

Fano Polytopes and Gorenstein Polytopes

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Introduction

Given a normal projective variety X over a field k with $H^0(X, \mathcal{O}_X) = k$ and an ample \mathbb{Q} -divisor D , namely rational coefficient Weil divisor, we consider a normal graded ring $R(X, D)$ defined by

$R(X, D) = \bigoplus_{n \geq 0} H^0(X, \mathcal{O}_X(nD)) T^n$ (T an indeterminate), where by $\mathcal{O}_X(nD)$ we denote the sheaves $\Gamma(U, \mathcal{O}_X(nD)) = \{f \in K(X); \text{div}_U(f) + nD|_U \geq 0\}$ for each open set U of X . These rings have been introduced by Demazure [De], to show that a normal graded domain over a field k is obtained in this way. For a Cohen-Macaulay graded ring $R(X, D)$, Watanabe [Wa] has given a necessary and sufficient condition for the Gorenstein property, in terms of an ample \mathbb{Q} -divisor D and a canonical divisor K_X on X . Therefore, it is natural to ask what kind of normal projective variety X has an ample \mathbb{Q} -divisor D such that $R(X, D)$ is Gorenstein.

This problem for ample Cartier divisors D has been treated by Goto and Watanabe [GW] and it has been shown that $R(X, D)$ is Gorenstein if and only if there exists an ample Cartier divisor D such that $H^i(X, \mathcal{O}_X(nD)) = 0$ for $0 < i < \dim X$ and for all $n \in \mathbb{Z}$ and $\mathcal{O}_X(aD)$ is isomorphic to the canonical sheaf ω_X on X for some integer a . However, as far as I know, there is not much known as yet about the answer to the problem for ample \mathbb{Q} -divisors, beyond the criterion of Watanabe [Wa]. Our purpose here is to give an answer to this problem in the case that X are normal projective torus embeddings, by

constructing $R(X,D)$ explicitly for T -stable ample \mathbb{Q} -divisors D on X , namely ample \mathbb{Q} -divisors which are stable under the torus action.

The first main result is a technical one, whose precise statement is given in (1.5). Roughly speaking, for a projective torus embedding X and a T -stable ample \mathbb{Q} -divisor D , we construct $R(X,D)$ as a numerical semigroup ring from the data of the fan and the support function associated to X and D . To this end, we firstly relate the pairs of r -dimensional projective torus embeddings and T -stable ample \mathbb{Q} -divisors on them with the r -dimensional rational convex polytopes P in \mathbb{R}^r , according to Oda [Od2, chapter2]. Consequently, when we define a graded ring $R(P)$ over a field k for a rational convex polytope P of dimension r in \mathbb{R}^r by $R(P) = \bigoplus_{n \geq 0} (\sum_{m \in P \cap \mathbb{Z}^r} k e(m)) T^n$ (T an indeterminate), it turns out that there is a natural isomorphism from $R(P)$ to a graded ring $R(X(P), D(P))$ for the T -stable ample \mathbb{Q} -divisor $D(P)$ on the normal projective torus embedding $X(P)$ associated to P .

This result provides us with some consequences, as well as the second main result, namely a vanishing theorem for T -stable ample \mathbb{Q} -divisors on normal projective torus embeddings (1.6), and an enumeration problem of integral points in rational convex polytopes (1.7).

The second main result is theorem (2.2), which is a criterion for a normal graded numerical semigroup ring $R(P)$ to be Gorenstein, in terms of a rational polytope P or the projective torus embedding $X(P)$ with the T -stable ample \mathbb{Q} -divisor $D(P)$. As immediate consequences, we have two results which provide us with an answer to our problems:

Corollary 2.5. *Let X be a normal projective torus embedding. Then*

there exists an ample Cartier divisor D such that $R(X, D)$ is Gorenstein if and only if the canonical sheaf ω_X on X is isomorphic to an invertible sheaf $\mathcal{O}_X(-aD)$ for some $a \in \mathbb{N}$.

Theorem 2.6. *Every normal projective torus embedding X over a field k has a T -stable ample \mathbb{Q} -divisor D such that $R(X, D)$ is Gorenstein.*

From two results above, for example, it turns out that minimal rational surfaces whose anticanonical divisors are not ample have ample \mathbb{Q} -divisors D such that $R(X, D)$ are Gorenstein but do not have such ample (integral) divisors, because every minimal rational surface is a normal projective torus embedding (c.f. [Od1, Theorem 8.2]). But, in general, the situation for our problem would be still obscure.

In another direction, by theorem (2.2) together with a theorem of Stanley [St1, (4.4)], we recover results of Hibi [Hi1, 2]. Our proof here makes clear why the condition (c2) in (2.2) is needed, in terms of Demazure's construction.

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§ 0. Preliminaries.

(0.1). $[a]$ denotes the greatest integer not greater than $a \in \mathbb{R}$. $\lceil a \rceil$ denotes $-[-a]$ for $a \in \mathbb{R}$.

(0.2). For notion of torus embeddings, we refer the reader to [Od2]. Let T be an r -dimensional algebraic torus over a field k . Let M, N be the group of characters and one-parameter subgroups, respectively. Set $M_{\mathbb{R}} = M \otimes_{\mathbb{Z}} \mathbb{R}$ and $N_{\mathbb{R}} = N \otimes_{\mathbb{Z}} \mathbb{R}$. Let $\langle \cdot, \cdot \rangle : M_{\mathbb{R}} \times N_{\mathbb{R}} \rightarrow \mathbb{R}$ represent the natural non-degenerate pairing. For a complete fan Δ of N , $\Delta(i)$ denotes the i -dimensional cones in Δ . A one-dimensional cone $\rho \in \Delta(1)$ is generated by a unique integral primitive vector $n(\rho)$. We denote by $SF(N, \Delta)$ the additive group consisting of Δ -linear support functions (see [Od2, p66] for the definition). Set $SF(N, \Delta, \mathbb{Q}) = SF(N, \Delta) \otimes_{\mathbb{Z}} \mathbb{Q}$. Its elements are also called Δ -linear support functions. Then we have two injections $M \rightarrow SF(N, \Delta)$ sending m to $\langle m, \cdot \rangle$, and $SF(N, \Delta) \rightarrow \mathbb{Z}^{\Delta(1)}$ sending h to $(h(n(\rho)))$. Let X be a normal complete torus embedding $T_N \text{emb}(\Delta)$. By $T\text{Div}(X)$, $TC\text{Div}(X)$ and $P\text{Div}(X)$ we denote the groups of T -stable Weil divisors, T -stable Cartier divisors and principal divisors on X . The one-dimensional cones ρ in $\Delta(1)$ are in a one-to-one correspondence with the irreducible T -stable closed subvarieties $V(\rho)$ of codimension one in X . Therefore the map $\mathbb{Z}^{\Delta(1)} \rightarrow T\text{Div}(X)$ sending g to $D_g = -\sum_{\rho} g_{\rho} \cdot V(\rho)$ ($\rho \in \Delta(1)$) is a bijection, and induces two isomorphisms of groups, $SF(N, \Delta) \rightarrow T\text{Div}(X)$ and $M \rightarrow P\text{Div}(X) \cap TC\text{Div}(X)$. As a result, we have two commutative diagrams:

$$\begin{array}{ccccccc}
 M & & \rightarrow & SF(N, \Delta) & \rightarrow & \mathbb{Z}^{\Delta(1)} & & SF(N, \Delta, \mathbb{Q}) & \rightarrow & \mathbb{Q}^{\Delta(1)} \\
 \downarrow & & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
 P\text{Div}(X) \cap TC\text{Div}(X) & \rightarrow & TC\text{Div}(X) & \rightarrow & T\text{Div}(X) & & TC\text{Div}(X, \mathbb{Q}) & \rightarrow & T\text{Div}(X, \mathbb{Q}).
 \end{array}$$

§ 1. Rational Polytopes and Projective Torus Embeddings.

In the present section, we shall describe the relation between

\mathbb{Q} -divisors on a normal complete torus embedding and support functions. And we shall give a relationship between rational polytopes and normal projective torus embeddings with T -stable ample \mathbb{Q} -divisors.

Lemma 1.1. *Let $X = T_N \text{emb}(\Delta)$ be an r -dimensional complete torus embedding over a field k . For $g \in \mathbb{Q}^{\Delta(1)}$, the set $\square_g = \{m \in M_{\mathbb{R}}; \langle m, n(\rho) \rangle \geq g_{\rho} \text{ for all } \rho \in \Delta(1)\}$ is a (possibly empty) convex polytope in $M_{\mathbb{R}}$. The set $H^0(X, \mathcal{O}_X(D_g))$ of global sections of the divisorial \mathcal{O}_X -module $\mathcal{O}_X(D_g)$ is a finite dimensional k -vector space with $\{e(m); m \in M \cap \square_g\}$ as a basis. Moreover $m \in \text{int}(\square_g)$ if and only if each coefficient a_{ρ} of a T -stable Weil divisor $V(\rho)$ in the \mathbb{Q} -divisor $\text{div}(e(m)) + D_g = \sum_{\rho} a_{\rho} V(\rho)$ ($\rho \in \Delta(1)$) is a positive rational number.*

Proof. The first part is the same as in the case of $g \in \mathbb{Z}^{\Delta(1)}$. Since $n(\rho)$ is a primitive vector and the pairing $\langle \cdot, \cdot \rangle$ is non-degenerate, we have $\square_g \cap M = \square_{[g]} \cap M$, where $[g]$ denotes the integral vector $([g_1], \dots, [g_{\#\Delta(1)}])$. On the other hand, we have $\mathcal{O}_X(D_g) = \mathcal{O}_X(D_{[g]})$ by definition. Hence we may assume that $g \in \mathbb{Z}^{\Delta(1)}$. In this case, the assertion follows from [TE, p41, theorem] (c.f. [Od2, lemma 2.3]). The rest is obvious. Q.E.D.

Recall that a Δ -linear support function $h \in \text{SF}(N, \Delta, \mathbb{Q})$ is said to be *strictly upper convex* with respect to Δ if h is upper convex, namely $h(n) + h(n') \leq h(n+n')$ for all $n, n' \in N_{\mathbb{R}}$, and Δ is the coarsest among the fans Δ' in N for which h is Δ' -linear.

Lemma 1.2. *Let $X = T_N \text{emb}(\Delta)$ be an r -dimensional complete torus embedding over a field k and $h \in \text{SF}(N, \Delta, \mathbb{Q})$. Then D_h is ample, that is,*

bD_h is an ample Cartier divisor for some positive integer b , if and only if h is strictly upper convex with respect to Δ .

Proof. See [Od2, corollary 2.14]. Q.E.D.

Proposition 1.3. *Let P be a rational r -polytope in $M_{\mathbb{R}} = \mathbb{R}^r$, namely, r -dimensional convex polytope in $M_{\mathbb{R}}$ whose vertices have rational coordinates. Then there exists a unique finite complete fan Δ_P in N such that support function $h_P: N_{\mathbb{R}} \rightarrow \mathbb{R}$ for P defined by $h_P(n) = \inf\{\langle m, n \rangle; m \in P\}$ ($n \in N_{\mathbb{R}}$) is a strictly upper convex Δ_P -linear support function with respect to Δ_P . We denote the corresponding r -dimensional projective torus embedding $T_N \text{emb}(\Delta_P)$ and the ample T -stable \mathbb{Q} -divisor D_{h_P} by $X(P)$ and D_{h_P} . Conversely, every pair of a normal projective torus embedding and a T -stable ample \mathbb{Q} -divisor on it is obtained from a rational r -polytope in $M_{\mathbb{R}}$ in this way.*

Proof. The first part follows from [Od2, A.18 & A.19]. Then, by (1.2), $D(P)$ is a T -stable ample \mathbb{Q} -divisor on $X(P)$. Conversely, given a normal projective torus embedding X with a T -stable ample \mathbb{Q} -divisor D , there exist a complete fan Δ and a Δ -linear support function h which is strictly upper convex with respect to Δ . Set $\square_h = \{u \in M_{\mathbb{R}}; \langle u, n(\rho) \rangle \geq h(n(\rho)) \text{ for all } \rho \in \Delta(1)\}$. By the construction and [Od2, A.18 & A.19], we have $X = X(\square_h)$ and $D = D(\square_h)$. Q.E.D.

Remark 1.4. In (1.3), $D(P)$ is a Cartier divisor if and only if P is integral. $D(P)$ is a Weil divisor if and only if P is *facet-reticular*, that is, each supporting hyperplane carried by a facet (face of the

maximal dimension) of P contains an element of M .

Proposition 1.5. *Let P be a rational r -polytope in $M_{\mathbb{R}}$. Then the graded semigroup ring $R(P) := \bigoplus_{n \geq 0} \{ \sum_m k e(m) \} T^n$ ($m \in P \cap M$) over a field k is isomorphic to the graded ring $R(X(P), D(P))$ associated to the projective torus embedding $X(P)$ over k and the ample \mathbb{Q} -divisor $D(P)$, as graded k -algebras. Consequently, $\text{Proj}(R(P))$ is isomorphic to $X(P)$ and the sheaf $\mathcal{O}(n) = R(P)(n)^\sim$ on $\text{Proj}(R(P))$ corresponds via this isomorphism to $\mathcal{O}_{X(P)}(nD(P))$ for all $n \in \mathbb{Z}$.*

Proof. Since $n \square_{nh_P} = nP$ and $D(nP) = D_{nh_P}$ for all $n \in \mathbb{N}$, we have

$$H^0(X(P), \mathcal{O}_{X(P)}(nD(P))) = \sum_m k e(m) \quad (m \in nP \cap M) \text{ by (1.1). This implies}$$

$R(P) \simeq R(X(P), D(P))$. The rest follows from a standard argument in the theory of Demazure's construction (c.f. [Wa, (2.1)]). Q.E.D.

Corollary 1.6. *For an r -dimensional normal projective torus embedding $X = T_N \text{emb}(\Delta)$ over a field k and a strict upper convex Δ -linear support function $h \in \text{SF}(N, \Delta, \mathbb{Q})$ with respect to Δ , we have:*

$$\begin{aligned} \text{(a) } \dim_k H^0(X, \mathcal{O}_X(nD_h)) &= \begin{cases} \#(n \square_h \cap M) & \text{if } n \geq 0 \\ 0 & \text{if } n < 0; \end{cases} \\ \text{(b) } \dim_k H^i(X, \mathcal{O}_X(nD_h)) &= 0 \quad \text{for } 0 < i < r \text{ and all } n \in \mathbb{Z}; \\ \text{(c) } \dim_k H^r(X, \mathcal{O}_X(nD_h)) &= \begin{cases} 0 & \text{if } n \geq 0 \\ \#(\text{int}(n \square_h) \cap M) & \text{if } n < 0, \end{cases} \end{aligned}$$

where $\#(n \square_h \cap M)$ is the number of the set $n \square_h \cap M$ of lattice points in the rational convex polytope $n \square_h$, and $\text{int}(n \square_h)$ denotes the interior of the convex polytope $n \square_h$.

Proof. (a): This follows from (1.1). (b): Since $R(X, D_h)$ is a normal

numerical semigroup ring by (1.3) and (1.5), $R(X, D_h)$ is normal and Cohen-Macaulay by a theorem of Hochster [Ho]. Therefore, this follows from corollary (2.2) in [Wa]. (c): By Serre duality (c.f. [Wa, (2.7)]), we have $\text{Hom}_k(H^r(X, \mathcal{O}_X(nD_h)), k) \simeq H^0(X, \mathcal{O}_X(-[nD_h] + K_X))$, where K_X denotes a canonical divisor on X . Since $K_X = -\sum_{\rho \in \Delta(1)} V(\rho)$ and n_ρ is a primitive vector for each $\rho \in \Delta(1)$, this follows from (1.1). Q.E.D.

Remark 1.7. Let P be a rational r -polytope in \mathbb{R}^r and $m = \min\{i \in \mathbb{N}; i > 0 \text{ and } iP \text{ is integral}\}$. By (1.3), (1.5) and (1.6), we have

$\#(nP \cap \mathbb{Z}^r) = \chi(X(P), \mathcal{O}_{X(P)}(nD(P)))$ for $n \geq 0$ and

$\#(\text{int}((-n)P) \cap \mathbb{Z}^r) = (-1)^r \chi(X(P), \mathcal{O}_{X(P)}(nD(P)))$ for $n < 0$, where

$\chi(X(P), \mathcal{O}_{X(P)}(nD(P)))$ denotes $\sum_{j=0}^r (-1)^j \dim_k H^j(X(P), \mathcal{O}(nD(P)))$. By a

result due to Snapper and Kleiman, for every $n \in \mathbb{Z}$, there exists a

polynomial $P_n(\lambda)$ with coefficients in \mathbb{Q} such that

$\chi(X(P), \mathcal{O}_{X(P)}((n+m\lambda)D(P))) = P_n(\lambda)$. Thus we recover the reciprocity

theorem and see that Ehrhart quasi-polynomial is indeed a

quasi-polynomial.

§ 2. Criteria for Gorenstein Property.

First, we prove the following lemma:

Lemma 2.1. *Let Δ be a complete fan in N and h be a strictly upper convex Δ -linear support function in $\text{SF}(N, \Delta, \mathbb{Q})$ with respect to Δ . Set*

$\square_h = \{u \in M_{\mathbb{R}}; \langle u, n(\rho) \rangle \geq h(n(\rho)) \text{ for each } \rho \in \Delta(1)\}$. *Suppose that h has*

negative values except at the origin or, equivalently, \square_h contains

the origin in its interior. Then the set of vertices of the polar

convex polyhedral set $(\square_h)^{\circ} := \{v \in N_{\mathbb{R}}; \langle u, v \rangle \geq -1 \text{ for all } u \in \square_h\}$ for \square_h is $\{- (1/h(n(\rho)))n(\rho); \rho \in \Delta(1)\}$.

Proof. By [Od2,A.19], there exists a bijection from $\Delta(1)$ to the set $\mathcal{F}^{r-1}(\square_h)$ of $(r-1)$ -dimensional faces of \square_h sending $\rho \in \Delta(1)$ to $Q_{\rho} = \{u \in \square_h; \langle u, n(\rho) \rangle = h(n(\rho))\}$. Also, by [Od2,A.17], there exists a bijection from $\mathcal{F}^{r-1}(\square_h)$ to the set of vertices of $(\square_h)^{\circ}$ sending an $(r-1)$ -dimensional face Q to $Q^* = \{v \in (\square_h)^{\circ}; \langle u, v \rangle = -1 \text{ for all } u \in Q\}$. Then we observe that $(Q_{\rho})^*$ is $- (1/h(n(\rho)))n(\rho)$. Q.E.D.

Theorem 2.2. For a rational r -polytope P in $M_{\mathbb{R}} = \mathbb{R}^r$ with $M = \mathbb{Z}^r$ and a positive integer δ , the following are equivalent:

(a) The semigroup ring $R(P) = \bigoplus_{n \geq 0} \{\sum_m k e(m)\} T^n$ ($m \in P \cap M$) over a field k is a Gorenstein ring with $a(R(P)) = -\delta$, where $a(R(P))$ is defined by $-\min\{m \in \mathbb{Z}; (K_{R(P)})_m \neq 0 \text{ for the canonical module } K_{R(P)} \text{ of } R(P)\}$. (For details concerning $a(\cdot)$, see [GW]).

(b) The normal projective torus embedding $X(P) = T_N \text{emb}(\Delta_P)$ over a field k , and the ample \mathbb{Q} -divisor $D(P) = \sum_{\rho} (p_{\rho}/q_{\rho})V(\rho)$ ($\rho \in \Delta_P(1)$, $q_{\rho} > 0$, p_{ρ} and q_{ρ} are coprime) satisfy the following:

(b1) There exist a positive integer r_{ρ} for each $\rho \in \Delta_P(1)$ and a character $m \in M$ such that $\delta D(P) + \text{div}(e(m)) = \sum_{\rho} (1/r_{\rho})V(\rho)$ ($\rho \in \Delta_P(1)$);

(b2) δ and q_{ρ} are coprime for each $\rho \in \Delta_P(1)$.

(c) P satisfies the following:

(c1) There exists a character $m \in M$ such that the polar polyhedral set $(\delta P - m)^{\circ} := \{v \in N_{\mathbb{R}}; \langle u, v \rangle \geq -1 \text{ for all } u \in \delta P - m\}$ for $\delta P - m = \{\delta p - m \in M_{\mathbb{R}}; p \in P\}$ is an integral r -polytope, namely, an r -polytope whose vertices have integral coordinates;

(c2) The convex hull \tilde{P} of the set $P \times \{0\} \cup \{(0, \dots, 0, 1/\delta)\}$ in

$M_{\mathbb{R}} \times \mathbb{R}$ is facet-reticular (c.f. (1.4)).

Proof. (a) \Leftrightarrow (b): By (1.5), $R(P)$ is isomorphic to $R(X(P), D(P))$ and, therefore, $R(X(P), D(P))$ is Cohen-Macaulay by a theorem of Hochster[Ho]. Since a canonical divisor $K_{X(P)}$ on $X(P)$ is $-\sum_{\rho} V(\rho)$ ($\rho \in \Delta_P(1)$) (see for example [TE, theorem 9, III.d]), it follows from a criterion of Watanabe [Wa, (2.9)] that $R(P)$ is a Gorenstein ring with $a(R(P)) = -\delta$ if and only if there exists a character $m \in M$ such that $\delta D(P) + \text{div}(e(m)) = \sum_{\rho} (1/q_{\rho}) \cdot V(\rho)$ ($\rho \in \Delta_P(1)$). Note that a semi-invariant rational function $f \in K(X(P))^*$ corresponds to some character $m \in M$. We assume that (a) holds. By the preceding remark, we have the relation above and, therefore, (b1) holds. Rewriting the relation, we have $\text{div}(e(m)) = \sum_{\rho} \{(1-\delta p_{\rho})/q_{\rho}\} V(\rho)$ ($\rho \in \Delta_P(1)$). Hence $(1-\delta p_{\rho})/q_{\rho}$ is an integer and, therefore, δ and q_{ρ} are coprime for each $\rho \in \Delta_P(1)$. Conversely, we assume that (b) holds. By the preceding remark, we claim that $r_{\rho} = q_{\rho}$ for each $\rho \in \Delta_P(1)$. Since r_{ρ} is a factor of q_{ρ} , $b_{\rho} := (q_{\rho}/r_{\rho})$ is a positive integer. Then, by (b1), $(b_{\rho} - \delta p_{\rho})/(r_{\rho} b_{\rho})$ is an integer and, therefore, b_{ρ} is a factor of δ or p_{ρ} . Hence we have $b_{\rho} = 1$ for each $\rho \in \Delta_P(1)$, as required, because neither δ nor p_{ρ} has any common factor with q_{ρ} .

(b1) \Rightarrow (c1): Set $g = \delta h_P - m \in \text{SF}(N, \Delta_P, \mathbb{Q})$. Since $D_g = \delta D(P) + \text{div}(e(m))$ and D_g is ample, g is strictly upper convex and $g(n(\rho)) = -(1/r_{\rho})$ for each $\rho \in \Delta_P(1)$. Therefore, by (2.1), the set of vertices of the polar convex polyhedral set $(\square_g)^{\circ}$ is $\{-(1/g(n(\rho)))n(\rho); \rho \in \Delta_P(1)\} = \{r_{\rho}n(\rho); \rho \in \Delta_P(1)\}$. On the other hand, $\square_g = \delta P - m$ by definition. Therefore $(\delta P - m)^{\circ}$ is an integral convex polytope.

(c1) \Rightarrow (b1): Set $g = \delta h_P - m \in \text{SF}(N, \Delta_P, \mathbb{Q})$. Since g is strictly upper convex with respect to Δ_P and $0 \in \text{int}(\delta P - m)$, it follows from (2.1) that the

vertices set of $(\delta P - m)^0$ is $\{-(1/g(n(\rho)))n(\rho); \rho \in \Delta_P(1)\}$. Hence, by assumption, $-(1/g(n(\rho)))n(\rho)$ is an integral vector. Since $n(\rho)$ is a primitive vector and $g \in SF(N, \Delta_P, \mathbb{Q})$ is negative-valued,

$r_\rho := -1/(g(n(\rho)))$ is a positive integer for each $\rho \in \Delta_P(1)$ and

$$\delta D(P) + \text{div}(e(m)) = D_g = \sum_{\rho} (1/r_\rho) V(\rho).$$

(b2) \Leftrightarrow (c2): Since a supporting hyperplane carried by a facet of P corresponding to $\rho \in \Delta_P(1)$ is $H_\rho = \{u \in M_{\mathbb{R}}; \langle u, n(\rho) \rangle = h_P(n(\rho))\}$, a supporting hyperplane carried by a facet of \tilde{P} is of the form

$$\tilde{H}_\rho := \{(u, x) \in M_{\mathbb{R}} \times \mathbb{R}; \delta x + (1/h_P(n(\rho))) \langle u, n(\rho) \rangle = 1\} \text{ or } \{(u, 0) \in M_{\mathbb{R}} \times \mathbb{R}\}.$$

Since $h_P(n(\rho)) = -(p_\rho/q_\rho)$ and $n(\rho)$ is a primitive vector, δ and q_ρ are coprime if and only if $\tilde{H}_\rho \cap M \times \mathbb{Z}$ is non-empty. Q.E.D.

Remark 2.3.1. Under the condition (b) in the theorem, suppose that there exist an integer $\delta' \leq \delta$, a character $m' \in M$ and a positive integer a_ρ for each $\rho \in \Delta_P(1)$ such that $\delta' D + \text{div}(e(m')) = \sum_{\rho} (a_\rho/q_\rho) \cdot V(\rho)$ ($\rho \in \Delta_P(1)$). Then we have $\delta' = \delta$, $m' = m$ and $a_\rho = 1$ for each $\rho \in \Delta_P(1)$. In other words, we have $\#(\mathbb{Z}^r \cap \text{int}(nP)) = 0$ for each $0 \leq n < \delta$ and $\#(\mathbb{Z}^r \cap \text{int}(\delta P)) = 1$. In fact, we observe that $(\delta' - \delta)D + \text{div}(e(m' - m)) = \sum_{\rho} (a_\rho - 1)/q_\rho \cdot V(\rho)$ ($\rho \in \Delta_P(1)$). But D is an ample \mathbb{Q} -Cartier divisor. So we have $\delta' = \delta$, $a_\rho = 1$ for each $\rho \in \Delta_P(1)$, and $m' = m$.

Remark 2.3.2. Combining the equivalence between (a) and (c) in (2.2) and a theorem of Stanley [St1, theorem 4.4], we recover theorem of Hibi [Hi1,2]. Our proof makes clear why the condition (c2) in (2.2) is needed, in terms of Demazure's construction. Indeed, let $R(X, D)$ be a Cohen-Macaulay graded ring obtained from a normal projective variety X and an ample \mathbb{Q} -divisor $D = \sum_V (p_V/q_V) V$ (V runs through irreducible subvarieties of codimension 1, $q_V > 0$ and p_V, q_V are coprime

for each V). Then it follows from [Wa, (2.9)] that $R(X, D)$ is Gorenstein if the Veronese subring $R(X, D)^{(d)}$ of order d is Gorenstein for an integer d such that $a \equiv 0 \pmod{d}$ and that d and q_V are coprime for each V .

Corollary 2.4. *For a rational r -polytope P in $M_{\mathbb{R}} = \mathbb{R}^r$ with $M = \mathbb{Z}^r$ and an integer δ , the following are equivalent:*

(a) P is integral and there exists a character $m \in M$ such that the polar polyhedral set $(\delta P - m)^{\circ}$ for $\delta P - m$ is an integral r -polytope;

(b) The \mathbb{Q} -divisor $D(P)$ on the normal projective torus embedding $X(P)$ over a field k is an ample Cartier divisor. And the invertible sheaf $\mathcal{O}_X(-\delta D(P))$ is isomorphic to the canonical sheaf $\omega_{X(P)}$.

Proof. It follows from (1.4) and (2.2) that (a) holds if and only if $D(P)$ is a Cartier divisor and there exists a character $m \in M$ such that $\delta D(P) + \text{div}(e(m)) = \sum_{\rho} V(\rho)$ ($\rho \in \Delta_P(1)$). Since a canonical divisor $K_{X(P)}$ on $X(P)$ is $-\sum_{\rho} V(\rho)$ ($\rho \in \Delta_P(1)$), (a) is equivalent to (b). Q.E.D.

Since every Cartier divisor on a normal complete torus embedding is linearly equivalent to a T -stable Cartier divisor (c.f. [Od1, (6.1)]), we have:

Corollary 2.5. *Let X be a normal projective torus embedding. Then there exists an ample Cartier divisor D such that $R(X, D)$ is Gorenstein if and only if the canonical sheaf ω_X on X is isomorphic to an invertible sheaf $\mathcal{O}_X(-aD)$ for some $a \in \mathbb{N}$.*

Theorem 2.6. *Every normal projective torus embedding X over a field k*

has a T -stable ample \mathbb{Q} -divisor D such that $R(X, D)$ is Gorenstein.

Proof. By assumption, we may assume that $X = T_N \text{emb}(\Delta)$ has a T -stable ample Cartier divisor D of the form $D = \sum_{\rho} a_{\rho} V(\rho)$, $a_{\rho} > 0$ ($\rho \in \Delta(1)$). Set $c = \text{L.C.M.}\{a_{\rho}; \rho \in \Delta(1)\}$. By (1.3), we may assume that $(X, (1/c)D)$ corresponds to a rational polytope P in $M_{\mathbb{R}}$. Then, by (1.5) and (2.2), $R(X, (1/c)D)$ is a Gorenstein ring with $a(R(X, (1/c)D)) = -1$, as required.

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