $(\|x\|_{W_{t+h}} - \|x\|_{W_t})/h$ is convergent as h tends to 0, uniformly with respect to x in each bounded subset of W_t. The limit above is denoted by $\frac{d}{dt}\|\cdot\|_{W_t}$.

We describe the assumptions throughout this section. Let $\{X_t; -1 \le t \le 1\}$ and $\{Y_t; -1 \le t \le 1\}$ be families of Hilbert spaces in Z such that Y_t is densely imbedded in X_t for each t, and that X_t (resp. Y_t) is equivalent to X (resp. Y) for each $t \in [-1,1]\setminus\{0\}$. Let $(\cdot,\cdot)_{X_t}$ (resp. $(\cdot,\cdot)_{Y_t}$) denote the inner products, and let $\|\cdot\|_{X_t}$ (resp. $\|\cdot\|_{Y_t}$) denote the norms of X_t (resp. Y_t).

(S.1) There are constants C_i $(i = 1,2,3)$ and $\theta \in (0,1]$ such that

$$\|\cdot\|_Z \le C_1\|\cdot\|_{X_t} \le C_2\|\cdot\|_{Y_t} \quad \text{and} \quad \|\cdot\|_{X_t} \le C_3\|\cdot\|_{Y_t}^{1-\theta}\|\cdot\|_Z^{\theta},$$

for $-1 \le t \le 1$.

(S.2) If t_n tends to $t \in [-1,1]$ from the left and if $\{y_n \in Y_{t_n}\}$ is a sequence such that

$\sup_n \|y_n\|_{Y_{t_n}} < \infty$ and y_n converges to y in the topology of Z, then y belongs to Y_t and satisfies

$$\|y\|_{X_t} \le \limsup_{n\to\infty} \|y_n\|_{X_{t_n}} \quad \text{and} \quad \|y\|_{Y_t} \le \limsup_{n\to\infty} \|y_n\|_{Y_{t_n}}.$$

(S.3) Let $\mathbf{I}$ be a closed interval in $(-1,1)\setminus\{0\}$, and let B_X (resp. B_Y) be a bounded set in X (resp. Y). Then the map $s \to \|x\|_{X_s}$ (resp. $\|x\|_{Y_s}$) is differentiable, and it satisfies

$$\sup\{\tfrac{d}{ds}\|x\|_{X_s} (\text{resp. } \tfrac{d}{ds}\|x\|_{Y_s}) \ ; \ x\in B, \ s\in\mathbf{I}\} < \infty.$$

(S.4) For every $\varepsilon > 0$, if $h > 0$ is sufficiently small, then there exists a linear operator $P \in B(X_0,X) \cap B(Y,Y_0)$ satisfying

$$\|P\|_{X_0,X_h}, \ \|P\|_{Y_0,Y_h} < C \quad \text{and} \quad \|I-P\|_{Y_0,Z} < \varepsilon,$$

where I is the identity mapping on Y_0 and C is a constant independent of ε and h.

(A.1) For each $t \in [-1,1]\setminus\{0\}$, the operator $A(t)$ is a closed operator in X with domain $D(A) \equiv Y$, and the resolvent set of $A(t)$ includes $(-\infty,c_0)$ for some real number c_0.

(A.2) (Weak stability condition) There is an integrable function ω on $(-1,1)$ which is continuous on $[-1,1]\setminus\{0\}$ and satisfies the following inequalities for every $t \in [-1,1]\setminus\{0\}$, $x, y \in Y_t$ with $A(t) \in Y_t$

$$\frac{d}{dt}\|x\|^2_{X_t} \le 2\mathrm{Re}(A(t)x,x)_{X_t} + \omega(t)\|x\|^2_{X_t},$$

$$\frac{d}{dt}\|y\|^2_{Y_t} \le 2\mathrm{Re}(A(t)y,y)_{Y_t} + \omega(t)\|x\|^2_{Y_t}.$$

(A.3) For each fixed $t \in [-1,1]\setminus\{0\}$ and $y \in Y$, the map: $s \to A(s)y$ is continuously differentiable at t in the topology of X.

(A.4) The norm $\|A(t)\|_{Y_t,X_t}$ is dominated by some integrable and continuous function on $[-1,1]\setminus\{0\}$.

(F) The map $t \to F(t)$ is Y-strongly continuous on $[-1,1]\setminus\{0\}$, and the norm $\|F(t)\|_{Y_t}$ is dominated by some integrable function on $(-1,1)$.

(F)′ The map $t \to F(t)$ is X-strongly continuously differentiable on $[-1,1]\setminus\{0\}$, and the norm $\|F(t)\|_{Y_t}$ is dominated by some integrable function on $(-1,1)$.

From the theorem of Kato [6] for nonsingular evolution equation and the theorem in [14] for singular one, the next theorem follows.

THEOREM A. *Assume* (S.1) ~ (S.4), (A.1) ~ (A.4), *and either* (F) *or* (F)′. *Then for every* $y \in Y_s$, *the Cauchy problem* (ACP) *has a unique solution*

$$U \in C([-1,1]\setminus\{0\};Y) \cap C^1([-1,1]\setminus\{0\};X) \cap C^1([-1,1];Z)$$

with $\sup_{-1\le t\le 1} \|U(t)\|_{Y_t} < \infty$ *in the following sense:*

(i) $U(t) \in Y_t$ *for every* $-1 \le t \le 1$, *and* $U(t_0) = U_0$,

(ii) *the equation* (2.1) *holds in* X *for every* t *in* $[-1,1]\setminus\{0\}$. *Furthermore,* U *belongs to* $AC([-1,1];Z)$.

§4. Sketch of the proof

In the proof, making use of the symbol calculus of pseudo-differential operators, we construct a family of

t-dependent norms with which the abstract theorem given in section 2 is applicable to the system obtained by rewriting the original equation.

We first define a smooth approximate function $\xi(t)$ of $\eta(t)^{-1/2}$.

DEFINITION. *Let* ρ *be a function in* $C_0^\infty(\mathbb{R}^+)$ *such that*

$$\rho(t) > 0 \quad on \quad (\tfrac{1}{2},2) \quad and \quad \rho(t) \equiv 0 \quad on \quad \mathbb{R}^+\setminus(\tfrac{1}{2},2),$$

$$\int_{1/2}^{2} \rho(t)\frac{dt}{t} = 1.$$

Put

$$\tilde{\eta}(t) = \begin{cases} \frac{1}{t}\int_0^t \eta(s)ds & \text{for} \quad 0 < t \le 1, \\ \tilde{\eta}(1)/t & \text{for} \quad t > 1, \end{cases}$$

$$(4.1) \qquad \xi(t) = \begin{cases} \int_{1/2}^{2} \tilde{\eta}(t/s)^{-1/2}\rho(s)(ds/s) & \text{for} \quad t > 0, \\ \xi(-t) & \text{for} \quad t < 0. \end{cases}$$

Then the function ξ belongs to $C^\infty(\mathbb{R}\setminus\{0\})$, and satisfies the following estimates;

$$(4.2) \qquad M_0^{-1}\tilde{\eta}(t)^{-1/2} \le \xi(t) \le M_0\tilde{\eta}(t)^{-1/2} \quad for \quad 0 < t \le 1,$$

$$(4.3) \qquad M_0^{-1}\eta(t)^{-1/2} \le \xi(t) \le M_0\eta(t)^{-1/2} \quad for \quad 0 < t \le 1,$$

$$\xi(s) \le M_0\xi(t)$$

$$for \quad s,t \in \mathbb{R}\setminus\{0\} \quad with \quad 1/2 \le |t/s| \le 2,$$

$$(4.4) \qquad t\xi'(t) \le \tfrac{1}{2}(1 - \tfrac{1}{M_\eta})\xi(t) \le \tfrac{1}{2}\xi(t) \qquad for \quad t > 0,$$

$$|t^k\xi^{(k)}(t)| \le M_k\xi(t) \quad (k = 1,2,\cdots) \quad for \quad t > 0,$$

where M_k $(k = 0,1,2,\cdots)$ *are positive constants.*

REMARK 4.1. From (4.4), it follows that $\xi(t)/t$ is a strictly decreasing C^∞-function on $\mathbb{R}^+$.

DEFINITION. Put

$$b(t,x,\xi) \equiv a_{2m}(t,x,\xi)^{-1/2}.$$

Then by assumption (A2m) and (4.2), we easily see that

$$\partial_t^k b(t,x,\xi) \in BS^{-m}(t^{-k}\xi(t)) \quad (k = 0,1,2),$$

and

$$b(t,x,\xi) \geq M_{a,0}^{-1/2} M_0^{-1} \xi(t) \langle\xi\rangle^{-m},$$

for every $t \in \mathbb{R}$. We put

$$M_b = \max\{M_{a,0}^{1/2} M_0,\ |\partial_t^k b(t,x,\xi)|^{(-m)}_{t^{-k}\xi(t),0} \ (k = 0,1)\}.$$

Since η is integrable, it follows from the assumption (η) that $\eta(t) \leq Ct^{-1}$ for some positive constant C. Thus by (4.3),

$$\xi(t) \geq C^{-1} t^{1/2} \quad \text{for every } t > 0,$$

and therefore $\xi(t)/t$ tends to ∞ as $t \to 0+$. By the definition of $\tilde{\eta}$ and (4.1),

$$\sup_{t>1} \xi(t) < \infty,$$

and therefore $\xi(t)/t$ tends to 0 as $t \to \infty$. From these facts and the monotonicity of $\xi(t)/t$ (see Remark 4.1), we can uniquely determine a C^∞-function τ on $\mathbb{R}^n_\xi$ by

(4.5) $\quad M^* \xi(\tau(\xi))/\tau(\xi) = \langle\xi\rangle^m,$

where $M^* = 16M_b^2$. Since $\tau(\xi)$ is determined only by $\langle\xi\rangle$, we can define a C^∞-function $\tilde{\tau}$ on $[0,\infty)$ by $\tilde{\tau}(\langle\xi\rangle) = \tau(\xi)$. Then from the monotonicity of $\xi(t)/t$, it

follows that the function $\tilde{\tau}$ is monotone decreasing and

$$\tilde{\tau}(\langle\xi\rangle) = \tau(\xi) \to 0 \quad \text{as} \quad \langle\xi\rangle \to \infty.$$

LEMMA 4.1. *The function* τ *satisfies* $\tau \in S(\tau)$, *and there is a positive constant* M_τ *such that*

$$M_\tau^{-1}\langle\xi\rangle^{-2m} \le \tau(\xi) \le M_\tau \quad \textit{for every} \quad \xi \in \mathbb{R}^n.$$

DEFINITION. Let χ be a $C_0^\infty(\mathbb{R})$-function with $0 \le \chi(t) \le 1$ such that

$$\chi(t) = \begin{cases} 1 & \text{for } |t| \le 1 \\ 0 & \text{for } |t| \ge 2. \end{cases}$$

We define functions $\lambda_q(t,\xi)$ and $\lambda_p(t,\xi)$ on $[-1,1] \times \mathbb{R}^n$ and $\mathring{p}(t,\xi)$, $\mathring{q}(t,\xi)$ and $\mathring{r}(t,\xi)$ on $[-1,1] \times \mathbb{R}^{2n}$ by

$$\lambda_q(t,\xi) = \{\xi(t)(1-\chi(t/\tau(\xi))+\xi(\tau(\xi))\chi(t/\tau(\xi))\}\langle\xi\rangle^{-m},$$

$$\lambda_p(t,\xi) = \{\lambda_q(t,\xi)\}^{-1},$$

$$\mathring{p}(t,x,\xi)$$

$$= b(t,x,\xi)^{-1}(1-\chi(t/\tau(\xi))) + b(\tau(\xi),x,\xi)^{-1}\chi(t/\tau(\xi)),$$

$$\mathring{q}(t,x,\xi)$$

$$= b(t,x,\xi)(1-\chi(t/\tau(\xi))) + b(\tau(\xi),x,\xi)\chi(t/\tau(\xi)),$$

$$\mathring{r}(t,x,\xi) = -\tfrac{1}{2}\partial_t b(t,x,\xi)(1-\chi(t/\tau(\xi))).$$

LEMMA 4.2. (i) *The following estimates hold for every* (t,x,ξ) *in* $[-1,1] \times \mathbb{R}^{2n}$;

$$\frac{1}{2M_b}\lambda_p(t,\xi) \le \mathring{p}(t,x,\xi) \le (1+M_0)M_b\lambda_p(t,\xi),$$

$$\frac{1}{M_b}\lambda_q(t,\xi) \le \mathring{q}(t,x,\xi) \le M_b\lambda_q(t,\xi),$$

$$|\mathring{r}(t,x,\xi)| \le \frac{1}{32M_b}.$$

(ii) $\lambda_p,\ \mathring{p} \in BS(\lambda_p)$ *and* $\lambda_q,\ \mathring{q} \in BS(\lambda_q)$.

(iii) $\lambda_p,\ \mathring{p} \in BS^{3m}(1)$ *and* $\lambda_q,\ \mathring{q},\ \mathring{r} \in BS(1)$.

DEFINITION. We put

$$g_{2m}(t,\xi) = M_g[t^{-2}\zeta(t)\langle\xi\rangle^{-m}(1-\chi(2t/\tau(\xi)))$$
$$+ \tau(\xi)^{-1}\{1+\zeta(\tau(\xi))^2\zeta(t)^{-2}\}\chi(t/2\tau(\xi)) + \eta(t)],$$

$$g_j = M_g\eta_j(t)\langle\xi\rangle^j\lambda_q(t,\xi) \quad \text{for} \quad j = 0,1,\cdots,m,$$

where M_g is a large positive constant such that the following hold:

$$|(\partial_t\mathring{p}-2\mathring{r}a_{2m})(t,x,\xi)| \le \tfrac{1}{2}(\mathring{p}g_{2m})(t,x,\xi),$$

$$|(\partial_t\mathring{q}+2\mathring{r})(t,x,\xi)| \le \tfrac{1}{2}(\mathring{q}g_{2m})(t,x,\xi),$$

$$2|(\partial_t\mathring{r})(t,x,\xi)|+|(\mathring{p}-\mathring{q}a_{2m})(t,x,\xi)| \le \frac{1}{64M_b}g_{2m}(t,x,\xi),$$

$$|(\mathring{r}a_j)(t,x,\xi)| \le \frac{1}{32M_b}(\lambda_p g_j)(t,x,\xi) \quad (j = 0,1,\cdots,m),$$

$$|(\mathring{q}a_j)(t,x,\xi)| \le \frac{1}{32M_b}g_j(t,x,\xi) \qquad (j = 0,1,\cdots,m).$$

Then we define a function g by

$$g(t,\xi) = \Sigma_{j=0,1,\cdots,m,2m}g_j(t,\xi).$$

We can prove that the function $g(t,\xi)$ is integrable with respect to t on $(-1,1)$ and

$$\int_{-1}^{t} g(s,\xi)ds \in BS(1).$$

Then we define a function $G(t,\xi)$ by

$$G(t,\xi) = \exp(-\int_{-1}^{t} g(s,\xi)ds) \quad \text{on} \quad [-1,1] \times \mathbb{R}^n.$$

LEMMA 4.3. *The function* G *belongs to* $BS(1)$ *and satisfies*

$$\inf_{t,\xi} G(t,\xi) > 0.$$

DEFINITION. For a real number κ, put

$$\tilde{p}_\kappa(t,x,\xi) = (\mathring{p}G)(t,x,\xi)\langle\xi\rangle^{2\kappa},$$

$$\tilde{q}_\kappa(t,x,\xi) = (\mathring{q}G)(t,x,\xi)\langle\xi\rangle^{2\kappa},$$

$$\tilde{r}_\kappa(t,x,\xi) = (\mathring{r}G)(t,x,\xi)\langle\xi\rangle^{2\kappa},$$

for $(t,x,\xi) \in [-1,1]\setminus\{0\} \times \mathbb{R}^n_x \times \mathbb{R}^n_\xi$. Then from the above Lemmas 4.1 ~ 4.3 together with Lemmas 1.1 and 1.2, we have positive constants L_κ and L'_κ for each κ such that the following inequalities hold,

$$\begin{aligned}
&\frac{1}{4M_b}((\lambda_p G)(t,D_x)\langle D_x\rangle^{2\kappa}u,\ u) \\
&\quad \le (\tilde{p}^w_\kappa(t,x,D_x)u,\ u) + L_\kappa\|u\|^2_\kappa \\
&\quad \le L'_\kappa((\lambda_p G)(t,D_x)\langle D_x\rangle^{2\kappa}u,\ u), \\
&\frac{1}{2M_b}((\lambda_q G)(t,D_x)\langle D_x\rangle^{2\kappa}v,\ v) \\
&\quad \le (\tilde{q}^w_\kappa(t,x,D_x)v,\ v) + L_\kappa\|v\|^2_{\kappa-2m} \\
&\quad \le L'_\kappa((\lambda_q G)(t,D_x)\langle D_x\rangle^{2\kappa}v,\ v),
\end{aligned}$$

$$|(\tilde{r}_k^w(t,x,D_x)u,\ v)|$$

$$\leq \frac{1}{32M_b}\{((\lambda_p G)(t,D_x)u,\ u) + ((\lambda_q G)(t,D_x)v,\ v)\}$$

$$+ \frac{1}{8}L_\kappa(\|u\|_\kappa^2 + \|v\|_{\kappa-2m}^2),$$

for every $u,\ v \in \mathcal{S}$.

Put

$$p_\kappa(t,x,\xi) = \tilde{p}_\kappa(t,x,\xi) + L_\kappa\langle\xi\rangle^\kappa,$$

$$q_\kappa(t,x,\xi) = \tilde{q}_\kappa(t,x,\xi) + L_\kappa\langle\xi\rangle^{\kappa-2m},$$

$$r_\kappa(t,x,\xi) = \tilde{r}_\kappa(t,x,\xi).$$

Then from the above estimates, it follows that

$$(4.6)\quad \frac{1}{M}\{(\lambda_p(t,D_x)\langle D_x\rangle^{2\kappa}u,\ u) + (\lambda_q(t,D_x)\langle D_x\rangle^{2\kappa}v,\ v)\}$$

$$\leq \frac{1}{2}\{(p_\kappa^w(t,x,D_x)u,\ u) + (q_\kappa^w(t,x,D_x)v,\ v)\}$$

$$\leq (p_\kappa^w(t,x,D_x)u,\ u) + (q_\kappa^w(t,x,D_x)v,\ v)$$

$$+ 2\mathrm{Re}(r_\kappa^w(t,x,D_x)u,\ v),$$

$$\leq 2\{(p_\kappa^w(t,x,D_x)u,\ u) + (q_\kappa^w(t,x,D_x)v,\ v)\}$$

$$\leq M\{(\lambda_p(t,D_x)\langle D_x\rangle^{2\kappa}u,\ u) + (\lambda_q(t,D_x)\langle D_x\rangle^{2\kappa}v,\ v)\},$$

for every $\kappa \in \mathbb{R}$, $t \in [-1,1]$ and $(u,v)^t \in X_{t,\kappa}$, where M is a positive constant independent of t, κ and u, v. From this and the fact that operators $p_\kappa^w(t,x,D_x)$ and $q_\kappa^w(t,x,D_x)$ are symmetric in L^2, we can define a Hilbert space $X_{t,\kappa} \subset \mathcal{D}' \times \mathcal{D}'$ for $\kappa \in \mathbb{R}$ as the closure of $\mathcal{S} \times \mathcal{S}$ with respect to the norm $\|\cdot\|_{X_{t,\kappa}}$ which associates with the inner product $(\cdot,\cdot)_{X_{t,\kappa}}$ defined by

$$\left(\begin{pmatrix}u_1\\v_1\end{pmatrix}, \begin{pmatrix}u_2\\v_2\end{pmatrix}\right)_{X_{t,\kappa}} = (p_\kappa^w(t,x,D_x)u_1,u_2) + (q_\kappa^w(t,x,D_x)v_1,v_2)$$

$$+ (r_\kappa^w(t,x,D_x)u_1,v_2) + (v_1,r_\kappa^w(t,x,D_x)u_2),$$

for every u_1, u_2, v_1, $v_2 \in \mathcal{S}$.

REMARK 4.2. From (4.6) and the definition of λ_p and λ_q, we have the following equivalences as Banach spaces,

$$X_{t,\kappa} \sim H^{m/2+\kappa} \times H^{-m/2+\kappa} \quad \text{for} \quad t \in [-1,1]\setminus\{0\},$$

$$X_{0,\kappa} \sim H^{m/2+\kappa}(\xi(\tau)^{-1/2}) \times H^{-m/2+\kappa}(\xi(\tau)^{1/2}).$$

REMARK 4.3. By (4.6) and (iii) of Lemma 4.3, we have

$$H^{3m/2+\kappa} \times H^{\kappa} \subset X_{t,\kappa} \subset H^{\kappa} \times H^{-3m/2+\kappa},$$

and there are positive constants C_1 and C_2 such that

$$\|u\|_\kappa^2 + \|v\|_{-3m/2+\kappa}^2 \le C_1\left\|\begin{pmatrix}u\\v\end{pmatrix}\right\|_{X_{t,\kappa}}^2 \le C_2(\|u\|_{3m/2+\kappa}^2 + \|v\|_\kappa^2),$$

for every $\kappa \in \mathbb{R}$, $t \in [-1,1]$ and $(u,v)^t \in X_{t,\kappa}$.

DEFINITION. For every $\kappa \in \mathbb{R}$, we put

$$Z_\kappa = H^{\kappa} \times H^{-3m/2+\kappa}.$$

We can now prove the theorem. For the Banach spaces in Theorem A, we take

$$X_t = X_{t,\kappa}, \quad Y_t = X_{t,m+\kappa} \quad \text{for} \quad -1 \le t \le 1,$$

$$X = \pi_1^{\kappa} = H^{(m/2)+\kappa} \times H^{-(m/2)+\kappa},$$

$$Y = \pi_1^{m+\kappa} = H^{(3m/2)+\kappa} \times H^{(m/2)+\kappa}, \quad Z = Z_\kappa.$$

We note that by Remark 4.3 and definition, we have $Y_t \subset X_t \subset Z$ with

$$\|U\|_Z \le C_1\|U\|_{X_t} \le C_2\|U\|_{Y_t},$$

for every $t \in [-1,1]$ and $U \in Y_t$. By Remark 4.2, we have the following equivalences as Banach spaces.

$$Y_t \sim \pi_t^{m+\kappa} \quad \text{for every} \quad t \in [-1,1],$$

$$X_t \sim X \quad \text{and} \quad Y_t \sim Y \quad \text{for every} \quad t \in [-1,1]\setminus\{0\}.$$

Take $A(t)$ and $F(t)$ as

$$A(t) = \begin{pmatrix} 0 & -I \\ a(t,x,D_x) & 0 \end{pmatrix} \quad \text{and} \quad F(t) = \begin{pmatrix} 0 \\ f(t) \end{pmatrix}.$$

We can prove that the assumptions of Theorem A are satisfied, by using lemmas 1.1 and 1.2 with $\lambda(t,\xi)$ being the function of t, $\lambda_p(t,\xi)$ and $\lambda_q(t,\xi)$. Then the theorem gives the solution $U(t) = \begin{pmatrix} u(t) \\ \partial_t u(t) \end{pmatrix}$ of (ACP) with bounded Y_t-norm, and $u(t)$ becomes a solution of (CP). The condition "bounded Y_t-norm" is stronger than the regularity condition of the solutions in the theorem. But the uniqueness part of the theorem is easily proved in the same way as in the proof of Proposition 3.3 of [14]. Thus the theorem is proved.

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