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Factorizations of the Orlik-Solomon Algebras

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1 Introduction.

Let $L$ be a finite geometric lattice with the top element $\hat{1}$ and the bottom element $\hat{0}$, and the rank function $r$. Let $r = r(\hat{1})$. The characteristic polynomial of $L$ is defined by
\[
\chi(L; t) = \sum_{X \in L} \mu(\hat{0}, X) t^{r - r(X)}.
\]
In the right handside $\mu$ is the M"obius function [6]. For certain geometric lattices including the supersolvable lattices [7], it is known that the characteristic polynomial $\chi(L; t)$ factors as
\[
\chi(L; t) = \prod_{i=1}^{r} (t - d_i) \quad \text{(each $d_i$ is a nonnegative integer)}.
\]
In this paper we prove a sufficient condition (2.9) of the factorization of this type. The condition is stated as the existence of a "nice" partition of the set $A = A(L)$ of atoms of $L$. It is not difficult to check that a supersolvable geometric lattice admits a "nice" partition (2.4).

In fact we will actually show a stronger result. Let us briefly explain about it. Let $K$ be an arbitrary field. In [4, p.171] the Orlik-Solomon algebra $OS(L)$ of $L$ over $K$ was introduced. It is a graded anticommutative $K$-algebra. One of the most important results concerning $OS(L)$ is [4]:
\[
Poin(OS(L); t) = \sum_{X \in L} \mu(\hat{0}, X)(-t)^{r(X)}.
\]
Here the left handside stands for the Poincaré series of the graded algebra $OS(L)$. Suppose that we have a partition $(\pi_1, \ldots, \pi_s)$ of the set $\mathcal{A}$ of atoms of $L$. Define

$$(\pi_i) := \text{the vector space over } K \text{ spanned by } 1 \text{ and the elements of } \pi_i$$

for $i = 1, 2, \ldots, s$.

Then the main theorem (2.8) in this paper is that there exists a natural graded vector space isomorphism

$$\kappa : (\pi_1) \otimes (\pi_2) \otimes \cdots \otimes (\pi_s) \rightarrow OS(L)$$

if and only if the partition $(\pi_1, \ldots, \pi_s)$ is "nice".

The above-mentioned sufficient condition easily follows from the main theorem.

2 Main Theorem and Its Corollaries.

Let $L, K, \mathcal{A} = \mathcal{A}(L), OS(L)$ be as in the previous section.

Definition 2.1 A partition $\pi = (\pi_1, \ldots, \pi_s)$ of $\mathcal{A}$ is called independent if atoms $H_1, \ldots, H_s$ are independent (i.e., $r(H_1 \lor \cdots \lor H_s) = s$) whenever $H_i \in \pi_i$ for $i = 1, \ldots, s$.

For $X \in L$, define

$$L_X := \{Y \in L \mid Y \leq X\}, \quad \mathcal{A}_X := \mathcal{A}(L_X) = \{H \in \mathcal{A} \mid H \leq X\}.$$  

Definition 2.2 Let $X \in L$. Let $\pi = (\pi_1, \ldots, \pi_s)$ be a partition of $\mathcal{A}$. Then the induced partition $\pi_X$ is a partition of $\mathcal{A}_X$ whose blocks are the subsets $\pi_i \cap \mathcal{A}_X$ for $i = 1, \ldots, s$ which are not empty.

Definition 2.3 A partition $\pi = (\pi_1, \ldots, \pi_s)$ of $\mathcal{A}$ is called nice if:

1) it is independent, and

2) the induced partition $\pi_X$ contains a block which is a singleton unless $\mathcal{A}_X \neq \emptyset$. 

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Remark. In [2], M. Falk and M. Jambu studied a similar partition. A major difference from ours lies in their assumption that the characteristic polynomial of $L$ factors completely in $\mathbb{Z}[t]$.

Example 2.4 Let $L$ be a supersolvable lattice. Then the set $\mathcal{A} = \mathcal{A}(L)$ admits a nice partition. In fact, define

$$\pi_i = \{H \in \mathcal{A} \mid a \leq X_i; H \not\leq X_{i-1}\}$$

for a chain of modular elements

$$\hat{0} = X_0 < X_1 < \cdots < X_r = \hat{1} \quad (r(X_i) = i).$$

Then it is not difficult to show that a partition $\pi = (\pi_1, \ldots, \pi_r)$ is a nice partition.

Example 2.5 Consider the lattice arising from the following matroid (the non-Fano matroid)

For this, $\{\{1\}, \{2,3,4\}, \{5,6,7\}\}$ is a nice partition.
For a partition $\pi = (\pi_1, \ldots, \pi_s)$ of $\mathcal{A}$, define a graded vector space

$$(\pi) := (\pi_1) \otimes (\pi_2) \otimes \cdots \otimes (\pi_s),$$

where each graded vector space $(\pi_i)$ is as in the Introduction. Agree that $(\pi) = K$ when $\mathcal{A} = \emptyset$. Since the Poincaré series $\text{Poin}((\pi_i); t)$ of each $(\pi_i)$ is equal to $(1 + |\pi_i| t)$, we obtain

$$\text{Poin}((\pi); t) = \prod_{i=1}^s (1 + |\pi_i| t).$$

**Definition 2.6** A $k$-tuple $I = (H_1, \ldots, H_k)$ ($k \geq 0$) of elements of $\mathcal{A}$ is called a $k$-section of $\pi$ if

$$H_i \in \pi_{n(i)} \ (i = 1, \ldots, k), \quad 1 \leq n(1) < n(2) < \cdots < n(k) \leq s.$$ 

For a $k$-section $I = (H_1, \ldots, H_k)$, define $p_I$ by

$$p_I := x_1 \otimes \cdots \otimes x_s \in (\pi).$$

Here

$$x_j = \begin{cases} H_i & \text{if } j = n(i) \\ 1 & \text{if } j \not\in \{n(1), \ldots, n(k)\} \end{cases}.$$ 

Then $p_I$ is homogeneous of degree $k$. The graded $K$-vector space $(\pi)$ has a basis $\{p_I \mid I \text{ is a section of } \pi\}$.

For the Orlik-Solomon algebra we keep the notation in [5]: For a $k$-tuple $I = (H_1, \ldots, H_k)$ ($k \geq 0$) of atoms, the notation $a_I \in \text{OS}(L)$ stands for the class of the exterior product $e_{H_1} \wedge \cdots \wedge e_{H_k}$. Recall that each element of the Orlik-Solomon algebra $\text{OS}(L)$ can be (not necessarily uniquely) expressed as a linear combination of $\{a_I \mid I \text{ is a tuple of atoms}\}$.

**Definition 2.7** Define

$$\kappa : (\pi) \longrightarrow \text{OS}(L)$$

as the homogeneous $K$-linear map of degree zero satisfying

$$\kappa(p_I) = a_I$$

for each section $I$ of $\pi$. 
The main theorem is:

**Theorem 2.8** The map $\kappa$ is an isomorphism (as graded vector spaces) if and only if the partition $\pi$ is nice.

We will prove this theorem in the next section.

**Corollary 2.9** If there exists a nice partition $\pi = (\pi_1, \ldots, \pi_s)$, we have $s = r$ and

$$
\chi(L; t) = \sum_{X \in L} \mu(\hat{0}, X)t^{|\pi(X)|} = \prod_{i=1}^{r}(t - |\pi_i|).
$$

**Corollary 2.10** If $\pi$ is a nice partition, then the multiset $\{|\pi_1|, \ldots, |\pi_s|\}$ depends only upon $L$.

**Corollary 2.11** If $\pi$ is a nice partition, then

$$
r(X) = |\{i \mid \pi_i \cap A_X \neq \emptyset\}|
$$

for all $X \in L$.

**Corollary 2.12** Let $A$ be an arrangement of hyperplanes in a vector space. Let $L$ be the intersection lattice of $A$. Suppose that there exists a partition $\pi = (\pi_1, \ldots, \pi_s)$ of $A$ such that

1) $\text{codim}(H_1 \cap \cdots \cap H_s) = s$ whenever $H_i \in \pi_i$ \ ($i = 1, \ldots, s$), and

2) For every $X \in L$, there exists a block $\pi_{iX}$ of $\pi$ such that the set

$\{H \in \pi_{iX} \mid X \subseteq H\}$

is a singleton.

Then $s = r(L)$ and

$$
\chi(L; t) = \prod_{i=1}^{s}(t - |\pi_i|).
$$

These corollaries, except 2.11 which will be proved in the next section, are immediate consequences from the main theorem.
3 Proof of Main Theorem

We keep the notation in the previous section. First we will review three results concerning the Orlik-Solomon algebra. Denote the homogeneous part of degree $d$ of the graded algebra $OS(L)$ by $OS_k(L)$:

$$OS(L) = \bigoplus_{k=0}^{r} OS_k(L).$$

For a tuple $I = (H_1, \ldots, H_k)$ of atoms, let

$$\bigvee I = H_1 \vee \cdots \vee H_k \in L.$$

For each $X \in L$, define a vector subspace $OS_X(L)$ of $OS(L)$ which is generated by $\{a_I | \bigvee I = X\}$. Agree that $OS_0(L) = OS_0(L) = K$.

Lemma 3.1 ([4, 2.11]) For each $k \geq 0$, we have

$$OS_k(L) = \bigoplus_{X \in L} OS_X(L).$$

Lemma 3.2 ([3, 1.7]) For $X, Y \in L$ with $Y \leq X$, there exists a natural isomorphism

$$OS_Y(L_X) \rightarrow OS_Y(L).$$

Define a boundary map

$$\partial : OS_k(L) \rightarrow OS_{k-1}(L) \ (k = 1, \ldots, r)$$

to be the $K$-linear map satisfying

$$\partial(a_I) = \sum_{j=1}^{k} (-1)^{j-1} a_{I_j}$$

for any $k$-tuple $I = (H_1, \ldots, H_k)$ of atoms. Here

$$I_j = (H_1, \ldots, H_{j-1}, H_{j+1}, \ldots, H_k)$$

for $1 \leq j \leq k$. 

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Lemma 3.3 ([4, 2.18]) The complex $(OS_\ast(L), \partial)$ is acyclic.

Next let $\pi = (\pi_1, \ldots, \pi_*)$ be a partition of the set $A = A(L)$. We study the graded vector space $(\pi)$. Denote the homogeneous part of degree $k$ of $(\pi)$ by $(\pi)_k$:

$$(\pi) = \bigoplus_{k=0}^{\delta} (\pi)_k.$$ 

For each $X \in L$, define a vector subspace $(\pi)_X$ of $(\pi)$ which has a basis \{p_I | I is a section with $\bigvee I = X$\}. Agree that $(\pi)_0 = (\pi)_0 = K$.

Lemma 3.4 Suppose that $\pi$ is an independent partition. For each $k \geq 0$, we have

$$(\pi)_k = \bigoplus_{X \in L} (\pi)_X.$$ 

Proof. By definition, the right handside is actually a direct sum. Note that $(\pi)_k$ has a basis

\{p_I | I is a $k$-section of $\pi$\}.

Put $X = \bigvee I$. Then $p_I \in (\pi)_X$. We have $r(X) = k$ because $\pi$ is independent. \hfill \Box

Lemma 3.5 For $X, Y \in L$ with $Y \leq X$, there exists a natural isomorphism

$$(\pi_X)_Y \rightarrow (\pi)_Y.$$ 

Proof. If $I$ is a section of $\pi$ with $\bigvee I = Y$, then $I \subseteq A_Y \subseteq A_X$. Thus $I$ is also a section of $\pi_X$. This shows:

$$\{I | I is a section of $\pi$ with $\bigvee I = Y$\} = \{I | I is a section of $\pi_X$ with $\bigvee I = Y$\}.$$ 

Therefore an isomorphism

$$p_I \in (\pi_X)_Y \mapsto p_I \in (\pi)_Y$$
is obtained by inserting "1⊗" \( r - r(X) \) times.

Define a \( K \)-linear map

\[
\partial : (\pi)_k \longrightarrow (\pi)_{k-1} \ (k = 1, \ldots, s)
\]

satisfying

\[
\partial(p_I) = \sum_{i=1}^{k} (-1)^{i-1} p_{I_i}
\]

for any \( k \)-section \( I \) of \( \pi \). Then it is easy to check \( \partial \circ \partial = 0 \).

Lemma 3.6 Suppose that a partition \( \pi \) of \( A \) contains a block which is a singleton. Then the complex \( ((\pi)_*, \partial) \) is acyclic.

**Proof.** We can assume that \( \pi_1 \) is a singleton: \( \pi_1 = \{a_1\} \). Suppose that \( x \in (\pi)_k \) is a cycle: \( \partial x = 0 \). Write \( x \) as

\[
x = a_1 \otimes x_1 + 1 \otimes x_2,
\]

where \( x_1, x_2 \in (\pi_2) \otimes \cdots \otimes (\pi_s) \). Then

\[
0 = \partial x = 1 \otimes x_1 - a_1 \otimes (\partial x_1) + 1 \otimes (\partial x_2) = 1 \otimes (x_1 + \partial x_2) - a_1 \otimes (\partial x_1).
\]

This implies

\[
x_1 = -\partial x_2.
\]

Define

\[
y = a_1 \otimes x_2 \in (\pi)_{k+1}.
\]

Then

\[
\partial y = 1 \otimes x_2 - a_1 \otimes (\partial x_2) = 1 \otimes x_2 + a_1 \otimes x_1 = x.
\]

Proof of Main Theorem.

**Sufficiency:**

Assume that \( \pi \) is a nice partition. We will prove by induction on \( r(L) = r(\bar{1}) \). When \( r(L) = 0, \ A = \emptyset \). Thus \( (\pi) = K = OS(L) \).
Assume that $r = r(L) > 0$. Note $s \leq r$ because $\pi$ is independent. Consider a diagram

$$
\begin{array}{cccccccc}
0 & \rightarrow & (\pi)_{0} & \varphi & (\pi)_{r-1} & \varphi & \cdots & \varphi & (\pi)_{1} & \varphi & (\pi)_{0} & \rightarrow & 0 \\
\downarrow \kappa_{0} & & \downarrow \kappa_{r-1} & & \downarrow \kappa_{1} & & \downarrow \kappa_{0} & & \downarrow \kappa_{0} & & \downarrow \kappa_{0} & & \downarrow \kappa_{0} & & 0 \\
0 & \rightarrow & OS_{r}(L) & \varphi & OS_{r-1}(L) & \varphi & \cdots & \varphi & OS_{1}(L) & \varphi & OS_{0}(L) & \rightarrow & 0.
\end{array}
$$

Here all of the vertical maps are induced from $\kappa : (\pi) \rightarrow OS(L)$. The top row is exact because of 3.6. The bottom row is exact because of 3.3. Note that

$$
(\pi)_{k} = \bigoplus_{Y \in L} (\pi)_{Y} \simeq \bigoplus_{Y \in L} (\pi_{Y})_{Y}
$$

by 3.4 and 3.5. Also note that

$$
OS_{k}(L) = \bigoplus_{Y \in L} OS_{Y}(L) \simeq \bigoplus_{Y \in L} OS_{Y}(L_{Y})
$$

by 3.1 and 3.2. By applying the induction assumption to $L_{Y}$ for $r(Y) < r$, we know that $\kappa_{i}$ ($i = 1, \ldots, r - 1$) are isomorphisms. Therefore $\kappa_{r}$ is also an isomorphism. Putting these together, we get an isomorphism

$$
\kappa : (\pi) \rightarrow OS(L).
$$

**Necessity:**

Suppose $\kappa$ is an isomorphism. First we will show that $\pi$ is independent. Let $I$ be a section of $\pi$. Then $p_{I} \neq 0$. So

$$
a_{I} = \kappa(p_{I}) \neq 0.
$$

This shows that $I$ is independent.

Next we will show that $\pi_{X}$ contains a block which is a singleton unless $X = \hat{0}$. Since

$$
(\pi) = \bigoplus_{Y \in L} (\pi)_{Y}, \quad OS(L) = \bigoplus_{Y \in L} OS_{Y}(L),
$$

$\kappa$ induces isomorphisms

$$
(\pi)_{Y} \rightarrow OS_{Y}(L).
$$
By 3.5 and 3.2, we obtain

$$(\pi_X) = \bigoplus_{Y \in L_X} (\pi_Y) \simeq \bigoplus_{Y \leq_X} OS_Y(L) \simeq \bigoplus_{Y \in L_X} OS_Y(L)$$

$= OS(L_X)$.

Let $X \neq \emptyset$. Then

$$0 = \sum_{1' \in L} \mu(0, Y) = Poin(OS(L_X); 1) = Poin((\pi_X); 1) = \prod_{i}(1 - |\pi_i \cap A_X|).$$

This implies that $\pi_X$ contains a block which is a singleton. □

**Remark.** In [1] A. Björner and G. Ziegler gave a sufficient condition for the map $\kappa$ to be an isomorphism. The condition is the existence of a rooting map $\rho$ for which the root complex $RC(L, \rho)$ factors completely. We do not know if the existence of a nice partition is enough to construct such a rooting map.

**Proof of Corollary 2.11.** As we saw in the proof of Main Theorem, the isomorphism $\kappa$ induces isomorphisms

$$\kappa_X : (\pi_X) \rightarrow OS(L_X)$$

for all $X \in L$. So $\pi_X$ is a nice partition of $A_X$. By 2.9, we have

$$r(X) = r(L_X) = |\pi_X| = |\{i \mid \pi_i \cap A_X \neq \emptyset\}|.$$ □

Since we have the factorization theorem for free arrangements [8], it is natural to pose

**Problem.** If an arrangement admits a nice partition, then is it free?

The converse is not true in general. (For example, the Coxeter arrangement $D_4$ has no nice partitions.)
References


