数理解析研究所講究録
第 730 巻 1990 年 21-40

On the behavior of solutions of quasilinear heat conduction equations

TADASHI KAWANAGO (川中子 正)

§0. Introduction

We shall consider the behavior of weak solutions of the following initial-boundary value problem:

$$(D)\begin{cases} u_t = \Delta\phi(u) & \text{in} \quad \Omega\times\mathbb{R}^+ \\ u(x,t)=0 & \text{on} \quad \partial\Omega\times\mathbb{R}^+ \\ u(x,0)=u_0(x) & \text{in} \quad \Omega. \end{cases}$$

Here, $\Omega \subset \mathbb{R}^N (N \geq 1)$ is a bounded domain with smooth boundary $\partial\Omega$. We assume throughout that

(0.1) $\phi : \mathbb{R} \to \mathbb{R}$ is a strictly increasing, continuous function with $\phi(0) = 0$.

Many authors have studied the problem (D) under the conditions: $\phi'(0) = 0$ and $\phi'(r) > 0$ if $r \neq 0$ (i.e. (D) is degenerate only at $u = 0$). See Bertsch and Peletier [3] and the references in [3]. We are interested in the case when (D) is nondegenerate, with applications to the degenerate case. In this situation, (D) arises, for example, in heat flow through solids and in diffusion of moleculars in mediums. There are not many works for the nondegenerate case. Berryman and Holland [2] and Nagasawa [9] studied the large time behavior of classical solutions of equations related to (D) with the dimension $N = 1$. The author [7][8] studied that of weak solutions of (D) (with applications to the degenerate case). And Alikakos and Rostamian [1] and Bertsch and Peletier [4] inverstigated in order to apply the degenerate case.

In §1 we mention basic results about (D) including a definition of weak solutions of (D). In §2 we shall introduce some results in [7] and [8] for the nondegenerate case. In §3 we shall give an up-to-date result for the nondegenerate case. In §4 we apply the results in §2 to the case when (D) is degenerate. (The results in §4 are parts of [7] and [8].)

Remark 0.1. We can obtain similar results for zero-Neumann boundary value problem and for the following type of equation: $u_t = \sum_{i,j=1}^{N} \frac{\partial}{\partial x_i}(a^{ij}(x,u)\frac{\partial u}{\partial x_j})$, which we omit in this article for want of space. See [8] for the details.

Remark 0.2. For want of space, We shall assume in proofs of all results the additional conditions: $\phi(r) \in C^\infty(\mathbb{R})$ and $u_0 \in C_0^\infty(\Omega)$, which make the solution $u(x,t)$ smooth. To prove the results for general weak solutions $u(x,t)$, we need arguments on smoothing technique. For the details, see [5] , [7] and [8].

Notation.

1. $\|\cdot\|_p$ denotes the norm of $L^p(\Omega)$.
2. $(\cdot,\cdot)_2$ denotes the inner product in $L^2(\Omega)$.
3. We denote by $\{\lambda_\nu\}_{\nu=1}^{\infty}$ $(0 < \lambda_1 < \lambda_2 < \cdots)$ all eigenvalues of $-\Delta$ with Dirichlet condition and by $\{e_\nu^{(i)}\}_i$ the normal orthogonal basis of the eigenspace corresponding to λ_ν. If the eigenspace corresponding to λ_ν is one-dimensional, we simply denote by e_ν the normal orthogonal base of it. And we choose e_1 such that $e_1 \geq 0$.

§1. Preliminary

We shall briefly describe a definition of weak solution by nonlinear semigroup theory. We define operator $A : L^1(\Omega) \to L^1(\Omega)$ by

$$Au = -\Delta\phi(u) \quad \text{for} \quad u \in D(A)$$

with $D(A) = \{u \in L^1(\Omega);\ \phi(u) \in W_0^{1,1}(\Omega),\ \Delta\phi(u) \in L^1(\Omega)\}$. The operator A is m-accretive in $L^1(\Omega)$ under the condition (0.1). Therefore A generates the contraction semigroup $S_A(t)$. Hence we can define a unique weak solution of (D) by $S_A(t)u_0$ for any $u_0 \in \overline{D(A)} = L^1(\Omega)$. For the details, see [5] and [1].

We shall mention basic known properties of weak solutions of (D). (For the proof, see [1] and [5] for example.)

Proposition 1.1. *We assume that ϕ satisfies* (0.1). *If $u(x,t)$ is the weak solution of (D), then the following hold:*
(1) *(The maximum principle) For any $u_0 \in L^p(\Omega)$ with $p \in [1,\infty]$, $u(t) \in L^p(\Omega)$ for $t \geq 0$, and $\|u(t)\|_p$ is non-increasing.*
(2) *(The order-preserving property) If $u_0, v_0 \in L^1(\Omega)$ and $u_0 \leq v_0$, then $S(t)u_0 \leq S(t)v_0$ a.e. in Ω for any $t \in \mathbb{R}^+$. Here $S(t)u_0$ and $S(t)v_0$ denote respectively the solution corresponding to u_0 and v_0.*

§2. The nondegenerate case I

Theorem 2.1. *We assume (0.1) and the following:*
(2.1) $\phi^{-1} : \mathbb{R} \to \mathbb{R}$ is a uniformly Lipschitz continuous function with a Lipschitz constant $1/k_0$ ($k_0 > 0$).
Let $u(x,t)$ be the weak solution of (D). Then for any $u_0 \in L^2(\Omega)$, $u(t) \in L^\infty(\Omega)$ for $t > 0$ with the estimate:

$$\|u(t)\|_\infty \leq \frac{C_1}{(k_0 t)^{N/4}} \|u_0\|_2 \quad \text{for} \quad t > 0, \tag{2.2}$$

$$\|u(t)\|_\infty \leq C(N, k_0, t_0) e^{-\lambda_1 k_0 t} \|u_0\|_2 \quad \text{for} \quad t \geq t_0, \tag{2.3}$$

$$\text{where} \quad C(N, k_0, t_0) = \frac{C_1 e^{\lambda_1 k_0 t_0}}{(k_0 t_0)^{N/4}},$$

where $t_0 > 0$ is an arbitrary time, and $C_1 > 0$ depends only on N.

Proof of Theorem 2.1. With the aid of Stokes' Theorem,

$$\frac{d}{dt}\int_\Omega |u|^p dx = p\int |u|^{p-1}\text{sign}\, u \cdot u_t dx$$

$$= -p(p-1)\int |u|^{p-2}\nabla u \cdot \nabla\phi(u)dx$$

$$\le -p(p-1)k_0 \int |u|^{p-2}|\nabla u|^2 dx \tag{2.4}$$

$$\le -2k_0 \int_\Omega |\nabla |u|^{p/2}|^2 dx \quad \text{for} \quad p \in [2,\infty). \tag{2.5}$$

We can prove (2.2) with the aid of (2.5) and Sobolev's inequality:

$$\|f\|_{2N/(N-1)} \le C\|\nabla f\|_2^{1/2}\|f\|_2^{1/2} \quad \text{for any } f \in H_0^1(\Omega). \tag{2.6}$$

(Here we set $2N/(N-1) = \infty$ for $N = 1$.) Indeed when $N = 1$, we set $p = 2$ in (2.5) (or (2.4)) and integrate in t :

$$\|u(t)\|_2^2 - \|u_0\|_2^2 \le -2k_0 \int_0^t \|\nabla u(s)\|_2^2 ds. \tag{2.7}$$

It follows from (2.6), (2.7) and (1) of Proposition 1.1 that

$$\|u_0\|_2^2 \ge 2k_0 \int_0^t \left(\frac{\|u(s)\|_\infty^2}{C\|u(s)\|_2}\right)^2 ds \ge Ck_0 t \times \frac{\|u(t)\|_\infty^4}{\|u_0\|_2^2}, \tag{2.8}$$

which implies (2.2). When $N \ge 2$, the proof is essentially the same, but we need Moser's iteration technique, which is used in §4 of Evans [6]. We omit the details because the argument is the same as in [6].

Next we shall derive (2.3). Following Alikakos and Rostamian [1] (the proof of Theorem 3.3), we shall get L^2-decay estimate. By substituting $p = 2$ into (2.4) and by the spectral resolution of $-\Delta$,

$$\frac{d}{dt}\int u^2 dx \le -2k_0\lambda_1 \int u^2 dx.$$

Therefore, we obtain that

$$\|u(t)\|_2 \le e^{-k_0\lambda_1 t}\|u_0\|_2 \quad \text{for} \quad t \ge 0. \tag{2.9}$$

Hence, with the aid of (2.2) and (2.9),

$$\|u(t)\|_\infty \le \frac{C_1}{(k_0 t_0)^{N/4}} e^{-k_0\lambda(t-t_0)}\|u_0\|_2.$$

This implies (2.3). ∎

We assume below in this section that

(2.11) $\phi : \mathbb{R} \to \mathbb{R}$ is a strictly increasing, C^1-class function with $\phi(0) = 0$.

(2.12) There exists $k_0 > 0$ such that $k(r) = \phi'(r) \ge k_0$ for any $r \in \mathbb{R}$.

and assume for simplicity that

(2.13) $k(0) = 1$.

We shall describe the main result in this section:

Theorem 2.2. *We assume that ϕ satisfies (2.11), (2.12) and (2.13). Let $u(x,t)$ be the weak solution of (D) with $u_0 \in L^\infty(\Omega)$. We also assume that*

(2.14) there exsist $\theta > 0$ and $\rho > 0$ such that $k(r) \ge 1 - \theta/(-\log|r|)^{1+\rho}$ for any $r \in (-1,1)$.

Then, the following estimate holds:

$$\|u(t)\|_\infty \le Ce^{-\lambda_1 t} \quad \textit{for} \quad t \ge 0 \tag{2.15}$$

where $0 < C = C(N, \Omega, \|u_0\|_\infty, \theta, \rho, k_0)$.

Proof of Theorem 2.2. By Theorem 2.1,

$$\|u(t)\|_\infty \le \theta_1 e^{-\theta_2 t} \quad \text{for} \quad t \ge 0, \tag{2.16}$$

$$\|u(t)\|_\infty \le \theta_3\|u(t-t_0)\|_2 \quad \text{for} \quad t \ge t_0, \tag{2.17}$$

where $t_0 > 0$ is an arbitrary but fixed time, and $\theta_1, \theta_2, \theta_3 > 0$ are some constants. θ_1 depends only on $N, \Omega, \|u_0\|_\infty$ and k_0, θ_2 only on N, Ω and k_0, and θ_3 only on N, k_0 and t_0.

In view of (2.16), we may assume without loss of generality that $\|u_0\|_\infty$ and $\theta_1 > 0$ is sufficiently small. By (2.17), the proof is complete if we prove that

$$\|u(t)\|_2 \le \theta_4 e^{-\lambda_1 t} \quad \text{for} \quad t \ge 0, \tag{2.18}$$

where $\theta_4 = \theta_4(N, \Omega, \|u_0\|_\infty, \theta, \rho, k_0) > 0$. With the aid of (2.14),

$$\begin{aligned} \frac{d}{dt}\int_\Omega u^2 dx &= -2\int k(u)|\nabla u|^2 dx \\ &\le -2\int \{1 - \frac{\theta}{(-\log|u|)^{1+\rho}}\}|\nabla u|^2] dx. \end{aligned} \tag{2.19}$$

Since $(-\log r)^{-(1+\rho)} (0 \le r < 1)$ is an increasing function, we obtain that

$$\frac{1}{(-\log|u|)^{1+\rho}} \le \frac{1}{(-\log\|u\|_\infty)^{1+\rho}}. \tag{2.20}$$

It follows from (2.19), (2.20) and the spectral resolution that

$$\frac{d}{dt}\int_\Omega u^2 dx \le -2[1 - \frac{\theta}{(-\log\|u\|_\infty)^{1+\rho}}]\int |\nabla u|^2 dx$$

$$= -2[1 - \frac{\theta}{(-\log\|u\|_\infty)^{1+\rho}}]\sum_{j=1}^\infty \lambda_j (u, e_j)_2^2 \tag{2.21}$$

$$\le -2\lambda_1 [1 - \frac{\theta}{(-\log\|u\|_\infty)^{1+\rho}}]\int_\Omega u^2 dx, \tag{2.22}$$

It follows from (2.22) that

$$\|u(t)\|_2 \le \|u_0\|_2 \exp\{-\lambda_1 t + \int_0^t \frac{\lambda_1\theta}{(-\log\|u(s)\|_\infty)^{1+\rho}} ds\}. \tag{2.23}$$

Here, we obtain from (2.16) that

$$\int_0^t \frac{1}{(-\log\|u(s)\|_\infty)^{1+\rho}}ds \leq \int_0^t \frac{ds}{(-\log\theta_1+\theta_2 s)^{1+\rho}}. \tag{2.24}$$

The right-hand side of (2.24) is less than some constant depending on θ_1 and θ_2 because we may assume that $\theta_1 \in (0,1)$. Therefore (2.18) holds. ∎

We shall concider under what condition the estimate (2.15) with $\leq$ replaced by $\geq$ hold.

Proposition 2.1. *Assume that ϕ satisfies* (2.11), (2.12) and (2.13). *We also assume that*

(2.25) *there exsists $k_1 > 0$ such that $k(r) \leq k_1$ for any $r \in \mathbb{R}$,*

(2.26) *there exsist $\theta, \rho > 0$ such that $|k(r)-1| \leq \theta/(-log\mid r\mid)^{1+\rho}$ for any $r \in (-1,1)$,*

(2.27) *$u_0 \geq 0$, $u_0(x)$ does not identically vanish in Ω and $u_0 \in L^\infty(\Omega)$.*

Let $u(x,t)$ be the solution of (D). Then, the following estimate holds:

$$C_1 e^{-\lambda_1 t} \leq \| u(t) \|_\infty \leq C_2 e^{-\lambda_1 t} \quad \text{for} \quad t \geq 0, \tag{2.28}$$

where C_1, $C_2 > 0$ depend only on N, Ω, $\|u_0\|_\infty$, $(u_0, e_1)_2$, ρ, θ, k_0 and k_1.

Proof. We can obtain Proposition 2.1 from similar calculations as in the proof of Theorem 2.2. We omit the proof. For readers who want to know it, see Theorem 2.4 and its proof in [8].

Remark 2.1. The condition (2.26) is not technical. Indeed the left-hand side of (2.28) does not always hold without the condition (2.26). Indeed if $k(r) = 1 + 1/(-\log|r|)^\rho$ for some $\rho \in (0,1)$ and $\|u_0\|_\infty < 1$, then the corresponding solution $u(x,t)$ satisfies the following estimate:

$$\|u(t)\|_\infty \leq C\exp(-\lambda_1 t - (\lambda_1 t)^{1-\rho}) \quad \text{for} \quad t \geq 0. \tag{2.29}$$

To obtain (2.29), we have only to substitute $\varepsilon = C\exp\{-\lambda_1 t - (\lambda_1 t)^{1-\rho}\}$ into (4.4) of Proposition 4.1 in §4.

Remark 2.2 The estimate (2.28) does not always hold without the condition: $u_0 \geq 0$ ((2.27)). We give a counterexample (Remark 2.3 in [8]). Assume that $\phi : \mathbb{R} \to \mathbb{R}$ is a smooth odd function with $\phi' > 0$. We assume that $N = 1$, $\Omega = (0, \pi)$ and $u_0(x) = \sin mx$ ($m \in \mathbb{N}$). Let $u(x,t)$ be the solution of (D). Then the following estimate holds:

(2.30) $$C_1 e^{-m^2 k(0) t} \leq \|u(t)\|_\infty \leq C_2 e^{-m^2 k(0) t} \quad \text{for} \quad t \geq 0.$$

We shall derive (2.30). We define by $v(x,t)$ the solution corresponding to $u_0(x) = \sin x$. Then, we obtain that

(2.31) $$u(x,t) = (-1)^j v(m(x - j\pi/m), m^2 t) \quad \text{if } x \in [\,j\pi/m, (j+1)\pi/m\,].$$
$$(j = 0, 1, 2, \cdots, m-1)$$

We immediately obtain (2.30) from (2.31) and Theorem 2.2.

§3. The nondegenerate case II

We again consider the case when (D) are nondegenerate. We can not know fully from known results how the solution with *any* initial data behave in large time. We shall show that every solution of (D) behaves completely in the same way as the solution of linear heat equation in large time. Throughout in this section we assume the conditions (2.11), (2.12) and (2.13), and also assume that

(3.1) There exist $\eta > 0$ and $\alpha \in (0, \infty)$ such that $|k(r) - 1| \leq \eta |r|^\alpha$ for any $r \in \mathbb{R}$.

Theorem 3.1. *We assume (2.11), (2.12), (2.13) and (3.1). Assume also that $u_0 \in L^2(\Omega)$ and that u_0 does not identically vanish in Ω. Let $u(x,t)$ be the weak solution of (D). Then there exist $\nu \in \mathbb{N}$ and $\{c_i\}_i \subset \mathbb{R}$ $[\sum_i (c_i)^2 > 0]$ such that*

(3.2) $$e^{\lambda_\nu t} u(t) \underset{t\to\infty}{\to} \sum_i c_i e_\nu^{(i)} \quad in \quad H_0^1(\Omega).$$

Remark 3.1. We can obtain (3.2) with the estimates: if $\lambda_{\nu+1} \neq (\alpha+1)\lambda_\nu$, then

$$\|e^{\lambda_\nu t}u(t) - \sum_i c_i e_\nu^{(i)}\|_{H_0^1} \leq C\exp[-\min\{\alpha\lambda_\nu, (\lambda_{\nu+1}-\lambda_\nu)\}t] \quad \text{for} \quad t \geq 0, \tag{3.3}$$

if $\lambda_{\nu+1} = (\alpha+1)\lambda_\nu$, then

$$\|e^{\lambda_\nu t}u(t) - \sum_i c_i e_\nu^{(i)}\|_{H_0^1} \leq C(t+1)e^{-\alpha\lambda_\nu t} \quad \text{for} \quad t \geq 0, \tag{3.4}$$

where $0 < C = C(N, |\Omega|, \|u_0\|_2, \eta, \alpha, K_0)$. When $\nu = 1$, such estimates like (3.3) were already obtained in Kawanago [8] (Theorem 2.4 in [8]) and Nagasawa [9] (Theorem 2.5 in [9]). The derivation of (3.3) and (3.4) follows closely [9]. See the proof of Theorem 3.1.

Remark 3.2. The condition (3.1) seems to be a technical one. Taking Proposition 2.1 into account, Theorem 3.1 is expected to be valid even if we assume the condition (2.26) instead of (3.1).

We need a lower estimate to prove Theorem 3.1.

Lemma 3.1. *(A lower estimate) We assume all conditions of Theorem 3.1. Then there exist constants C, $\eta > 0$ such that*

$$\|u(t)\|_2 \geq Ce^{-\eta t} \quad \textit{for} \quad t \geq 0. \tag{3.5}$$

This lemma is one of the most difficult parts to establish Theorem 3.1. A similar lower estimate was obtained by Alikakos and Rostamian [1] (Theorem 4.1 in [1]) when $\phi(r) = |r|^{m-1}r$, $m \in (1, \infty)$. But our proof is much different from that in [1] which uses the monotonicity of an appropriate Liapunov functional. It seems unlikely to prove Lemma 3.1 in a similar way as in [1].

Proof of Lemma 3.1 With the aid of Theorem 2.1, we can assume, without generality, that $\|u_0\|_\infty > 0$ is sufficiently small. By Stokes' theorem, we obtain that

$$\frac{d}{dt}\int_\Omega u^2 dx = -2\int \frac{|\nabla\phi(u)|^2}{k(u)}dx, \tag{3.6}$$

$$\frac{d}{dt}\int_\Omega |\nabla\phi(u)|^2 dx = -2\int k(u)\{\Delta\phi(u)\}^2 dx. \tag{3.7}$$

We set

$$q(t) = \int \frac{|\nabla\phi(u)|^2}{k(u)}dx \Big/ \int u^2 dx. \tag{3.8}$$

It follows from (3.6) and (3.8) that

$$\|u(t)\|_2^2 = \|u_0\|_2^2 \exp(-2\int_0^t q(s)ds). \tag{3.9}$$

Therefore, the proof is complete if we can show that

(3.10) there exists a constant $\eta > 0$ such that $q(t) \leq \eta$ for every $t \geq 0$.

With the aid of Stokes' theorem and Schwarz's inequality, we obtain that

$$\begin{aligned}\int \frac{|\nabla\phi(u)|^2}{k(u)}dx &= -\int u\cdot\Delta\phi(u)dx \\ &\leq (\int \frac{u^2}{k(u)}dx)^{1/2}(\int k(u)\{\Delta\phi(u)\}^2 dx)^{1/2},\end{aligned} \tag{3.11}$$

$$\begin{aligned}\int_\Omega |\nabla\phi(u)|^2 dx &= -\int \Delta\phi(u)\cdot\phi(u)dx \\ &\leq (\int k(u)\{\Delta\phi(u)\}^2 dx)^{1/2}(\int \frac{\phi(u)^2}{k(u)}dx)^{1/2}.\end{aligned} \tag{3.12}$$

It follows from (3.11) and (3.12) that

(3.13) $$\int k(u)\{\Delta\phi(u)\}^2 dx \geq r(t)\int |\nabla\phi(u)|^2 dx.$$

Here we set

(3.14) $$r(t) = \int \frac{|\nabla\phi(u)|^2}{k(u)}dx \Big/ [(\int \frac{u^2}{k(u)}dx)(\int \frac{\phi(u)^2}{k(u)}dx)]^{1/2}$$

We obtain from (3.7) and (3.13) that

(3.15) $$\|\nabla\phi(u)\|_2^2 \leq \|\nabla\phi(u_0)\|_2^2 \exp(-2\int_0^t r(s)ds).$$

By (3.9) and (3.15),

(3.16) $$\frac{\|\nabla\phi(u)\|_2^2}{\|u(t)\|_2^2} \leq \frac{\|\nabla\phi(u_0)\|_2^2}{\|u_0\|_2^2}\exp[2\int_0^t \{q(s) - r(t)\}ds].$$

Hence, if we can prove that

(3.17) There exists a constant $C > 0$ such that $\int_0^t |q(s) - r(s)|ds \leq C$ for $t \geq 0$,

then we obtain (3.10). Therefore, we shall prove (3.17) from now on.

(3.18) $$|q(t) - r(t)| = q(t)\cdot|1 - \frac{\sqrt{c}}{\sqrt{a}\sqrt{b}}| \leq \frac{1}{\sqrt{ab}}\{\frac{\sqrt{a}|b-c|}{\sqrt{b}+\sqrt{c}} + \frac{\sqrt{c}|a-c|}{\sqrt{a}+\sqrt{c}}\},$$

where we set $a = \int u^2/k(u)\,dx$, $b = \int \phi(u)^2/k(u)\,dx$ and $c = \int u^2 dx$. It follows from (3.1), (2.12) and (3.18) that

(3.19) $$|q(t) - r(t)| \leq Cq(t)\int |u|^{2+\alpha}dx \Big/ \|u\|_2^2.$$

We shall consider the following four cases: (a) $N = 1$, (b) $N = 2$, (c) $N \geq 3$ and $2+\alpha \leq 2N/(N-2)$, and (d) $N \geq 3$ and $2+\alpha > 2N/(N-2)$. But we shall describe the proof only for the cases (c) and (d). (The proof is similar in other cases.)

The case (c). By Sobolev's inequality,

$$
\begin{aligned}
\int |u|^{2+\alpha} dx &\leq (C\|\nabla u\|_2^{N\alpha/2(2+\alpha)}\|u\|_2^{1-N\alpha/2(2+\alpha)})^{2+\alpha} \\
&\leq C\|\nabla\phi(u)\|_2^{N\alpha/2}\|u\|_2^{2-(N-1)\alpha/2}. \qquad (3.20)
\end{aligned}
$$

With the aid of (3.19) and (3.20),

$$
|q(t)-r(t)| \leq Cq(t)\|\nabla\phi(u)\|_2^{N\alpha/2}\|u\|_2^{-(N-1)\alpha/2}. \qquad (3.21)
$$

It follows from (3.8), (3.9), (3.14), (3.15), (3.21) and $k(u) \approx 1$ that

$$
\begin{aligned}
|q(t)-r(t)| &\leq Cq(t)(\|\nabla\phi(u_0)\|_2^N \,/\, \|u_0\|_2^{N-1})^{\alpha/2}\exp\{-2\eta\int_0^t q(s)ds\} \\
&= -\frac{C}{2\eta}(\frac{\|\nabla\phi(u_0)\|_2^N}{\|u_0\|_2^{N-1}})^{\alpha/2}\frac{d}{dt}[\exp\{-2\eta\int_0^t q(s)ds\}], \qquad (3.22)
\end{aligned}
$$

where η is a positive constant such that $\eta \approx N\cdot\alpha/2-(N-1)\cdot\alpha/2=\alpha/2$. Hence, we obtain that

$$
\int_0^t |q(s)-r(s)|ds \leq \frac{C}{2\eta}(\frac{\|\nabla\phi(u_0)\|_2^N}{\|u_0\|_2^{N-1}})^{\alpha/2}.
$$

Therefore, we have proved (3.17).

The case (d). Using the following inequality:

$$
\int |u|^{2+\alpha}dx \leq \|u_0\|_\infty^{2+\alpha-2N/(N-2)}\int |u|^{2N/(N-2)}dx,
$$

the argument is similar to the case (c). ∎

Proof of Theorem 3.1. We use the spectral resolution of $-\Delta$, calculations in [8] and [9], and iteration technique. For simplicity we assume throughout that the eigenspace corresponding to $\lambda_i > 0$ is one-dimensional for all $i \in \mathbb{N}$. If $u(x,t)$ satisfies

$$
\limsup_{t\to\infty} e^{\lambda_\nu t}\|u(t)\|_\infty < \infty \quad \text{for some} \quad \nu \in \mathbb{N}, \qquad (3.23)
$$

then

$$(3.24) \qquad A_\nu \equiv e^{\lambda_\nu t}(u(t), e_\nu) - \lambda_\nu \int_t^\infty e^{\lambda_\nu s}(\phi(u(s)) - u(s), e_\nu)ds$$

is a constant not depending on t (Nagasawa [9]). (Indeed we can easily verify this by $\frac{d}{dt} A_\nu = 0$.) We claim that if we assume the condition (3.23), then

$$(3.25) \qquad A_\nu = 0 \Rightarrow \limsup_{t\to\infty} e^{\lambda_{\nu+1}t}\|u(t)\|_\infty < \infty,$$

$$(3.26) \qquad A_\nu \neq 0 \Rightarrow \text{(3.2), (3.3) and (3.4) hold.}$$

In view of Theorem 2.2 and Lemma 3.1, the proof is complete if we prove (3.25) and (3.26). First we consider the case when $A_\nu = 0$. Since $A_i = 0$ $(i = 1, 2, \cdots, \nu - 1)$ also holds, we obtain that for $i = 1, 2, \cdots, \nu$,

$$(3.27) \qquad (u(t), e_i) = \lambda_i e^{-\lambda_i t} \int_t^\infty e^{\lambda_i s}(\phi(u) - u, e_i)ds.$$

It follows from (3.23), (3.27) and (3.1) that for $i = 1, 2, \cdots, \nu$,

$$(3.28) \qquad |(u(t), e_i)| \leq Ce^{-(1+\alpha)\lambda_\nu t} \quad \text{for} \quad t \geq 0.$$

With the aid of Theorem 2.1, we can assume without generality that $\|u_0\|_\infty > 0$ is sufficiently small. By (3.1) and the spectral resolution of $-\Delta$,

$$\begin{aligned}
\frac{d}{dt}\int_\Omega u(t)^2 dx &= -2\int k(u)|\nabla u|^2 dx \\
&\leq -2(1 - \eta\|u\|_\infty^\alpha)\int |\nabla u|^2 dx \\
&= -2(1 - \eta\|u\|_\infty^\alpha)\sum_{i=1}^\infty \lambda_i (u, e_i)^2 \\
(3.29) \qquad &\leq -2\lambda_{\nu+1}\int u^2 dx + C\|u\|_\infty^\alpha \int u^2 dx + C\sum_{i=1}^\nu (u, e_i)^2.
\end{aligned}$$

With the aid of Theorem 2.1, (3.23), (3.28) and (3.29), we can easily verify that

$$\|u(t)\|_\infty \le C \exp[-\min\{\lambda_{\nu+1}, (1+\alpha/2)\lambda_\nu\}t] \quad \text{for} \quad t \ge 0. \tag{3.30}$$

Speaking precisely, (3.30) holds when $\lambda_{\nu+1} \neq (1+\alpha/2)\lambda_\nu$. Below, we shall describe the outline of the proof in the case when

(3.31) there exists $n \in \mathbb{N}$ such that $\lambda_{\nu+1} > (1+\alpha/2)^{n-1}\lambda_\nu$ and $\lambda_{\nu+1} < (1+\alpha/2)^n\lambda_\nu$.

(If otherwise, i.e. there exists $n \in \mathbb{N}$ such that $\lambda_{\nu+1} = (1+\alpha/2)^n\lambda_\nu$, then we need slightly modify the argument, but it is easy.) When $n = 1$, the claim (3.25) holds in view of (3.30). When $n \ge 2$, we shall iterate the argument above. It follows from (3.30), (3.27) and (3.1) that for $i = 1, 2, \cdots, \nu$,

$$|(u(t), e_i)| \le Ce^{-(1+\alpha)(1+\alpha/2)\lambda_\nu t} \quad \text{for} \quad t \ge 0. \tag{3.32}$$

The estimate (3.30) is a sharp version of (3.23), and (3.32) is of (3.28). Using these estimates and (3.29), we can easily verify that

$$\|u(t)\|_\infty \le C \exp[-\min\{\lambda_{\nu+1}, (1+\alpha/2)^2\lambda_\nu\}t] \quad \text{for} \quad t \ge 0.$$

Iterating the argument above, we obtain that for any $n \in \mathbb{N}$,

$$\|u(t)\|_\infty \le C \exp[-\min\{\lambda_{\nu+1}, (1+\alpha/2)^n\lambda_\nu\}t] \quad \text{for} \quad t \ge 0. \tag{3.33}$$

The claim (3.25) follows (3.31) and (3.33). Next we concider the case: $A_\nu \neq 0$. We shall evaluate $\|e^{\lambda_\nu t}u(t) - A_\nu e_\nu\|_{H_0^1}$ to obtain (3.2) with (3.3) and (3.4). The calculations below is essentially the same as in Nagasawa [9] (The proof of Theorem 2.5 in [9]). With the aid of (3.24),

(3.34)

$$\|e^{\lambda_\nu t}u(t) - A_\nu e_\nu\|_{H_0^1} \le e^{\lambda_\nu t}\|u(t) - (u, e_\nu)e_\nu\|_{H_0^1} + \lambda_\nu \left| \int_t^\infty e^{\lambda_\nu s}(\phi(u(s)) - u(s), e_\nu)ds \right| \|e_\nu\|_{H_0^1}.$$

With the aid of (3.34), (3.23) and (3.1), we can verify that for $t \geq 0$,

(3.35) $$\|e^{\lambda_\nu t}u(t) - A_\nu e_\nu\|_{H_0^1} \leq e^{\lambda_\nu t}\|\phi(u) - (\phi(u), e_\nu)e_\nu\|_{H_0^1} + Ce^{-\alpha\lambda_\nu t}.$$

We set $v = \phi(u) - (\phi(u), e_\nu)e_\nu$ and we shall obtain a gradient estimate of v. With the aid of Stokes' theorem, the condition (3.1) and

(3.36) $$(\Delta v, e_\nu) = -\lambda_\nu(v, e_\nu) = 0,$$

we obtain that

$$\begin{aligned}
&\frac{d}{dt}\int_\Omega |\nabla\phi(v)|^2 dx \\
&= -2\int k(u)(\Delta v)^2 dx - \lambda_\nu(\phi(u), e_\nu)\int (k(u) - 1)\Delta v \cdot e_\nu \, dx \\
&\leq -2(1 - \eta\|u\|_\infty^\alpha)\|\Delta v\|_2^2 + Ce^{-(\alpha+1)\lambda_\nu t}\|\Delta v\|_2 \\
&\leq (-2 + 2\eta\|u\|_\infty^\alpha + \delta)\|\Delta v\|_2^2 + \frac{C}{\delta}e^{-2(\alpha+1)\lambda_\nu t} \qquad (3.37)
\end{aligned}$$

for any $\delta > 0$ and $t \geq 0$. We shall concider three cases: (a) $(\alpha + 1)\lambda_\nu > \lambda_{\nu+1}$, (b) $(\alpha + 1)\lambda_\nu < \lambda_{\nu+1}$ and (c) $(\alpha + 1)\lambda_\nu = \lambda_{\nu+1}$.

The case (a). We fix $\varepsilon > 0$ sufficiently small such that

(3.38) $$(\alpha + 1)\lambda_\nu > \lambda_{\nu+1} + \varepsilon/2.$$

Substituting $\delta = e^{-\varepsilon t}$ into (3.37) and using (3.28) and the spectral resolution of $-\Delta$, we obtain that

(3.39)
$$\begin{aligned}
\frac{d}{dt}\int_\Omega |\nabla v|^2 dx \leq -2\lambda_{\nu+1}\|\nabla v\|_2^2 + C(e^{-\alpha\lambda_\nu t} + e^{-\varepsilon t})e^{-2(\alpha+1)\lambda_\nu t} \\
+ C\exp[\{-2(\alpha+1)\lambda_\nu + \varepsilon\}t].
\end{aligned}$$

It follows from (3.38) and (3.39) that

(3.40) $$\|\nabla v\|_2 \le Ce^{-\lambda_{\nu+1}t} \quad \text{for} \quad t \ge 0.$$

We obtain (3.3) from (3.35) and (3.40).

The case (b). The argument is similar to that in the case (a).

The case (c). We obtain (3.4) by substituting $\delta = 1/(t+1)$ into (3.37) and by a similar argument as in the case (a). ∎

§4. Applications to the degenerate case

Throughout this section, we assume that

(4.1) $\phi : \mathbb{R} \to \mathbb{R}$ is a strictly increasing C^1-class function with $\phi(0) = 0$,

(4.2) There exists a strictly increasing function $K : [0, \infty) \to \mathbb{R}$ with $K(0) \ge 0$ such that $k(r) = \phi'(r) \ge K(|r|)$ for any $r \in \mathbb{R}$.

We begin with a result about the smoothing effect:

Proposition 4.1. *Assume that ϕ satisfies (4.1) and (4.2) and that $u_0 \in L^2(\Omega)$. Let $u(x,t)$ be the solution of (D). Then $u(t) \in L^\infty(\Omega)$ for $t > 0$ and $u(t) \underset{t\to\infty}{\to} 0$ in $L^\infty(\Omega)$ with the estimates:*

(4.3)
$$\|u(t)\|_\infty \le \varepsilon + \frac{C_1}{(K(\varepsilon)t)^{N/4}}\|u_0\|_2 \quad \textit{for any} \quad \varepsilon > 0 \quad \textit{and} \quad t > 0.$$

(4.4)
$$\|u(t)\|_\infty \le \varepsilon + \frac{C_2 e^{-\lambda_1 K(\varepsilon)(t-t_0)}}{(K(\varepsilon)t_0)^{N/4}}\|u_0\|_2 \quad \textit{for any} \quad \varepsilon > 0 \quad \textit{and} \quad t > t_0.$$

Here, $t_0 > 0$ is an arbitrary time, and C_1, $C_2 > 0$ are some constants dependent only on N.

Proof. Following Bertsch and Peletier [4], we compare $u(x,t)$ with the solution $v(x,t)$ of the following (I_ε):

$$(I_\varepsilon)\begin{cases} v_t = \Delta\phi(v) & \text{in} \quad \Omega\times\mathbb{R}^+ \\ v(x,t) = \varepsilon > 0 & \text{on} \quad \partial\Omega\times\mathbb{R}^+ \\ v(x,0) = \sup(u_0(x),\varepsilon) & \text{in} \quad \Omega. \end{cases}$$

With the aid of the comparison principle,

$$u(x,t) \le v(x,t) \quad \text{in} \quad \Omega\times\mathbb{R}^+. \tag{4.5}$$

On the other hand, by (2.2) of Theorem 2.1,

$$\|v(t)-\varepsilon\|_\infty \le \frac{C}{(K(\varepsilon)t)^{N/4}}\|v(0)-\varepsilon\|_2 \le \frac{C}{(K(\varepsilon)t)^{N/4}}\|u_0\|_2 \quad \text{for} \quad t>0. \tag{4.6}$$

It follows from (4.5) and (4.6) that

$$u(x,t) \le \varepsilon + \frac{C}{(K(\varepsilon)t)^{N/4}}\|u_0\|_2 \quad \text{in} \quad \Omega\times\mathbb{R}^+. \tag{4.7}$$

Replacing in (I_ε) ε by $-\varepsilon$ and 'sup' by 'inf' respectively and from the same argument as above, we obtain that

$$u(x,t) \ge -\varepsilon - \frac{C}{(K(\varepsilon)t)^{N/4}}\|u_0\|_2 \quad \text{in} \quad \Omega\times\mathbb{R}^+.$$

Hence we obtain (4.3). We similarly obtain (4.4) from (2.3) of Theorem 2.1. ∎

If (D) is degenerate at $u=0$, then, as is expected, the solution $u(x,t)$ *never* satisfy the estimate (2.15).

Corollary 4.1. *Assume that ϕ satisfies* (4.1) *and* (4.2) *with* $k(0) = \phi'(0) = 0$. *We also assume that* $u_0 \in L^2(\Omega)$, $u_0 \geq 0$ *and* $u_0(x)$ *does not identically vanish in* Ω. *Let* $u(x,t)$ *be the solution of* (D). *Then, for all* $\eta > 0$ *there exsists a time* $T > 0$ *such that*

$$\|u(t)\|_\infty \geq e^{-\eta t} \quad \text{for} \quad t \geq T. \tag{4.8}$$

Proof. It follows from Proposition 4.1 that there exsists a time $T > 0$ such that

$$\|u(t)\|_\infty \leq R \quad \text{for} \quad t \geq T, \tag{4.9}$$

where $R > 0$ is a constant such that $\max_{0 \leq |r| \leq R} k(r) \leq \eta/\lambda_1$. On the other hand, by Stokes' Theorem,

$$\frac{d}{dt}(u(t), e_1)_2 = (\phi(u), \Delta e_1) \geq -\eta/\lambda_1 \cdot \lambda_1(u, e_1) = -\eta(u, e_1),$$

which implies (4.8). ∎

However, the solutions of some of degenerate equations decay fairly fast.

Corollary 4.2. *Assume that ϕ satisfies* (4.1) *and that there exist* $r_0 \in (0,1)$ *and* η, k_0, $\theta > 0$ *such that*

$$k(r) \geq \frac{\theta}{(-log|r|)^\eta} \quad \text{for} \quad r \in [-r_0, r_0], \tag{4.10}$$

$$k(r) \geq k_0 \quad \text{for} \quad r \in \mathbb{R} \setminus [-r_0, r_0]. \tag{4.11}$$

Let $u(x,t)$ *be the solution of* (D) *with* $u_0 \in L^2(\Omega)$. *Then the following estimate holds:*

$$\|u(t)\|_\infty \leq C(t+1)^{\frac{N\eta}{4(\eta+1)}} exp\{-(\theta\lambda_1 t)^{\frac{1}{\eta+1}}\} \quad \text{for} \quad t \geq 0, \tag{4.12}$$

where $C > 0$ depends only on $\|u_0\|_2$, r_0, k_0, η , θ, N and Ω.

Proof. We assume, by Proposition 4.1, without loss of generality that $u_0 \in L^\infty(\Omega)$ and $\|u_0\|_\infty \le r_0$. Substituting $\epsilon = C\exp\{-(\theta\lambda t)^{1/(\eta+1)}\}$ to (4.4), we immediately obtain (4.12). ∎

Remark 4.1. The estimate (4.12) is fairly sharp. Assume that there exist $r_0 \in (0,1)$ and η, $\theta > 0$ such that $\phi(r) = \theta r/(-\log|r|)^\eta$ for $r \in [-r_0, r_0]$. We assume for simplicity that $u_0 \in L^\infty(\Omega)$ with $\|u_0\|_\infty \le r_0$ and $\inf_{x\in\Omega} u_0(x) \ge \delta$ for some $\delta \in (0,1)$. Then the following lower estimate holds:

$$\|u(t)\|_1 \ge C\exp\{-(\eta+1)(\theta\lambda_1 t)^{\frac{1}{\eta+1}}\} \quad \text{for} \quad t \ge 0.$$

Here $C > 0$ depends only on r_0, η, θ, δ, Ω and N. We omit the proof. For the details, see Remark 4.1 and its proof in [8].

References

[1] N.D.Alikakos and R.Rostamian, *Large time behavior of solutions of Neumann boundary value problem for the porous medium equation*, Indiana University Math. Journal **30** (1981), 749–785.

[2] J. G. Berryman and C. J. Holland, *Asymptotic behavior of the nonlinear diffusion equation $n_t = (n^{-1}n_x)_x$*, J. Math. Phys. **23** (1982), 983–987.

[3] M. Bertsch and L.A. Peletier, *The Asymptotic Profile of Solutions of Degenerate Diffusion Equations*, Arch. Rat. Mech. Anal. (1986), 207–229.

[4] M. Bertsch and L.A. Pelitier, *A positivity property of solutions of nonlinear diffusion equations*, J. Diff. Eqns. **53** (1984), 30–47.

[5] L.C. Evans, *Differentiability of a nonlinear semigroup in L^1*, J. Math. Analysis and Application **60** (1977), 703–715.

[6] L.C.Evans, *Regulality properties for the heat equation subject to nonlinear boundary constraints*, Nonlinear Analysis **1** (1977), 593–602.

[7] T. Kawanago, Thesis for master's degree, Osaka University (1989).

[8] T. Kawanago, *The behavior of solutions of quasi-linear heat equations*, Preprint.

[9] T. Nagasawa, *Boundary value problems to a certain class of nonlinear diffusion equations*, Preprint.