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On the existence of weak solutions of stationary Boussinesq equation

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§1. Notations and results.

In this paper, we discuss the existence of weak solutions of equations which describe the motion of fluid of natural convection (Boussinesq approximation) in a bounded domain \( \Omega \) in \( \mathbb{R}^n \), \( 2 \leq n \). We consider the following system of differential equations which is called stationary Boussinesq equation:

\[
\begin{align*}
(u \cdot \nabla)u &= -\frac{1}{\rho} \nabla p + \nu \Delta u + \beta g \theta, \\
\text{div } u &= 0, \\
(u \cdot \nabla)\theta &= \kappa \Delta \theta,
\end{align*}
\]

where \( u \cdot \nabla = \sum_j u_j \frac{\partial}{\partial x_j} \). Here \( u \) is the fluid velocity, \( p \) is the pressure, \( \theta \) is the temperature, \( g \) is the gravitational vector function, and \( \rho \) (density), \( \nu \) (kinematic viscosity), \( \beta \) (coefficient of volume expansion), \( \kappa \) (thermal conductivity) are positive constants. We study this system of equations with mixed boundary condition for \( \theta \).

In the previous paper[ 8], we treated this problem only for the case \( n = 3 \). By using the Galerkin method, we can show the existence of weak solution, for any integer \( n \) greater than or equal to 2. Some uniqueness result is also obtained.

Let \( \partial \Omega \) (the boundary of \( \Omega \)) be divided into two parts \( \Gamma_1, \Gamma_2 \) such that
\[ \varnothing \Omega = \Gamma_1 \cup \Gamma_2, \quad \Gamma_1 \cap \Gamma_2 = \phi. \]

The boundary conditions are as follows.

(1-2) \[
\begin{align*}
u &= 0, \quad \vartheta = \xi, \quad \text{on } \Gamma_1, \\
u &= 0, \quad \frac{\partial \vartheta}{\partial n} = 0, \quad \text{on } \Gamma_2,
\end{align*}
\]

where \( \xi \) is a given function on \( \Gamma_1 \), \( n \) is the outer normal vector to \( \varnothing \Omega \). If we can find a function \( \vartheta_0 \) defined on \( \Omega \), of class \( C^2(\Omega) \cap C^1(\overline{\Omega}) \), satisfying

\( \vartheta_0 = \xi \) on \( \Gamma_1 \) and \( \frac{\partial \vartheta}{\partial n} \vartheta_0 = 0 \) on \( \Gamma_2 \),

then we can transform the equations (1-1),(1-2) for \( u \) and \( \vartheta = \vartheta - \vartheta_0 \) and we obtain the following:

(1-3) \[
\begin{align*}
(u \cdot \nabla)u &= -\frac{1}{\rho} \nabla p + \nu \Delta u + \beta u \vartheta + \beta u \vartheta_0, \quad \text{in } \Omega, \\
\text{div } u &= 0, \quad \text{in } \Omega, \\
(u \cdot \nabla)\vartheta &= \kappa \Delta \vartheta - (u \cdot \nabla)\vartheta_0 + \kappa \Delta \vartheta_0, \quad \text{in } \Omega, \\
u &= 0, \quad \vartheta = 0 \quad \text{on } \Gamma_1, \\
u &= 0, \quad \frac{\partial \vartheta}{\partial n} = 0 \quad \text{on } \Gamma_2.
\end{align*}
\]

For the domain \( \Omega \), we assume:

**Condition (H)**

\( \Omega \) is a bounded domain in \( \mathbb{R}^n \) with \( C^2 \) boundary. The boundary \( \varnothing \Omega \) of \( \Omega \) is devided as follows:

\( \varnothing \Omega = \Gamma_1 \cup \Gamma_2, \quad \Gamma_1 \cap \Gamma_2 = \phi \), measure of \( \Gamma_1 \neq 0 \),

and the intersection \( \overline{\Gamma}_1 \cap \overline{\Gamma}_2 \) is a \( n-1 \) dimensional \( C^1 \) manifold.

In order to state the definition of weak solution and our result, we introduce some

**Function spaces:**

\[ D_\sigma = \{ \text{vector function } \varphi \in C^\infty(\Omega) | \text{supp } \varphi \subset \Omega, \text{ div } \varphi = 0 \text{ in } \Omega \} \]

\[ H = \text{completion of } D_\sigma \text{ under the } L^2(\Omega)-\text{norm} \]
\[ \mathbb{V} = \text{completion of } D_\sigma \text{ under the } H^1(\Omega)\text{-norm} \]
\[ \mathbb{V} = \text{completion of } D_\sigma \text{ under the norm } \| u \|_{H^1(\Omega)} + \| u \|_{L^n(\Omega)}. \]
\[ D_0 = \{ \text{scalar function } \varphi \in C^\infty(\overline{\Omega}) | \]
\[ \varphi \equiv 0 \text{ in a neighborhood of } \Gamma_1 \}, \]
\[ \mathbb{W} = \text{completion of } D_0 \text{ under the } H^1(\Omega)\text{-norm}. \]
\[ \mathbb{W} = \text{completion of } D_0 \text{ under the norm } \| u \|_{H^1(\Omega)} + \| u \|_{L^n(\Omega)}. \]

Consider \( L^2 \) inner product of the first equation of (1-3) with \( \nu \) in \( \mathbb{V} \), and the third equation of (1-3) with \( \tau \) in \( \mathbb{W} \). Then we obtain:

**Auxiliary problem:** Find \( u \in \mathbb{V} \) and \( \theta \in \mathbb{W} \) satisfying

\[
\begin{align*}
\nu(\nabla u, \nabla v) + B(u, u, v) - (\beta g \theta, v) - (\beta g \theta_0, v) &= 0, \\
&\quad \text{for all } v \text{ in } \mathbb{V}, \\
\kappa(\nabla \theta, \nabla \tau) + b(u, \theta, \tau) + b(u, \theta_0, \tau) + \kappa(\nabla \theta_0, \nabla \tau) &= 0, \\
&\quad \text{for all } \tau \text{ in } \mathbb{W},
\end{align*}
\]

where

\[
B(u, v, w) = ((u \cdot \nabla)v, w) = \sum_{i,j=1}^{n} u_j(x) \frac{\partial v_i(x)}{\partial x_j} w_i(x) \, dx,
\]

and

\[
b(u, \theta, \tau) = ((u \cdot \nabla)\theta, \tau) = \sum_{j=1}^{n} u_j(x) \frac{\partial \theta(x)}{\partial x_j} \tau(x) \, dx.
\]

Now, we define the weak solution of (1-1), (1-2).

**Definition 1.** The pair of functions \((u, \theta)\) is called a weak solution of (1-1), (1-2), if there exists a function \( \theta_0 \) in \( C^1(\overline{\Omega}) \).
such that \( u \in V, \theta - \theta_0 \in W, \theta_0 = \xi \) on \( \Gamma_1 \), \( \frac{\partial}{\partial n} \theta_0 = 0 \) on \( \Gamma_2 \), and \((u, \mathcal{G}) \) \( (\mathcal{G} = \theta - \theta_0) \) satisfies (1-4).

Now, we state our results.

**Theorem 1**

Let \( \Omega \) be a bounded domain in \( \mathbb{R}^n \) with \( C^2 \) boundary satisfying the condition (H). If the function \( g(x) \) is in \( L^\infty(\Omega) \) and \( \xi \) is of class \( C^1(\overline{\Gamma_1}) \), then there exists a weak solution of (1-1), (1-2).

**Remark 1**

Generally, \( \tilde{V} \subset V \cap L^1(\Omega) \) and \( \tilde{W} \subset W \cap L^1(\Omega) \). For \( 2 \leq n \leq 4 \), \( \tilde{V} = V \) and \( \tilde{W} = W \) (c.f. Masuda[7], Giga[3]). Therefore our theorem contains the result of [8].

Let \( g_\infty = \| g \|_{L^\infty(\Omega)} \), and \( c, c_1, c_2 \) be constants in Lemma 3 (§2). As for the uniqueness, we have:

**Theorem 2**

The weak solution \((u, \theta)\) of (1-1), (1-2) satisfying

(i) \( u \in L^n(\Omega), \theta \in L^n(\Omega) \),

(ii) \( c \| u \|_n + \frac{\beta g_\infty c_1 c_2}{\kappa} \| \theta \| n < \nu \), when \( n \geq 3 \),

(iii) \( c \| u \|_p + \frac{\beta g_\infty c_1 c_2}{\kappa} \| \theta \|_p < \nu \), for some \( p > 2 \), when \( n = 2 \)

is, if it exists, unique.

**Remark 2**

The condition (i) is automatically satisfied when \( 2 \leq n \leq 4 \).

**Remark 3**

If we set

\[
\text{Re} = \frac{c}{\nu} \| u \|_n
\]

(Reynolds number),
\[ Ra = \frac{\beta g \sigma c}{\nu k} \cdot \frac{c c_1 c_2 - \| \theta \|}{n} \quad \text{(Rayleigh number)}, \]

then the condition (ii) reads as

\[ \text{Re} + Ra < 1. \]

See also Joseph[5].

\[ §2. \quad \text{Some lemmas.} \]

In this section, we prepare some lemmas.

**Lemma 1**

\( V \) and \( \hat{W} \) are separable Banach spaces.

**Proof.** A subset of separable metric space is separable (e.g. Brezis[2]). If we show \( V \cap L^n(\Omega) \) is separable, Lemma is proved. We can identify \( V \cap L^n(\Omega) \) as a subset

\[ F = \{(v, \frac{\partial}{\partial x_1} v, \ldots, \frac{\partial}{\partial x_n} v) ; v \in V \cap L^n(\Omega)\} \]

of \( L^n(\Omega) \times L^2(\Omega) \times \cdots \times L^2(\Omega) \). Since the latter space is separable, the set \( F \) is also separable and Lemma 1 is proved.

**Lemma 2** (Sobolev)

Sobolev space \( H^1(\Omega) \) is continuously imbedded in \( L^q(\Omega) \), where \( q = \frac{2n}{n-2} \) for \( n \geq 3 \), and \(* \circ \circ > q \geq 1 \) for \( n = 2 \).

For the proof, see Adams[1].

**Lemma 3** (Poincaré)

There exist constants \( c_1, c_2, c \) depending on \( \Omega \) and \( n \) such that
(i) \[ \|u\| \leq c_1 \|\nabla u\| \quad \text{for} \quad \forall u \in V, \]

(ii) \[ \|u\|_q \leq c_2 \|\nabla u\| \quad \text{for} \quad \forall u \in V, \quad q = \frac{2n}{n-2} \quad (n \geq 3), \]
\[ \quad q = 4 \quad (n = 2), \]

(iii) \[ \|\Theta\| \leq c_2 \|\nabla \Theta\| \quad \text{for} \quad \forall \Theta \in W. \]

These constants are used in the statement of Theorem 2. For the proof of (i), (iii), see Morimoto[8]. (ii) follows from (i) and Lemma 2.

By Hölder’s inequality and Lemmas 2, 3, we have:

**Lemma 4**

Let \( n \geq 3 \). There exists a constant \( c_B \) depending on \( \Omega \) and \( n \) such that

\[ |B(u,v,w)| \leq c_B \|\nabla u\| \|\nabla v\| \|w\|_n \quad \text{for} \quad \forall u \in V, \forall v \in H^1(\Omega), \forall w \in L^n(\Omega), \]

\[ |b(u,\Theta,\tau)| \leq c_B \|\nabla u\| \|\nabla \Theta\| \|\tau\|_n \quad \text{for} \quad \forall u \in V, \forall \Theta \in H^1(\Omega), \forall \tau \in L^n(\Omega), \]

hold.

Using the integration by parts, we obtain:

**Lemma 5**

(i) \( B(u,v,w) = -B(u,w,v) \) for \( \forall u \in V, \forall v,w \in H^1 \cap L^n \)

holds. In particular,

\[ B(u,v,v) = 0 \quad \text{for} \quad \forall u \in V, \forall v \in H^1 \cap L^n. \]

(ii) \( b(u,\Theta,\tau) = -b(u,\tau,\Theta) \) for \( \forall u \in V, \forall \Theta,\tau \in H^1 \cap L^n \),

holds. In particular,

\[ b(u,\Theta,\Theta) = 0 \quad \text{for} \quad \forall u \in V, \forall \Theta \in H^1 \cap L^n. \]
Lemma 6 (Whitney)

Let \( \Omega \) be a bounded domain in \( \mathbb{R}^n \) with \( C^2 \) boundary \( \partial \Omega \). If \( \xi \) is a \( C^1 \) function defined on \( \partial \Omega \), then for any positive number \( \varepsilon \) and any \( p \geq 1 \), there exists an extension \( \theta_0 \) of \( \xi \) such that

\[
\theta_0 \in C^1(\mathbb{R}^n), \quad \theta_0 = \xi, \quad \frac{\partial \theta_0}{\partial n} = 0 \text{ on } \partial \Omega, \quad \| \theta_0 \|_p < \varepsilon.
\]

Proof. It is well known as Whitney's extension theorem (see Malgrange[6]). In the case \( n = 3 \), we can prove it directly (Morimoto[8]), and it is easy to extend to the general case.

§3. Proof of Theorem 1.

Under our assumptions on \( \partial \Omega \) and \( \xi \), we have an extension \( \theta_0 \) of \( \xi \) (Lemma 6), and we study the equation (1-4). Using the Galerkin method, we construct approximate solutions of (1-4).

Let \( (\varphi_j) \) be a sequence of functions in \( D_0 \), linearly independent and total in \( \tilde{V} \). We can assume \( (\varphi_j, \varphi_k) = \delta_{jk} \) without loss of generality. Let \( (\psi_j) \) be a sequence of functions in \( D_0 \), linearly independent and total in \( \tilde{W} \). We can assume \( (\varphi_j, \psi_k) = \delta_{jk} \). Since \( \bar{V} \) (resp. \( \bar{W} \)) is separable and \( D_\sigma \) (resp. \( D_0 \)) is dense there, we can find these functions. We put

\[
u(m) = \sum_{j=1}^m \xi_j \varphi_j, \quad \theta(m) = \sum_{j=1}^m \xi_{m+j} \psi_j,
\]

and we consider the following system of equations:

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(3-1) $\nu(\nabla u^{(m)}, \nabla \varphi_j) + ((u^{(m)} \cdot \nabla) u^{(m)}, \varphi_j) - (\beta g \theta^{(m)}, \varphi_j) - (\beta g \theta_0, \varphi_j) = 0, \quad 1 \leq j \leq m.$

(3-2) $\kappa(\nabla \theta^{(m)}, \nabla \psi_j) + ((u^{(m)} \cdot \nabla) \theta^{(m)}, \psi_j) + ((u^{(m)} \cdot \nabla) \theta_0, \psi_j) + \kappa(\nabla \theta_0, \nabla \psi_j) = 0, \quad 1 \leq j \leq m.$

Substituting $u^{(m)}, \theta^{(m)}$ into these equations, we obtain:

(3-3) $\xi_j + \frac{1}{\nu} \sum_{k,l} \xi_k \xi_l ((\phi_k \cdot \nabla \varphi_l, \varphi_j) - \frac{1}{\nu} \sum_k \xi_{m+k} (\beta g \psi_k, \varphi_j) - \frac{1}{\nu} (\beta g \theta_0, \varphi_j) = 0, \quad 1 \leq j \leq m,$

(3-4) $\xi_{m+j} + \frac{1}{\kappa} \sum_{k,l} \xi_k \xi_l ((\phi_k \cdot \nabla \psi_l, \psi_j) + \frac{1}{\kappa} \sum_k \xi_k ((\phi_k \cdot \nabla) \theta_0, \psi_j) + (\nabla \theta_0, \nabla \psi_j) = 0, \quad 1 \leq j \leq m.$

The left hand side of (3-3), (3-4) determines a polynomial which we denote by

$$P_j(\xi_1, \xi_2, \ldots, \xi_{2m}), \quad 1 \leq j \leq 2m.$$ 

$P_j$ is a polynomial in $\xi = (\xi_1, \ldots, \xi_{2m})$ of degree 2. Let $P$ be a mapping from $\mathbb{R}^{2m}$ to $\mathbb{R}^{2m}$ defined by $P(\xi) = (P_1(\xi), \ldots, P_{2m}(\xi)).$

Then the fixed point $\xi$ of $P$, if it exists, is a solution of (3-3), (3-4). We show the existence of a fixed point of $P$.

Let $\xi = \xi(\lambda)$ be any solution of $\xi = \lambda P(\xi), \ 0 \leq \lambda \leq 1$. First we treat the case $n \geq 3$.

$$\sum_{j=1}^{m} |\xi_j|^2 = \|\nabla u^{(m)}\|^2 = \lambda \sum_{j=1}^{m} P_j(\xi) \xi_j$$

$$= -\frac{\lambda}{\nu} \sum_{j,k,l} \xi_j \xi_k \xi_l ((\phi_k \cdot \nabla) \varphi_l, \varphi_j) + \frac{\lambda}{\nu} \sum_k \xi_{m+k} (\beta g \psi_k, \varphi_j) + \frac{\lambda}{\nu} \sum_j (\beta g \theta_0, \varphi_j)$$

$$= -\frac{\lambda}{\nu} ((u^{(m)} \cdot \nabla) u^{(m)}, u^{(m)}) + \frac{\lambda}{\nu} ((g \theta^{(m)}, u^{(m)}) + (g \theta_0, u^{(m)}))$$

$$\leq \frac{\lambda}{\nu} (\| \theta^{(m)} \| + \| \theta_0 \|) \| u^{(m)} \|$$

$$\leq \frac{\lambda}{\nu} c_2 \| \nabla \theta^{(m)} \| + \| \theta_0 \| \| \nabla u^{(m)} \|,$$

where we have used Lemmas 4, 5. Thereby,
(3-5) \( \| \nabla u^{(m)} \| \leq \frac{\lambda \theta \| c_1 \} \|}{0} \left( c_2 \| \nabla \theta^{(m)} \| + \| \theta_0 \| \right) . \]

Similarly,

\[
\sum_{j=1}^{m} |\xi_{m+j}|^2 = \| \nabla \theta^{(m)} \|^2 = \lambda \sum_{j=1}^{m} P_{m+j}(\xi) \xi_{m+j}
\]

\[
= -\frac{\lambda}{\kappa} \sum_{j,k} \xi_k \xi_{m+j} \left( (\psi_k \cdot \nabla) \psi_j, \psi_j \right) + \frac{\lambda}{\kappa} \sum_{j} \xi_j \xi_{m+j} \left( (\nabla \theta_0, \nabla \psi_j) - \lambda \sum_{j} \xi_j \nabla \theta_0, \nabla \psi_j \right)
\]

\[
= -\frac{\lambda}{\kappa} \left( (u^{(m)} \cdot \nabla) \theta^{(m)}, \theta^{(m)} \right) - (u^{(m)} \cdot \nabla) \theta^{(m)}, \theta_0) \right) - \lambda (\nabla \theta_0, \nabla \theta^{(m)})
\]

\[
\leq \frac{\lambda}{\kappa} \| u^{(m)} \|_2 n/(n-2) \| \nabla \theta^{(m)} \| \| \theta_0 \|_n + \lambda \| \nabla \theta^{(m)} \| \| \nabla \theta_0 \| \]

(by Hölder's inequality)

\[
\leq \frac{\lambda C}{\kappa} \| u^{(m)} \| \| \nabla \theta^{(m)} \| \| \theta_0 \|_n + \lambda \| \nabla \theta_0 \| \| \nabla \theta^{(m)} \|
\]

(by Lemma 3).

For \( n = 2 \), we have

\[
\| \nabla \theta^{(m)} \|^2 \leq \frac{\lambda C}{\kappa} \| u^{(m)} \| \| \nabla \theta^{(m)} \| \| \theta_0 \|_4 + \lambda \| \nabla \theta_0 \| \| \nabla \theta^{(m)} \|
\]

Thereby,

(3-6) \( \| \nabla \theta^{(m)} \| \leq \frac{\lambda C}{\kappa} \| \theta_0 \|_p \| u^{(m)} \| + \lambda \| \nabla \theta_0 \|. \)

where \( p = n \) when \( n \geq 3 \), and \( p = 4 \) when \( n = 2 \). Substituting

(3-6) into (3-5), we obtain:

\[
1 - \frac{c c_1 c_2 \theta |k \nu |^2 \| \theta_0 \|_p}{2} \| u^{(m)} \| \leq \frac{\lambda C c \theta |k \nu |}{0} \left( c_2 \| \nabla \theta_0 \| + \| \theta_0 \| \right).
\]

According to Lemma 6, we can choose \( \theta_0 \) such that

(3-7) \( 1 - \frac{c c_1 c_2 \theta |k \nu |}{2} \| \theta_0 \|_p > \frac{1}{2} \)
holds. Then, we have

(3-8) \( \| u^{(m)} \| \leq \frac{2 \lambda c \theta |k \nu |}{0} \left( c_2 \| \nabla \theta_0 \| + \| \theta_0 \| \right) \]

\[
\leq \frac{2 \lambda c \theta |k \nu |}{0} \left( c_2 \| \nabla \theta_0 \| + \| \theta_0 \| \right) = \rho_1.
\]

Similarly, using (3-7), we have:

(3-9) \( \| \nabla \theta^{(m)} \| \leq \frac{2 \| \nabla \theta_0 \| + \frac{1}{c_2} \| \theta_0 \| \| \theta_0 \| \equiv \rho_2 .
\]

Note that \( \rho_1 \) and \( \rho_2 \) are constants independent of \( \lambda \) and \( m \).
Thereby the solution $\xi$ of $\xi = \lambda P(\xi)$ satisfies:

$$\sum_{j=1}^{2m} |\xi_j|^2 \leq \rho_1^2 + \rho_2^2 = \rho^2 , \text{ for } 0 \leq \lambda \leq 1.$$ 

Leray-Schauder's Theorem[4] tells us the existence of a fixed point of the mapping $P$: $\xi = P(\xi)$, such that $|\xi| \leq \rho$. Thus we have obtained the solutions $u^{(m)}, \theta^{(m)}$ of (3-1), (3-2).

Moreover, they satisfy the estimates:

$$\|\nabla u^{(m)}\| \leq \rho_1 , \|\nabla \theta^{(m)}\| \leq \rho_2.$$ 

Since $V$ (resp. $W$) is compactly imbedded in $H_\sigma$ (resp. $L^2$), we can choose subsequences of $(u^{(m)}), (\theta^{(m)})$ which we denote by the same symbols, and elements $u \in V$, $\theta \in W$ such that the following convergences hold:

(3-10) $u^{(m)} \longrightarrow u$ weakly in $V$, strongly in $H_\sigma$

(3-11) $\theta^{(m)} \longrightarrow \theta$ weakly in $W$, strongly in $L^2(\Omega)$.

For these convergent sequences, the following lemma holds:

**Lemma 7**

$$B(u^{(m)}, u^{(m)}, v) \longrightarrow B(u, u, v) , \text{ for } \forall v \in D_\sigma$$

$$b(u^{(m)}, \theta^{(m)}, \tau) \longrightarrow b(u, \theta, \tau) , \text{ for } \forall \tau \in D_0.$$ 

The proof is found in [9] and omitted. Using this lemma for (3-1), (3-2), we find

(3-12) $\nu(\nabla u, \nabla v) + B(u, u, v) - (\beta g \theta, v) - (\beta g \theta_0, v) = 0$,

(3-13) $\kappa(\nabla \theta, \nabla \tau) + b(u, \theta, \tau) + b(u, \theta_0, \tau) + \kappa(\nabla \theta_0, \nabla \tau) = 0$,

hold for $v = \varphi_j$, $\tau = \psi_j$, $\forall j$. By Lemma 4, we see the linear functional

$$v \longrightarrow B(u, u, v) \text{ (resp. } \tau \longrightarrow b(u, \theta, \tau))$$

is continuous in $L^n$. Thereby the linear functional

$$v \longrightarrow \text{the left hand side of (3-12)}$$

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(resp. $\tau \rightarrow$ the left hand side of (3-13))
is continuous in $\tilde{V} \cap L^2$ (resp. $W \cap L^2$). Since $\varphi_j$
(resp. $\psi_j$) is total in $\tilde{V}$ (resp. $\tilde{W}$), (3-12) (resp. (3-13)) holds
for any $v$ in $\tilde{V}$ (resp. $\tilde{W}$). Thereby $(u, \theta)$ is a required weak
solution.

§4. Proof of Theorem 2.

Let $(u_i, \theta_i)$, $i=1,2$, be weak solutions of (1-1), (1-2)
satisfying (i), (ii). For $i = 1, 2$, there is a function $\theta_0^{(i)}$
satisfying the condition in Definition 1. Then $u_i$ and
$\theta_i - \theta_0^{(i)}$ satisfy (1-4). Since $\theta_0^{(1)} - \theta_0^{(2)}$ is 0 on $\Gamma_1$, it
belongs to $W$. Thereby, $\theta_1 - \theta_2$ is also in $W$. Put $u = u_1 - u_2$
$\theta = \theta_1 - \theta_2$. Then, they satisfy the following:

\[ \begin{align*}
v(\nabla u, \nabla v) + B(u, u_1, v) + B(u_2, u, v) - (\beta g u, v) &= 0, \quad \forall v \in \tilde{V}, \\
k(\nabla \theta, \nabla \tau) + b(u, \theta_1, \tau) + b(u_2, \theta, \tau) &= 0, \quad \forall \tau \in \tilde{W}.
\end{align*} \tag{4-1} \]

Here we have used Lemma 5. From the condition (i), we see
$u \in \tilde{V}$, $\theta \in \tilde{W}$.

Therefore, we can take $v = u, \tau = \theta$, and we have

\[ \begin{align*}
v \| \nabla u \|^2 + B(u, u_1, u) - \beta g(\theta, u) &= 0, \\
k \| \nabla \theta \|^2 + b(u, \theta_1, \theta) &= 0. \tag{4-2} \end{align*} \]

Let $n \geq 3$. Making use of the Hölder's inequality and
Lemma 5 to estimate (4-2), we have

\[ \begin{align*}
v \| \nabla u \|^2 &\leq \| u \|_2^2 n/(n-2) \| \nabla u \|_n + \beta \| \theta \|_n \| u \|_n, \\
k \| \nabla \theta \|^2 &\leq \| u \|_2^2 n/(n-2) \| \nabla \theta \|_n + \| \theta_1 \|_n. \tag{4-3} \end{align*} \]
By Lemma 3, we estimate the right hand side of the above equations, and we obtain:
\[ \nu \| \nabla u \| \leq c \| u \|_{n} \| \nabla u \| + \beta g_n c_1 c_2 \| \nabla \theta \|, \]
\[ \kappa \| \nabla \theta \| \leq c \| \theta \|_{n} \| \nabla u \|. \]
Thereby,
\[ \nu \| \nabla u \| \leq (c \| u \|_{n} + \frac{\beta g_n c_1 c_2}{\kappa} \| \theta \|_{n}) \| \nabla u \| \]
holds. Since \( u_1, \theta_1 \) satisfy the condition (ii):
\[ c \| u_1 \|_{n} + \frac{\beta g_n c_1 c_2}{\kappa} \| \theta_1 \|_{n} \leq \nu, \]
therefore \( \| \nabla u \| = \| \nabla \theta \| = 0. \) Since \( u = 0 \) on \( \partial \Omega \) and \( \theta = 0 \) on \( \Gamma_1, \)
we see \( u = 0, \theta = 0 \) in \( \Omega. \) Therefore \( u_1 = u_2, \theta_1 = \theta_2 \) in \( \Omega. \)

When \( n = 2, \) we have
\[ \nu \| \nabla u \|^{2} \leq \| u \|_{p}, \| \nabla u \| \| u \|_{p} + \beta g_n \| \theta \| \| u \|, \]
\[ \kappa \| \nabla \theta \|^{2} \leq \| u \|_{p}, \| \nabla \theta \| \| \theta \|_{p}. \]
where \( 1/p + 1/p' = 1/2. \) We discuss in a similar way to the case \( n \geq 3, \) and we have \( u = 0, \theta = 0. \) Theorem is proved.

References.


