

On the Nonlinear Mean Ergodic Theorems for Asymptotically
Nonexpansive Mappings in Banach Spaces

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1. Introduction.

Throughout this note X denotes a uniformly convex real Banach space and C is a closed convex subset of X . The value of $x^* \in X^*$ at $x \in X$ will be denoted by (x, x^*) .

The duality mapping J (multi-valued) from X into X^* will be defined by $J(x) = \{x^* \in X^* : (x, x^*) = \|x\|^2 = \|x^*\|^2\}$ for $x \in X$.

We say that X is (F) if the norm of X is Fréchet differentiable,

i. e., for each $x \in X$ with $x \neq 0$, $\lim_{t \rightarrow 0} t^{-1}(\|x+ty\| - \|x\|)$ exists

uniformly in $y \in B_1$, where $B_r = \{z \in X : \|z\| \leq r\}$ for $r > 0$. It is easily seen that X is (F) if and only if for any bounded set $B \subset X$

and any $x \in X$, $\lim_{t \rightarrow 0} (2t)^{-1}(\|x+ty\|^2 - \|x\|^2) = (y, J(x))$ uniformly in

$y \in B$. We say that X satisfies Opial's condition if $w\text{-}\lim_{n \rightarrow \infty} x_n = x$

implies that $\limsup_{n \rightarrow \infty} \|x_n - x\| < \limsup_{n \rightarrow \infty} \|x_n - y\|$ for all $y \in X$ with $y \neq x$.

A mapping $T : C \rightarrow C$ is said to be asymptotically nonexpansive if for each $n = 1, 2, \dots$

$$(1.1) \quad \|T^n x - T^n y\| \leq (1 + \alpha_n) \|x - y\| \text{ for any } x, y \in C,$$

where $\lim_{n \rightarrow \infty} \alpha_n = 0$. In particular, if $\alpha_n = 0$ for $n \geq 1$, T is said to be nonexpansive. The set of fixed points of T will be denoted by $F(T)$.

Throughout the rest of this note let $T : C \rightarrow C$ be an asymptotically nonexpansive mapping satisfying (1.1).

A sequence $\{x_n\}_{n \geq 0}$ in C is called an almost-orbit of T if

$$(1.2) \quad \lim_{n \rightarrow \infty} \left[\sup_{m \geq 0} \|x_{n+m} - T^m x_n\| \right] = 0.$$

A sequence $\{z_n\}$ in X is said to be strongly (or weakly) almost convergent to $z \in X$ if $\frac{1}{n} \sum_{i=0}^{n-1} z_{i+k}$ converges strongly (or weakly) as $n \rightarrow \infty$ to z uniformly in $k \geq 0$. The convex hull of a set E ($\subset X$) is denoted by $\text{co } E$, the closed convex hull by $\text{clco } E$, and $\omega_w(\{x_n\})$ denotes the set of weak subsequential limits of $\{x_n\}$ as $n \rightarrow \infty$.

We get the following (nonlinear) mean ergodic theorems.

Theorem 1. Suppose that $\{x_n\}_{n \geq 0}$ is an almost-orbit of T and C is bounded. If X satisfies Opial's condition or if X is (F), then $\{x_n\}$ is weakly almost convergent to an element of $F(T)$.

Theorem 2. Suppose that $\{x_n\}_{n \geq 0}$ is an almost-orbit of T and C is bounded. If $\lim_{n \rightarrow \infty} \|x_n - x_{n+1}\|$ exists uniformly in $i \geq 0$, then $\{x_n\}$ is strongly almost convergent to an element of $F(T)$.

Theorem 1 is an extension of [5, Theorem 1.], [1, Corollary 2.1], [4, Theorem 2.1] and Theorem 2 is an extension of [6, Theorem 1].

2. Lemmas.

Throughout this section, we assume that C is bounded. By Bruck's inequality [2, Theorem 2.1], we get

Lemma 1. There exists a strictly increasing, continuous, convex function $\gamma : [0, \infty) \rightarrow [0, \infty)$ with $\gamma(0) = 0$ such that

$$\begin{aligned} & \|T^k(\sum_{i=1}^n \lambda_i x_i) - \sum_{i=1}^n \lambda_i T^k x_i\| \\ & \leq (1+\alpha_k)^{-1} \left(\max_{1 \leq i, j \leq n} [\|x_i - x_j\| - \frac{1}{1+\alpha_k} \|T^k x_i - T^k x_j\|] \right) \end{aligned}$$

for any $k, n \geq 1$, any $\lambda_1, \dots, \lambda_n \geq 0$ with $\sum_{i=1}^n \lambda_i = 1$, and any

$$x_1, \dots, x_n \in C.$$

Hereafter, let γ be as in Lemma 1.

Lemma 2. Suppose that $\{x_n\}_{n \geq 0}$ and $\{y_n\}_{n \geq 0}$ are almost-orbits of T . Then $\{\|x_n - y_n\|\}$ converges as $n \rightarrow \infty$.

Proof. Put $a_n = \sup_{m \geq 0} \|x_{n+m} - T^m x_n\|$ and $b_n = \sup_{m \geq 0} \|y_{n+m} - T^m y_n\|$ for $n \geq 0$. Then $a_n \rightarrow 0$ and $b_n \rightarrow 0$ as $n \rightarrow \infty$.

Since

$$\begin{aligned} \|x_{n+m} - y_{n+m}\| & \leq \|x_{n+m} - T^m x_n\| + \|T^m x_n - T^m y_n\| + \|T^m y_n - y_{n+m}\| \\ & \leq a_n + b_n + (1+\alpha_m) \|x_n - y_n\|, \text{ we have} \end{aligned}$$

$$\limsup_{m \rightarrow \infty} \|x_m - y_m\| \leq a_n + b_n + \|x_n - y_n\| \text{ for every } n \geq 0.$$

Taking the \liminf as $n \rightarrow \infty$,

we obtain $\limsup_{m \rightarrow \infty} \|x_m - y_m\| \leq \liminf_{n \rightarrow \infty} \|x_n - y_n\|$ and so the conclusion

holds.

Q. E. D.

We now put $D = \text{diameter } C$ and $M = \sup_{n \geq 1} (1+\alpha_n)$.

Lemma 3. Suppose that $\{x_j^{(p)}\}_{j \geq 1}$ ($p = 1, 2, \dots$) are almost-orbits of T . Then for any $\varepsilon > 0$ and $n \geq 1$ there exist $N_\varepsilon \geq 1$ and $i_n(\varepsilon) \geq 1$, where N_ε is independent of n , such that

$$\|T^k(\sum_{p=1}^n \lambda_p x_i^{(p)}) - \sum_{p=1}^n \lambda_p T^k x_i^{(p)}\| < \varepsilon \text{ for any } k \geq N_\varepsilon, \text{ any } i \geq i_n(\varepsilon),$$

and any $\lambda_1, \dots, \lambda_n \geq 0$ with $\sum_{p=1}^n \lambda_p = 1$.

Proof. For any $\varepsilon > 0$ choose $\delta > 0$ so that $\gamma^{-1}(\delta) < \varepsilon/M$. Then there exists $N_\varepsilon \geq 1$ such that $\alpha_k < \delta/4D$ for $k \geq N_\varepsilon$.

Since $\{\|x_j^{(p)} - x_j^{(q)}\|\}_{j \geq 1}$ converges as $j \rightarrow \infty$ by Lemma 2,

for each $p, q \geq 1$ there exists $i_0(\varepsilon, p, q) \geq 1$ such that

$$\|x_i^{(p)} - x_i^{(q)}\| - \|x_{i+k}^{(p)} - x_{i+k}^{(q)}\| < \delta/4 \text{ if } i \geq i_0(\varepsilon, p, q) \text{ and } k \geq 0.$$

Moreover, there is $i_1(\varepsilon, p) \geq 1$ such that $a_i^{(p)} < \delta/4$

for all $i \geq i_1(\varepsilon, p)$, where $a_i^{(p)} = \sup_{j \geq 0} \|x_{i+j}^{(p)} - T^j x_i^{(p)}\|$.

Put $i_n(\varepsilon) = \max\{i_0(\varepsilon, p, q), i_1(\varepsilon, p) : 1 \leq p, q \leq n\}$ for $n \geq 1$.

If $i \geq i_n(\varepsilon)$ and $k \geq N_\varepsilon$, then

$$\begin{aligned} & \|x_i^{(p)} - x_i^{(q)}\| - \frac{1}{1+\alpha_k} \|T^k x_i^{(p)} - T^k x_i^{(q)}\| \\ & \leq \|x_i^{(p)} - x_i^{(q)}\| - \|x_{i+k}^{(p)} - x_{i+k}^{(q)}\| + a_i^{(p)} + a_i^{(q)} + \alpha_k \|x_i^{(p)} - x_i^{(q)}\| < \delta \end{aligned}$$

for $1 \leq p, q \leq n$ and by Lemma 1,

$$\|T^k(\sum_{p=1}^n \lambda_p x_i^{(p)}) - \sum_{p=1}^n \lambda_p T^k x_i^{(p)}\| < \varepsilon$$

for any $\lambda_1, \dots, \lambda_n \geq 0$ with $\sum_{p=1}^n \lambda_p = 1$.

Q. E. D.

For any $\varepsilon > 0$ and $k \geq 1$, we put $F_\varepsilon(T^k) = \{x \in C : \|T^k x - x\| \leq \varepsilon\}$. Since C is bounded, $F(T) \neq \emptyset$. (For example, see [3, Theorem 1].)

Lemma 4. Suppose that $\{x_i\}_{i \geq 0}$ is an almost-orbit of T . Then for any $\varepsilon > 0$ there exists $N_\varepsilon \geq 1$ such that for each $k \geq N_\varepsilon$, there is $N_k (=N_k(\varepsilon)) \geq 1$ satisfying

$$\frac{1}{n} \sum_{i=0}^{n-1} x_{i+\alpha} \in F_\varepsilon(T^k) \text{ for all } n \geq N_k \text{ and all } \alpha \geq 0.$$

Proof. Let $\varepsilon > 0$ be arbitrarily given and σ be the inverse function of $t \mapsto My^{-1}(3t) + t$. Put $\delta = \min\{\sigma(\frac{\varepsilon}{3}), \frac{\varepsilon}{3MD}\}$ and $M' = M+1$. Choose $n > 0$ and $N_{1,\varepsilon} \geq 1$ so that $\gamma^{-1}(n) < \frac{\delta^2}{2M}$ and $\alpha_k < \sigma(\frac{\varepsilon}{3})/D$ for $k \geq N_{1,\varepsilon}$. Furthermore, by Lemma 3, there exists $N_{2,\varepsilon} \geq 1$ such that for any $p \geq 1$ there is $i_p(\varepsilon) \geq 1$ satisfying

$$(2.1) \quad \|T^k(\frac{1}{p} \sum_{j=0}^{p-1} x_{i+j+\alpha}) - \frac{1}{p} \sum_{j=0}^{p-1} T^k x_{i+j+\alpha}\| < \delta^2/8$$

for any $k \geq N_{2,\varepsilon}$, any $i \geq i_p(\varepsilon)$, and any $\alpha \geq 0$.

Put $N_\varepsilon = \max(N_{1,\varepsilon}, N_{2,\varepsilon})$ and let $k \geq N_\varepsilon$ be fixed. By Lemma 1 and the choice of δ , we get

$$(2.2) \quad \text{clco } F_\delta(T^k) \subset F_{\varepsilon/3}(T^k).$$

Next, choose $p \geq 1$ so that $\frac{Dk}{p} \leq \frac{\delta^2}{2}$ and let p be fixed. Since $\{x_i\}_{i \geq 0}$ is an almost-orbit of T , there exists $N \geq 1$ such that

$$\sup_{q \geq 0} \|x_{m+q} - T^q x_m\| < \frac{\delta^2}{8} \text{ for } m \geq N. \text{ Set } w_i = \frac{1}{p} \sum_{j=0}^{p-1} x_{i+j} \text{ for } i \geq 0.$$

If $i \geq i_p(\varepsilon) + N$, by (2.1),

$$\|w_{i+k+q} - T^k w_{i+q}\|$$

$$\leq \left\| \frac{1}{p} \sum_{j=0}^{p-1} (x_{i+j+k+q} - T^k x_{i+j+q}) \right\| + \left\| \frac{1}{p} \sum_{j=0}^{p-1} T^k x_{i+j+q} - T^k \left(\frac{1}{p} \sum_{j=0}^{p-1} x_{i+j+q} \right) \right\| < \frac{\delta^2}{4}$$

for all $q \geq 0$. Choose $N_3(k) \geq i_p(\varepsilon) + N + 1$ such that $\frac{D(i_p(\varepsilon) + N)}{n} < \frac{\delta^2}{4}$ for all $n \geq N_3(k)$. If $n \geq N_3(k)$, then

$$(2.3) \quad \frac{1}{n} \sum_{i=0}^{n-1} \|w_{i+q} - T^k w_{i+q}\| \leq \frac{1}{n} \sum_{i=0}^{n-1} \|w_{i+q} - w_{i+k+q}\|$$

$$+ \frac{1}{n} \left(\sum_{i=0}^{i_p + N - 1} + \sum_{i=i_p + N}^{n-1} \right) \|w_{i+k+q} - T^k w_{i+q}\| \leq \frac{Dk}{p} + \frac{(i_p + N)D}{n} + \frac{\delta^2}{4} \leq \delta^2$$

for all $q \geq 0$, where $i_p = i_p(\varepsilon)$. Finally, choose $N_4(k) \geq 1$ so that $\frac{(p-1)D}{2n} < \frac{\varepsilon}{3M}$ for all $n \geq N_4(k)$. Put $N_k = \max(N_3(k), N_4(k))$ and let $n \geq N_k$ be fixed and $q \geq 0$.

Set $A(k, n, q) = \{i \in Z : 0 \leq i \leq n-1 \text{ and } \|w_{i+q} - T^k w_{i+q}\| \geq \delta\}$ and $B(k, n, q) = \{0, 1, \dots, n-1\} \setminus A(k, n, q)$. By (2.3), $\#A(k, n, q) \leq n\delta$, where $\#$ denotes cardinality. Let $f \in F(T)$. Then,

$$\frac{1}{n} \sum_{i=0}^{n-1} x_{i+q} = \frac{1}{n} \sum_{i=0}^{n-1} w_{i+q} + \frac{1}{np} \sum_{i=1}^{p-1} (p-i) (x_{i+q-1} - x_{i+q+n-1})$$

$$= \left[\frac{1}{n} (\#A(k, n, q)) \cdot f + \frac{1}{n} \sum_{i \in B(k, n, q)} w_{i+q} \right] + \left[\frac{1}{n} \sum_{i \in A(k, n, q)} (w_{i+q} - f) \right]$$

$$+ \frac{1}{np} \sum_{i=1}^{p-1} (p-i) (x_{i+q-1} - x_{i+q+n-1}).$$

The first term on the right side of the above equality is contained in $\text{clco } F_\delta(T^k)$, and the rest term in $B_{2\varepsilon/3M}$. By (2.2), we get

$$\frac{1}{n} \sum_{i=0}^{n-1} x_{i+q} \in F_\varepsilon(T^k) \text{ for all } q \geq 0.$$

Q. E. D.

Lemma 5. Let $\{x_n\}$ in C be such that $w\text{-}\lim_{n \rightarrow \infty} x_n = x$. Suppose that for any $\varepsilon > 0$ there exists $N(\varepsilon) \geq 1$ such that for $k \geq N(\varepsilon)$ there is $N_k \geq 1$ satisfying $\|T^k x_n - x_n\| < \varepsilon$ for all $n \geq N_k$. Then $x \in F(T)$.

Proof. We shall show that $\lim_{k \rightarrow \infty} \|T^k x - x\| = 0$. For any $\varepsilon > 0$ choose $\delta > 0$ so that $\gamma^{-1}(\delta) < \frac{\varepsilon}{4M}$ and take $N_1(\varepsilon) \geq 1$ such that $\alpha_k < \frac{\delta}{3D}$ for all $k \geq N_1(\varepsilon)$. Put $\delta' = \min(\frac{\delta}{3}, \frac{\varepsilon}{4})$. By the assumption, there exists $N(\varepsilon) \geq 1$ such that for each $k \geq N(\varepsilon)$ there is $N_k \geq 1$ satisfying $\|T^k x_n - x_n\| < \delta'$ for all $n \geq N_k$.

Put $N_2(\varepsilon) = \max(N_1(\varepsilon), N(\varepsilon))$ and let $k \geq N_2(\varepsilon)$ be arbitrarily fixed. Since $x \in \text{clco}\{x_n : n \geq N_k\}$, there exists a sequence

$$\left\{ \sum_{i=1}^{\varrho_n} \lambda_n^{(i)} x_{\psi_n(i)} \right\} \subset \text{co}\{x_n : n \geq N_k\} \text{ such that } \lim_{n \rightarrow \infty} \sum_{i=1}^{\varrho_n} \lambda_n^{(i)} x_{\psi_n(i)} = x.$$

Therefore there is $N_3(k) \geq 1$ such that $\| \sum_{i=1}^{\varrho_n} \lambda_n^{(i)} x_{\psi_n(i)} - x \| < \frac{\varepsilon}{4M}$ for

all $n \geq N_3(k)$ and hence if $n \geq N_3(k)$, $\|T^k x - T^k(\sum_{i=1}^{\varrho_n} \lambda_n^{(i)} x_{\psi_n(i)})\| < \frac{\varepsilon}{4}$.

On the other hand, by Lemma 1 and the choice of δ and k , we get

$$\|T^k(\sum_{i=1}^{\varrho_n} \lambda_n^{(i)} x_{\psi_n(i)}) - \sum_{i=1}^{\varrho_n} \lambda_n^{(i)} T^k x_{\psi_n(i)}\| < \frac{\varepsilon}{4} \text{ for all } n \geq 1.$$

Consequently, $\|T^k x - x\| \leq \|T^k x - T^k(\sum_{i=1}^{\varrho_n} \lambda_n^{(i)} x_{\psi_n(i)})\|$

$$+ \|T^k(\sum_{i=1}^{\varrho_n} \lambda_n^{(i)} x_{\psi_n(i)}) - \sum_{i=1}^{\varrho_n} \lambda_n^{(i)} T^k x_{\psi_n(i)}\|$$

$$+ \left\| \sum_{i=1}^{\varrho_n} \lambda_n^{(i)} (T^k x_{\psi_n(i)} - x_{\psi_n(i)}) \right\| + \left\| \sum_{i=1}^{\varrho_n} \lambda_n^{(i)} x_{\psi_n(i)} - x \right\| < \varepsilon,$$

where $n \geq N_3(k)$.

This shows that $\|T^k x - x\| < \varepsilon$ for $k \geq N_2(\varepsilon)$.

Q. E. D.

Lemma 6. Suppose that X is (F) and $\{x_n\}$ is an almost-orbit of T . Then the following hold:

- (i) $\{(x_n, J(f-g))\}$ converges for every $f, g \in F(T)$.
- (ii) $F(T) \cap \text{clco } \omega_w(\{x_n\})$ is at most a singleton.

Proof. Let $\lambda \in (0, 1)$ and $f, g \in F(T)$. By Lemma 3, for any $\varepsilon > 0$ there exist $N_\varepsilon \geq 1$ and $i_2(\varepsilon) \geq 1$ such that if $k \geq N_\varepsilon$ and $n \geq i_2(\varepsilon)$,

$$\|T^k(\lambda x_n + (1-\lambda)f) - \lambda T^k x_n - (1-\lambda)f\| < \varepsilon.$$

Since $\|\lambda x_{n+m} + (1-\lambda)f - g\| \leq \lambda \|x_{n+m} - T^m x_n\|$

$$+ \|T^m(\lambda x_n + (1-\lambda)f) - \lambda T^m x_n - (1-\lambda)f\| + (1+\alpha_m) \|\lambda x_n + (1-\lambda)f - g\|$$

$$\leq \sup_{\varrho \geq 0} \|x_{n+\varrho} - T^\varrho x_n\| + \varepsilon + (1+\alpha_m) \|\lambda x_n + (1-\lambda)f - g\|$$

for $m \geq N_\varepsilon$ and $n \geq i_2(\varepsilon)$, we have

$$\limsup_{m \rightarrow \infty} \|\lambda x_m + (1-\lambda)f - g\| \leq \sup_{\varrho \geq 0} \|x_{n+\varrho} - T^\varrho x_n\| + \varepsilon + \|\lambda x_n + (1-\lambda)f - g\|$$

for $n \geq i_2(\varepsilon)$. Letting $n \rightarrow \infty$ and then $\varepsilon \downarrow 0$, we get

$$\limsup_{m \rightarrow \infty} \|\lambda x_m + (1-\lambda)f - g\| \leq \liminf_{n \rightarrow \infty} \|\lambda x_n + (1-\lambda)f - g\|$$

and so $\|\lambda x_n + (1-\lambda)f - g\|$ converges as $n \rightarrow \infty$.

The boundedness of $\{\|x_n - f\|\}_{n \geq 0}$ and the Fréchet differentiability of X imply that $a(\lambda, n) = (2\lambda)^{-1} (\|f - g + \lambda(x_n - f)\|^2 - \|f - g\|^2)$ converges to $(x_n - f, J(f-g))$ as $\lambda \downarrow 0$ uniformly in $n \geq 0$.

Hence $\lim_{n \rightarrow \infty} (x_n - f, J(f-g)) = \lim_{\lambda \rightarrow 0+, n \rightarrow \infty} a(\lambda, n)$ exists. This proves (i).

It follows from (i) that $(u-v, J(f-g)) = 0$ for all $u, v \in \omega_w(\{x_n\})$ and hence for all $u, v \in \text{clco } \omega_w(\{x_n\})$. Therefore, $F(T) \cap \text{clco } \omega_w(\{x_n\})$ is at most a singleton. Q. E. D.

We set

$$s(n; m) = \frac{1}{n} \sum_{i=0}^{n-1} x_{i+m} \quad (n \geq 1; m \geq 0)$$

for an almost-orbit $\{x_n\}$ of T .

Lemma 7. Let $\{x_n\}$ be an almost-orbit of T . Then there exists a sequence $\{i_n\}$ of nonnegative integers with $i_n \rightarrow \infty$ as $n \rightarrow \infty$ satisfying the following:

Let $\{k_n\}$ be a sequence of nonnegative integers with $k_n \geq i_n$ for all n . Then, we have the following:

- (i) $\|s(n; k_n) - f\|$ is convergent as $n \rightarrow \infty$ for every $f \in F(T)$.
- (ii) If X satisfies Opial's condition or if X is (F), then there exists an element f of $F(T)$ such that $w\text{-}\lim_{n \rightarrow \infty} s(n; k_n) = f$.

Moreover, $F(T) \cap \text{clco } \omega_w(\{x_n\}) = \{f\}$ in case X is (F).

Proof. By Lemma 3, there exist divergent sequences $\{N_n\}$ and $\{i_n\}$ of nonnegative integers such that if $k \geq N_n$ and $i \geq i_n$,

$$(2.4) \quad \|T^k \left(\frac{1}{n} \sum_{p=0}^{n-1} x_{p+i} \right) - \frac{1}{n} \sum_{p=0}^{n-1} T^k x_{p+i}\| < \frac{1}{n}.$$

Let $f \in F(T)$ and $\{k_n\}$ be a sequence of nonnegative integers with $k_n \geq i_n$ for all n . By (2.4),

$$\begin{aligned} & \left\| \frac{1}{n+m} \left(\sum_{p=0}^{k_n+N_n-1} + \sum_{p=k_n+N_n}^{n+m-1} \right) \left(\frac{1}{n} \sum_{q=0}^{n-1} x_{p+q+k_{n+m}} - f \right) \right\| \\ & \leq \frac{(k_n+N_n)D}{n+m} + \frac{1}{n+m} \sum_{p=k_n+N_n}^{n+m-1} \left\| \frac{1}{n} \sum_{q=0}^{n-1} (x_{p+q+k_{n+m}} - T^{p+k_{n+m}-k_n} x_{q+k_n}) \right. \\ & \quad \left. + \left(\frac{1}{n} \sum_{q=0}^{n-1} T^{p+k_{n+m}-k_n} x_{q+k_n} - T^{p+k_{n+m}-k_n} \left(\frac{1}{n} \sum_{q=0}^{n-1} x_{q+k_n} \right) \right) \right. \\ & \quad \left. + \left(T^{p+k_{n+m}-k_n} \left(\frac{1}{n} \sum_{q=0}^{n-1} x_{q+k_n} \right) - f \right) \right\| \\ & \leq \frac{(k_n+N_n)D}{n+m} + \frac{1}{n} \sum_{q=0}^{n-1} \sup_{\rho \geq 0} \|x_{\rho+q+k_n} - T^\rho x_{q+k_n}\| + \frac{1}{n} + \|s(n; k_n) - f\| \\ & \quad + \frac{1}{n+m} \sum_{p=k_n+N_n}^{n+m-1} \alpha_{p+k_{n+m}-k_n} D \quad \text{whenever } n+m \geq k_n+N_n+1. \end{aligned}$$

Therefore,

$$\begin{aligned} & \|s(n+m; k_{n+m}) - f\| \\ & \leq \left\| \frac{1}{n+m} \left(\sum_{p=0}^{k_n+N_n-1} + \sum_{p=k_n+N_n}^{n+m-1} \right) \left(\frac{1}{n} \sum_{q=0}^{n-1} x_{p+q+k_{n+m}} - f \right) \right\| \\ & \quad + \frac{1}{n(n+m)} \sum_{p=1}^{n-1} (n-p) \|x_{p+k_{n+m}-1} - x_{p+k_{n+m}+n+m-1}\| \\ & \leq \frac{(k_n+N_n)D}{n+m} + \frac{1}{n} \sum_{q=0}^{n-1} \sup_{\rho \geq 0} \|x_{\rho+q+k_n} - T^\rho x_{q+k_n}\| + \frac{1}{n} + \|s(n; k_n) - f\| \end{aligned}$$

$$+ \frac{1}{n+m} \sum_{p=k_n+N_n}^{n+m-1} \alpha_{p+k_{n+m}-k_n} D + \frac{(n-1)D}{2(n+m)} \quad \text{for } n+m \geq k_n + N_n + 1.$$

Hence

$$\limsup_{m \rightarrow \infty} \|s(m; k_m) - f\| \leq \liminf_{n \rightarrow \infty} \|s(n; k_n) - f\|.$$

This proves (i).

Now, let W be the set of weak subsequential limits of $\{s(n; k_n)\}$ as $n \rightarrow \infty$. Since X is reflexive and $\{s(n; k_n)\}$ is bounded, W is nonempty. To prove (ii) it suffices to show that $W \subset F(T)$ and W is a singleton. By Lemmas 4 and 5, $W \subset F(T)$ and so $\{\|s(n; k_n) - v\|\}$ converges as $n \rightarrow \infty$ for every $v \in W$ by (i).

First, suppose that X satisfies Opial's condition and let $v_i \in W$, $i = 1, 2$ and $v_i = w\text{-}\lim_{n(i) \rightarrow \infty} s(n(i); k_{n(i)})$, where $\{n(i)\}$, $i = 1, 2$, are subsequences of $\{n\}$. Suppose $v_1 \neq v_2$. Then, by Opial's condition,

$$\begin{aligned} \lim_{n \rightarrow \infty} \|s(n; k_n) - v_1\| &= \lim_{n(1) \rightarrow \infty} \|s(n(1); k_{n(1)}) - v_1\| \\ &< \lim_{n(1) \rightarrow \infty} \|s(n(1); k_{n(1)}) - v_2\| \\ &= \lim_{n \rightarrow \infty} \|s(n; k_n) - v_2\|. \end{aligned}$$

In the same way we have $\lim_{n \rightarrow \infty} \|s(n; k_n) - v_2\| < \lim_{n \rightarrow \infty} \|s(n; k_n) - v_1\|$.

This is a contradiction. Consequently, $v_1 = v_2$ and W is a singleton.

Next, suppose that X is (F). We can easily see that

$$W \subset \bigcap_{i=0}^{\infty} \text{clco} \{x_n : n \geq i\} = \text{clco } \omega_w(\{x_n\}).$$

Thus $W \subset F(T) \cap \text{clco } \omega_w(\{x_n\})$ and hence W is a singleton by Lemma 6

(ii).

Q. E. D.

Lemma 8. Let $\{x_n\}$ be an almost-orbit of T and $\{k_n\}$ a sequence of nonnegative integers. If $\{s(n; k_n + \varrho)\}$ converges weakly (or strongly) as $n \rightarrow \infty$, uniformly in $\varrho \geq 0$, to an element y of X , then $\{s(n; \varrho)\}$ converges weakly (or strongly) as $n \rightarrow \infty$, uniformly in $\varrho \geq 0$, to y .

Proof. Suppose that $\lim_{n \rightarrow \infty} s(n; k_n + \varrho) = y$ uniformly in $\varrho \geq 0$. Then, for any $\varepsilon > 0$ there is $N \geq 1$ such that $\|s(N; k_N + \varrho) - y\| < \varepsilon$ for all $\varrho \geq 0$.

$$\begin{aligned} \|s(n; \varrho) - y\| &\leq \frac{1}{n} \left(\sum_{i=0}^{k_N-1} + \sum_{i=k_N}^{n-1} \right) \|s(N; i + \varrho) - y\| \\ &\quad + \frac{1}{nN} \sum_{i=1}^{N-1} (N-i) \|x_{i+\varrho-1} - x_{i+\varrho+n-1}\| \\ &\leq \frac{k_N D}{n} + \varepsilon + \frac{(N-1)D}{2n} \text{ for } n \geq k_N + 1 \text{ and } \varrho \geq 0. \end{aligned}$$

This shows that $\lim_{n \rightarrow \infty} s(n; \varrho) = y$ uniformly in $\varrho \geq 0$.

In a similar way we can prove the weak case.

Q. E. D.

Throughout the rest of this section, we assume that $\{x_n\}$ is an almost-orbit of T satisfying

$$(2.5) \quad \lim_{n \rightarrow \infty} \|x_n - x_{n+i}\| \text{ exists uniformly in } i \geq 0.$$

Lemma 9. The following holds:

$$\lim_{\varrho, m, n \rightarrow \infty} \|T^\varrho \left(\frac{1}{2n} \sum_{i=0}^{n-1} x_{i+n} + \frac{1}{2m} \sum_{i=0}^{m-1} x_{i+m} \right) - \left(\frac{1}{2n} \sum_{i=0}^{n-1} T^\varrho x_{i+n} + \frac{1}{2m} \sum_{i=0}^{m-1} T^\varrho x_{i+m} \right)\| = 0.$$

In particular, $\lim_{\varrho, n \rightarrow \infty} \|T^\varrho \left(\frac{1}{n} \sum_{i=0}^{n-1} x_{i+n} \right) - \frac{1}{n} \sum_{i=0}^{n-1} T^\varrho x_{i+n}\| = 0.$

Proof. By Lemma 1,

$$(2.6) \quad \|T^\varrho \left(\frac{1}{2n} \sum_{i=0}^{n-1} x_{i+n} + \frac{1}{2m} \sum_{i=0}^{m-1} x_{i+m} \right) - \left(\frac{1}{2n} \sum_{i=0}^{n-1} T^\varrho x_{i+n} + \frac{1}{2m} \sum_{i=0}^{m-1} T^\varrho x_{i+m} \right)\|$$

$$\leq M\gamma^{-1} \left(\max \{ \|x_{i+n} - x_{j+n}\| - \frac{1}{1+\alpha_\varrho} \|T^\varrho x_{i+n} - T^\varrho x_{j+n}\|, \|x_{i+n} - x_{p+m}\| \right.$$

$$\left. - \frac{1}{1+\alpha_\varrho} \|T^\varrho x_{i+n} - T^\varrho x_{p+m}\|, \|x_{p+m} - x_{q+m}\| - \frac{1}{1+\alpha_\varrho} \|T^\varrho x_{p+m} - T^\varrho x_{q+m}\| : \right.$$

$$\left. 0 \leq i, j \leq n-1, 0 \leq p, q \leq m-1 \right) \quad \text{for any } n, m \geq 1 \text{ and } \varrho \geq 0.$$

For any $\varepsilon > 0$ choose $\delta > 0$ such that $\gamma^{-1}(\delta) < \varepsilon/M$. By the assumption, there exists $N \geq 1$ such that $\sup_{i \geq 0} | \|x_n - x_{n+i}\| - \|x_m - x_{m+i}\| | < \delta/4$, $\sup_{r \geq 0} \|x_{n+r} - T^r x_n\| < \delta/4$, and $\alpha_\varrho < \delta/4D$ for every $\varrho, m, n \geq N$.

$$\text{If } \varrho, m, n \geq N, \|x_{i+n} - x_{j+m}\| - \frac{1}{1+\alpha_\varrho} \|T^\varrho x_{i+n} - T^\varrho x_{j+m}\|$$

$$\leq \|x_{i+n} - x_{j+m}\| - \|x_{i+\varrho+n} - x_{j+\varrho+m}\| + \|x_{i+\varrho+n} - T^\varrho x_{i+n}\|$$

$$+ \|x_{j+\varrho+m} - T^\varrho x_{j+m}\| + \alpha_\varrho \|x_{i+n} - x_{j+m}\| < \delta \text{ for every } i, j \geq 0.$$

Combining this with (2.6),

$$\|T^\varrho \left(\frac{1}{2n} \sum_{i=0}^{n-1} x_{i+n} + \frac{1}{2m} \sum_{i=0}^{m-1} x_{i+m} \right) - \left(\frac{1}{2n} \sum_{i=0}^{n-1} T^\varrho x_{i+n} + \frac{1}{2m} \sum_{i=0}^{m-1} T^\varrho x_{i+m} \right)\| < \varepsilon$$

for every $\varrho, m, n \geq N$.

Q. E. D.

Lemma 10. $\{s(n;n)\}$ is strongly convergent as $n \rightarrow \infty$ to an element y of $F(T)$.

Proof. Take $f \in F(T)$ and set $u_n = s(n;n) - f$ for $n \geq 1$. Similarly as the proof of Lemma 7 (i), using Lemma 9, we can see that $\|u_n\| = \|s(n;n) - f\|$ converges as $n \rightarrow \infty$. Put $d = \lim_{n \rightarrow \infty} \|u_n\|$.

Then, we have

$$(2.7) \quad \lim_{n \rightarrow \infty} \|u_n + u_{n+i}\| = 2d \text{ for every } i \geq 1$$

because $\|u_n - u_{n+1}\| \rightarrow 0$ as $n \rightarrow \infty$.

Since

$$s(n+k;n+k) = \frac{1}{n+k} \sum_{i=0}^{n+k-1} s(n;n+k+i) + v(n,k), \quad \|v(n,k)\| \leq \frac{(n-1)D}{2(n+k)},$$

$$\text{where } v(n,k) = \frac{1}{n(n+k)} \sum_{i=1}^{n-1} (n-i) (x_{i+n+k-1} - x_{i+2(n+k)-1}),$$

it follows that

$$\begin{aligned} \|u_{n+k} + u_{m+k}\| &\leq \left\| \frac{1}{n+k} \sum_{i=0}^{n+k-1} (s(n;n+k+i) + s(m;m+k+i) - 2f) \right\| \\ &+ \left\| \frac{m-n}{(m+k)(n+k)} \sum_{i=0}^{n+k-1} (s(m;m+k+i) - f) \right\| \\ &+ \left\| \frac{1}{m+k} \sum_{i=n+k}^{m+k-1} (s(m;m+k+i) - f) \right\| + \|v(n,k)\| + \|v(m,k)\| \\ &\leq \frac{2}{n+k} \sum_{i=0}^{n+k-1} \|2^{-1} (s(n;n+k+i) + s(m;m+k+i)) - f\| + \frac{2(m-n)D}{m+k} \\ &+ \frac{(n-1)D}{2(n+k)} + \frac{(m-1)D}{2(m+k)} \text{ for } m \geq n \geq 1 \text{ and } k \geq 0. \end{aligned}$$

Moreover,

$$\|2^{-1} (s(n;n+k+i) + s(m;m+k+i)) - f\|$$

$$\begin{aligned} &\leq \frac{1}{2n} \sum_{j=0}^{n-1} \sup_{\varrho \geq 0} \|x_{j+n+\varrho} - T^{\varrho} x_{j+n}\| + \frac{1}{2m} \sum_{j=0}^{m-1} \sup_{\varrho \geq 0} \|x_{j+m+\varrho} - T^{\varrho} x_{j+m}\| \\ &+ \left\| \left(\frac{1}{2n} \sum_{j=0}^{n-1} T^{i+k} x_{j+n} + \frac{1}{2m} \sum_{j=0}^{m-1} T^{i+k} x_{j+m} \right) - T^{i+k} \left(\frac{1}{2n} \sum_{j=0}^{n-1} x_{j+n} + \frac{1}{2m} \sum_{j=0}^{m-1} x_{j+m} \right) \right\| \\ &+ (1 + \alpha_{i+k}) \|2^{-1} s(n;n) + 2^{-1} s(m;m) - f\| \end{aligned}$$

for $m, n \geq 1$ and $i, k \geq 0$.

By Lemma 9, for any $\varepsilon > 0$ there exists $N \geq 1$ such that

$$\|T^k \left(\frac{1}{2n} \sum_{i=0}^{n-1} x_{i+n} + \frac{1}{2m} \sum_{i=0}^{m-1} x_{i+m} \right) - \left(\frac{1}{2n} \sum_{i=0}^{n-1} T^k x_{i+n} + \frac{1}{2m} \sum_{i=0}^{m-1} T^k x_{i+m} \right)\| < \varepsilon,$$

$$\sup_{r \geq 0} \|x_{n+r} - T^r x_n\| < \varepsilon, \text{ and } \alpha_k < \varepsilon/D \text{ for every } k, m, n \geq N.$$

Consequently, we obtain

$$\|u_{n+k} + u_{m+k}\| \leq 6\varepsilon + \|u_n + u_m\| + \frac{2(m-n)D}{m+k} + \frac{(n-1)D}{2(n+k)} + \frac{(m-1)D}{2(m+k)}$$

for every $m \geq n \geq N$ and $k \geq N$. Letting $k \rightarrow \infty$, it follows from (2.7)

that $2d \leq 6\varepsilon + \|u_n + u_m\|$ for every $m, n \geq N$. Hence

$$2d \leq \liminf_{n, m \rightarrow \infty} \|u_n + u_m\| \leq \limsup_{n, m \rightarrow \infty} \|u_n + u_m\| \leq 2d$$

and so $\lim_{n, m \rightarrow \infty} \|u_n + u_m\| = 2d$. By uniform convexity of X

$$\text{and } \lim_{n \rightarrow \infty} \|u_n\| = d, \lim_{m \rightarrow \infty} \|s(n;n) - s(m;m)\| = \lim_{n, m \rightarrow \infty} \|u_n - u_m\| = 0,$$

whence $\{s(n;n)\}$ converges strongly. Put $y = \lim_{n \rightarrow \infty} s(n;n)$.

Then we have

$$\|y - T^{\varrho} y\| \leq \|y - s(n;n)\| + \|s(n;n) - s(n;n+\varrho)\|$$

$$\begin{aligned}
& + \left\| \frac{1}{n} \sum_{i=0}^{n-1} (x_{i+n+q} - T^q x_{i+n}) \right\| + \left\| \frac{1}{n} \sum_{i=0}^{n-1} T^q x_{i+n} - T^q \left(\frac{1}{n} \sum_{i=0}^{n-1} x_{i+n} \right) \right\| \\
& + \|T^q s(n;n) - T^q y\| \\
& \leq (M+1) \|y - s(n;n)\| + 2\varepsilon + \frac{q}{n} D \quad \text{for all } n, q \geq N.
\end{aligned}$$

Hence $\lim_{q \rightarrow \infty} \|T^q y - y\| = 0$ and so $y \in F(T)$.

Q. E. D.

3. Proof of Theorems.

Proof of Theorem 1. Let $\{x_n\}$ be an almost-orbit of T . First, suppose that X is (F). By Lemma 7 (ii), there exist a sequence $\{i_n\}$ of nonnegative integers and an element y of $F(T)$ such that $\{y\} = F(T) \cap \text{clco } \omega_w(\{x_n\})$ and $w\text{-}\lim_{n \rightarrow \infty} s(n; k_n) = y$ for any sequence $\{k_n\}$ with $k_n \geq i_n$ for all n . This implies that $w\text{-}\lim_{n \rightarrow \infty} s(n; i_n + q) = y$ uniformly in $q \geq 0$. Hence $\{x_n\}$ is weakly almost convergent to y by Lemma 8.

Next, suppose that X satisfies Opial's condition. We denote by Λ the set of sequences $\{k_n\}$ of nonnegative integers with $k_n \geq i_n$ for all n , where $\{i_n\}$ is as in Lemma 7. It follows from Lemma 7 (ii) that $\|s(n; k_n) - f\|$ converges as $n \rightarrow \infty$ for every $\{k_n\} \in \Lambda$ and $f \in F(T)$. Define $r(\{k_n\}; f)$, $r(\{k_n\})$, and r by

$$r(\{k_n\}; f) = \lim_{n \rightarrow \infty} \|s(n; k_n) - f\| \quad \text{for } \{k_n\} \in \Lambda \text{ and } f \in F(T),$$

$$r(\{k_n\}) = \inf \{r(\{k_n\}; f) : f \in F(T)\} \quad \text{for } \{k_n\} \in \Lambda,$$

and

$$r = \inf \{r(\{k_n\}) : \{k_n\} \in \Lambda\},$$

respectively. Now, choose $\{k_n^{(i)}\} \in \Lambda$, $i = 1, 2, \dots$, such that

$\lim_{i \rightarrow \infty} r(\{k_n^{(i)}\}) = r$, and let $h_n = \max \{k_n^{(i)} : 1 \leq i \leq n\} + N_n$ for $n \geq 1$,

where $\{N_n\}$ is as in the proof of Lemma 7. Clearly $\{h_n\} \in \Lambda$.

Moreover, we obtain

$$(3.1) \quad r(\{h_n\}) = r.$$

To show this, let $n \geq i \geq 1$ and $f \in F(T)$. Then,

$$(3.2) \quad \begin{aligned} \|s(n; h_n) - f\| &\leq \frac{1}{n} \sum_{j=0}^{n-1} \|x_{j+h_n} - T^{h_n - k_n^{(i)}} x_{j+k_n^{(i)}}\| \\ &+ \left\| \frac{1}{n} \sum_{j=0}^{n-1} T^{h_n - k_n^{(i)}} x_{j+k_n^{(i)}} - T^{h_n - k_n^{(i)}} \left(\frac{1}{n} \sum_{j=0}^{n-1} x_{j+k_n^{(i)}} \right) \right\| \\ &+ \left\| T^{h_n - k_n^{(i)}} \left(\frac{1}{n} \sum_{j=0}^{n-1} x_{j+k_n^{(i)}} \right) - f \right\| \leq \frac{1}{n} \sum_{j=0}^{n-1} \sup_{\alpha \geq 0} \|x_{j+k_n^{(i)} + \alpha} - T^\alpha x_{j+k_n^{(i)}}\| + \frac{1}{n} \\ &+ (1 + \alpha_{h_n - k_n^{(i)}}) \|s(n; k_n^{(i)}) - f\|. \end{aligned}$$

Letting $n \rightarrow \infty$, it follows that $r(\{h_n\}; f) \leq r(\{k_n^{(i)}\}; f)$

for all $f \in F(T)$ and so $r(\{h_n\}) \leq \lim_{i \rightarrow \infty} r(\{k_n^{(i)}\}) = r$.

But $r \leq r(\{h_n\})$ by the definition of r . Thus (3.1) holds.

Since $F(T)$ is closed convex (For example, see [3, Theorem 2].) and $\{s(n; h_n)\}$ is bounded, the reflexivity of X implies that there is an element y of $F(T)$ such that $r(\{h_n\}; y) = r(\{h_n\}) (= r)$.

Set $h'_n = h_n + N_n$. Then we shall show

$$(3.3) \quad w\text{-}\lim_{n \rightarrow \infty} s(n; h'_n + \alpha) = y \text{ uniformly in } \alpha \geq 0.$$

If this is shown, the conclusion follows from Lemma 8.

To show (3.3) let $\{\varrho_n\}$ be an arbitrary sequence such that $\varrho_n \geq h'_n$ for all n . $\{\varrho_n\} \in \Lambda$ and by Lemma 7 (ii) there exists $z \in F(T)$ such that $w\text{-}\lim_{n \rightarrow \infty} s(n; \varrho_n) = z$. Suppose $z \neq y$.

Then Opial's condition implies that

$$r(\{\varrho_n\}) \leq \lim_{n \rightarrow \infty} \|s(n; \varrho_n) - z\| < \lim_{n \rightarrow \infty} \|s(n; \varrho_n) - y\| = r(\{\varrho_n\}; y).$$

But, by the same way as in (3.2), we have

$r(\{\varrho_n\}; y) \leq r(\{h_n\}; y) \leq r(\{h_n\}) = r$. Thus $r(\{\varrho_n\}) < r$ and this contradicts the definition of r . Hence $z = y$ and so $w\text{-}\lim_{n \rightarrow \infty} s(n; \varrho_n) = y$.

Clearly, this implies (3.3).

Q. E. D.

Proof of Theorem 2. Let $\{x_n\}$ be an almost-orbit of T and suppose that $\lim_{n \rightarrow \infty} \|x_n - x_{n+i}\|$ exists uniformly in $i \geq 0$.

We shall show that there exists an element y of $F(T)$ such that $\lim_{n \rightarrow \infty} s(n; 2n+\varrho) = y$ uniformly in $\varrho \geq 0$. By Lemma 9, for any $\varepsilon > 0$ there exists $N \geq 1$ such that

$$\left\| \frac{1}{n} \sum_{i=0}^{n-1} T^{n+\varrho} x_{i+n} - T^{n+\varrho} \left(\frac{1}{n} \sum_{i=0}^{n-1} x_{i+n} \right) \right\| < \varepsilon \text{ and } \sup_{r \geq 0} \|x_{n+r} - T^r x_n\| < \varepsilon$$

for every $n \geq N$ and $\varrho \geq 0$.

By Lemma 10, there exists an element y of $F(T)$ such that

$\lim_{n \rightarrow \infty} s(n; n) = y$. Then we have

$$\begin{aligned} \|s(n; 2n+\varrho) - y\| &\leq \frac{1}{n} \sum_{i=0}^{n-1} \|x_{i+2n+\varrho} - T^{n+\varrho} x_{i+n}\| \\ &+ \left\| \frac{1}{n} \sum_{i=0}^{n-1} T^{n+\varrho} x_{i+n} - T^{n+\varrho} \left(\frac{1}{n} \sum_{i=0}^{n-1} x_{i+n} \right) \right\| + \left\| T^{n+\varrho} \left(\frac{1}{n} \sum_{i=0}^{n-1} x_{i+n} \right) - y \right\| \end{aligned}$$

$\leq 2\varepsilon + M \|s(n;n) - y\|$ for every $n \geq N$ and $\varrho \geq 0$.

Hence $\lim_{n \rightarrow \infty} s(n;2n+\varrho) = y$ uniformly in $\varrho \geq 0$ and so the conclusion

follows from Lemma 8.

Q. E. D.

Remark. The assumption "C is bounded" in Theorems 1 and 2 may be replaced by " $F(T) \neq \emptyset$ ".

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