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<tr>
<td>Author(s)</td>
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<tr>
<td>Citation</td>
<td>数理解析研究所講究録 (1990), 735: 28-42</td>
</tr>
<tr>
<td>Issue Date</td>
<td>1990-12</td>
</tr>
<tr>
<td>URL</td>
<td><a href="http://hdl.handle.net/2433/102027">http://hdl.handle.net/2433/102027</a></td>
</tr>
<tr>
<td>Type</td>
<td>Departmental Bulletin Paper</td>
</tr>
<tr>
<td>Textversion</td>
<td>publisher</td>
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A Word Problem In Coxeter Semigroups

by

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Abstract.
The braid group, with generators 1, 2, ..., n, has the presentation
\[
\begin{align*}
\ ij & = ji \quad \text{if } |i - j| = 1 \\
\ ij & = ji \quad \text{if } |i - j| > 1.
\end{align*}
\]
In this paper, we prove that the two products \( P = 1 \cdot 3 \cdot 5 \cdot \ldots \) and \( Q = 2 \cdot 4 \cdot 6 \cdot \ldots \) satisfy \( PQPQ \ldots = QPQP \ldots \) with \( n + 1 \) letters on each side of the equals sign. This is an affirmative answer to a problem of Coxeter.

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1. Introduction

In [2], Coxeter posed the following problem:

Problem. The braid group, with generators 1, 2, ..., \( n \), has the presentation

\[
\begin{align*}
iji &= ji \quad \text{if } |i-j| = 1 \\
ij &= ji \quad \text{if } |i-j| > 1.
\end{align*}
\]

Prove or disprove that the two products \( P = 1 \cdot 3 \cdot 5 \cdot \ldots \) and \( Q = 2 \cdot 4 \cdot 6 \cdot \ldots \) satisfy \( PQQPQ \ldots = QQPQP \ldots \) with \( n+1 \) letters on each side of the equals sign. \( \blacksquare \)

In this paper, we give an affirmative answer to the problem.

Here, we consider a topological feature of this problem. A generator \( i \) of the braid group corresponds to the following state of braids:

\[
\begin{array}{cccc}
1&2&i&n \\
\end{array}
\]

For example, the problem for \( n = 4 \) corresponds to the following state of braids:

\[
\begin{array}{c|c|c}
P & & Q \\
\hline
Q & & P \\
\hline
P & & Q \\
\hline
Q & & P \\
\hline
P & & Q \\
\end{array}
\]

All proofs in this paper are combinatorial, but these topological images will be helpful to see some of the proofs.

2. Definitions and notation

Let \( \Gamma = (V,E) \) be a multigraph. We define the semigroup \( S \) corresponding to \( \Gamma \) as the following. \( S \) is generated by the elements of \( V \) subject to the relations...
\[i j i \ldots = j i j \ldots \text{ for } i, j \in V\]

with \(e(i, j) + 2\) letters on each side of the equals sign, where \(e(i, j)\)
denotes the number of edges which join \(i\) and \(j\). Now we define Coxeter
semigroups by the following graphs:

\[
\begin{align*}
A_n & \quad \begin{array}{cccc}
1 & 2 & 3 & \cdots & n \\
\end{array} \\
B_n & \quad \begin{array}{cccc}
1 & 2 & 3 & \cdots & n \\
\end{array} \\
D_n & \quad \begin{array}{cccc}
1 & \\
3 & 4 & \cdots & n \\
2 & \\
\end{array} \\
E_n & \quad \begin{array}{cccc}
5 & 3 & 2 & 4 & 6 & n \\
\end{array} \quad (n = 6, 7, 8) \\
F_4 & \quad \begin{array}{cccc}
1 & 2 & 3 & 4 \\
\end{array} \\
H_n & \quad \begin{array}{cccc}
1 & 2 & \cdots & n \\
\end{array} \quad (n = 3, 4)
\end{align*}
\]

Let \(X_n\) be a Coxeter semigroup with generators \(1, 2, \ldots, n\). We use
the following notation for elements of \(X_n\):

- For \(X = x_1x_2 \cdots x_k \in X_n\), \(tX = x_kx_{k-1} \cdots x_1\).
- \([i,j] = \begin{cases} i \cdot i + 1 \cdot \ldots \cdot j & \text{if } i \leq j \\ t_{[j,i]} & \text{otherwise}. \end{cases}\)
- \([i,j][j-1,k] = \begin{cases} [i,j][j+1,k] & \text{if } j > k \\ [i,j] & \text{if } j = k. \end{cases}\)
- \(x_1x_2 \cdots x_k = x_1x_2 \cdots x_k\). (\(x_i\) will be defined later.)
- \(S_n = \{ x_1x_2 \cdots x_n \in X_n: \{ x_1, x_2, \ldots, x_n \} = \{ 1, 2, \ldots, n \} \} \).
- \(k(X_n) = \min\{ k \in N: A^k = [1,n]^k \} \) for all \(A \in S_n \subset X_n\).
- \(\Delta\{ 1, 2, \ldots, n \} = \Delta_n = [1,n][1,n-1][1,2][1,1].\)
- For \(A, B \in X_n\), we denote \(A \equiv B\) if \(A\) and \(B\) are identical, letter
by letter.

- The Coxeter group corresponding to \( X_n \) is denoted by \( \tilde{X}_n \), i.e.,
  \[ \tilde{X}_n: = \langle 1, 2, \ldots, n \mid t_i^2 = 1 \text{ for } 1 \leq i \leq n \rangle. \]

- The Coxeter number of \( \tilde{X}_n \) is denoted by \( h(\tilde{X}_n) \), that is
  \[ h(\tilde{X}_n): = \min \{ h \mid x^h = 1 \text{ for all } X \in S_n \subset \tilde{X}_n \}. \]

Remark. Note that \( 2k(X_n) \geq h(\tilde{X}_n) \).

3. The cancellation property of Coxeter semigroups

In this section, we show that the cancellation law holds in Coxeter semigroups. This property will be used in sections 4, 5.

Lemma 3.1 Let \( \Gamma \) be a multigraph with edge multiplicities \( \leq 3 \), and contains no \( K_3 \) (the complete graph of order 3) nor \( 
\text{edges} \). Let \( S \) be the semigroup corresponding to \( \Gamma \). Suppose that \( iX = jY \) holds for \( i, j \in V(\Gamma) \) and \( X, Y \in S \). Then, it follows that

(i) if \( i = j \), then \( X = Y \),

(ii) if \( i \neq j \) and \( e(i, j) = 0 \), then \( X = jZ \), \( Y = iZ \) for some \( Z \),

(iii) if \( e(i, j) = 1 \), then \( X = jiZ \), \( Y = ijZ \) for some \( Z \),

(iv) if \( e(i, j) = 2 \), then \( X = jijZ \), \( Y = ijiZ \) for some \( Z \).

(v) if \( e(i, j) = 3 \), then \( X = jijiZ \), \( Y = iijiZ \) for some \( Z \).

Proof. The lemma for words \( X, Y \) of word-length \( s \) will be referred to as \( L_s \). If \( iX \) is transformed \( jY \) by a sequence of \( t \) single applications of the defining relations, then the whole transformation will be said to be chain-length \( t \). We prove the lemma by induction on \( s \). We assume

(a) \( L_s \) is true for \( 0 \leq s \leq r \) for transformations of all chain-length, and

(b) \( L_{r+1} \) is true for all chain-length \( \leq t \).

Let \( X, Y \) be of word-length \( r+1 \), and \( iX = jY \) through a transformation of chain-length \( t+1 \). Let the successive words of the transformation be

\[ W_1 = iX, \ldots, W_g = kW, \ldots, W_{t+2} = jY, \]

with \( 2 \leq g \leq t+1 \). The transformations \( iX \to kW \), \( kW \to jY \) are each of chain-length \( \leq t \), and we can apply (b) to them.

We distinguish some cases according to the induced subgraph \( \langle \{i, j, k\} \rangle \). The general pattern of the proof is, however, exactly the
same for each case, and it will be sufficient here to deal with one case only, as a typical example of the common method of proof. See also [3].

Case. \( \langle i, j, k \rangle \cap = i \overline{k} \overline{j} \)

By (b) and

\[ iX = kW = jY, \]

we have

\[ X = kikP \]
\[ W = ik iP = jkQ \]
\[ Y = k j Q \]

for some \( P, Q \). By (a) and (2),

\[ k iP = j R \]
\[ kQ = i R \]

for some \( R \). By (4),

\[ iP = j k S \]
\[ R = k j S \]

for some \( S \). By (6),

\[ P = j T \]
\[ kS = iT \]

for some \( T \). By (9),

\[ S = ik i U \]
\[ T = k ik U \]

for some \( U \). On the other hand, by (5) and (7),

\[ Q = ikikM \]
\[ R = kikM = k j S \]

for some \( M \). By (a) and (13),

\[ ikM = j S \]

and so

\[ kM = j N \]
\[ S = i N \]

for some \( N \). By (14),

\[ M = j k O \]
\[ N = k j O \]

for some \( O \). By (10) and (15),

\[ S = ik i U = i N. \]

By (17) and (18),

- 4 -
\[ N = kjO = kiU, \]

and so
\[ iU = jO. \] (19)

By (19),
\[ U = jA \] (20)
\[ O = iA \] (21)

for some \( A \). By (1), (8), (11), and (20),
\[ X = kikP \]
\[ = kjkt \]
\[ = kjkjkikU \]
\[ = kjkjkikjA \]
\[ = kijkjikjA \]
\[ = kjikijkA \]
\[ = kjikikjkA \]
\[ = jkikijkA \]
\[ = jkijkjkA \]
\[ = j \cdot kikjikkA. \] (22)

By (3), (12), (16), and (21),
\[ Y = kijQ \]
\[ = kjikim \]
\[ = kjikijkO \]
\[ = kjikijkiA \]
\[ = kijkjkikA \]
\[ = kijkikikA \]
\[ = kijkjkjkA \]
\[ = kjikjkjkA \]
\[ = i \cdot kikjikkA. \] (23)

By (22) and (23), we have \( X = jZ, Y = iZ \) for \( Z := kikjikkA. \)

Lemma 3.2 \quad In Coxeter semigroups, it follows that
\begin{enumerate}
\item[(i)] if \( iX = iY \), then \( X = Y \),
\item[(ii)] if \( Xi = Yi \), then \( X = Y \).
\end{enumerate}

Proof. \quad Follows from Lemma 3.1(i) and the fact that \( A = B \) holds iff \( t_A = t_B \). \[ \blacksquare \]
4. Type $A_n$

In $A_n$, we define $\overline{i}^{(n)} = \overline{i} := n + 1 - i$. Note that this operator is considered as a graph isomorphism of $A_n$. In this section, we prove the following theorem which gives an affirmative answer to the problem of Coxeter.

Theorem 4.1 In $A_n$, the following hold.

1. For $A \in A_n$, $A^{n+1} = \Delta_n^2$ holds iff $A \in S_n$.
2. Suppose that $AB \in A_n$, $n = 2m$. Then, $(AB)^mA = \Delta_n$ holds iff $AB \in S_n$ and $B = \overline{A}$.
3. Suppose that $A \in A_n$, $n = 2m + 1$. Then, $A^{m+1} = \Delta_n$ holds iff $A \in S_n$ and $\overline{A} = A$.

Remark. We verify that Theorem 4.1 answers to the question of Coxeter. In $A_n$, define $P := 1 \cdot 3 \cdot 5 \cdot \ldots$, $Q := 2 \cdot 4 \cdot 6 \cdot \ldots$.

1. If $n = 2m$, then $\overline{Q} = P$, $\overline{P} = Q$, and so,
   \[ PQ \overline{P} \overline{Q} \ldots (n + 1 \text{ letters}) = (PQ)^mP \]
   \[ = \Delta_n \]
   \[ = (QP)^mQ \]
   \[ = QPQ \overline{P} \overline{Q} \ldots (n + 1 \text{ letters}). \]

2. If $n = 2m + 1$, then $\overline{PQ} = PQ$, $\overline{QP} = QP$, and so,
   \[ PQ \overline{P} \overline{Q} \ldots (n + 1 \text{ letters}) = (PQ)^{m+1} \]
   \[ = \Delta_n \]
   \[ = (QP)^{m+1} \]
   \[ = QPQ \overline{P} \overline{Q} \ldots (n + 1 \text{ letters}). \]

To prove the theorem, we need several lemmas.

Lemma 4.2 For all $A \in S_n$, $GA = [1,n]G$ holds for some $G \in A_{n-1}$.

Proof. We prove this lemma by induction on $n$. We can write $A$ in the form as $A = Xn$, or $A = nX$. If $A = nX$, we have $XA = XnX$. In both cases, we have $XA = YnX$ for some $X \in A_{n-1}$, $Y \in S_{n-1}$. By induction hypothesis, we have $ZY = [1,n-1]Z$ for some $Z \in A_{n-2}$. So,

\[ ZXA = ZYnX = [1,n-1]ZnX = [1,n]ZX. \]

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Formulae 4.3  In $A_n$, the following hold.

(i) $k[1,n] = [1,n] k - 1$ for $2 \leq k \leq n$,
(ii) $1[1,n]^2 = [1,n]^2 n$,
(iii) $k[1,n]^{n+1} = [1,n]^{n+1} k$ for $1 \leq k \leq n$,
(iv) $A^{n+1} = [1,n]^{n+1}$ for $A \in S_n$,
(v) $k\Delta_n = \Delta_n k$ for $1 \leq k \leq n$,
(vi) $\frac{\Delta_n}{\Delta} = t\Delta_n = \Delta_n$.

Proof. (i)-(iii) follow from the definition of $A_n$. We prove (iv). Suppose that $A \in S_n$. By Lemma 4.2, there exists $G \in A_n$ such that $GA = [1,n]G$. So, we have

$$GA^{n+1} = [1,n]^{n+1} G = [1,n]^{n+1}.$$

This and Lemma 3.2 give $A^{n+1} = [1,n]^{n+1}$. For the proof of (v) and (vi), see [3].

Lemma 4.4  Suppose that $A \tilde{A} \in S_n$, $n = 2m$.

(1) If $A = 1 B$, then there exist $x_1, \ldots, x_m, y_1, \ldots, y_m$ such that $A \tilde{A} = x_1 \ldots x_m y_1 \ldots y_m$, $x_1 < \ldots < x_m$, $y_1 > \ldots > y_m$, $x_1 = 1$, $y_1 = n$.

(2) If $A = B 1$, then there exist $x_1, \ldots, x_m, y_1, \ldots, y_m$ such that $A \tilde{A} = x_1 \ldots x_m y_1 \ldots y_m$, $x_1 < \ldots < x_m$, $y_1 > \ldots > y_m$, $y_1 = n$, $y_m = 1$.

Proof. This lemma is easily proved by induction on $m$.

We say that $A = a_1 a_2 \ldots a_k \in A_n$ is increasing (resp. decreasing) if $a_1 < \ldots < a_k$ (resp. $a_1 > \ldots > a_k$).

Lemma 4.5  Suppose that $D 1 U \in S_{n-1}$, $D$ decreasing, and $U$ increasing. Then, it follows that $[1,n]D 1 U = U n D[1,n]$.

Proof. Induction on $n$.

Case 1. $D = D' 2$

$$[1,n]D 1 U = [1,2,n]D' 2 1 U$$

$$= 1([2,n]D' 2 U)1$$

$$= 1(U n D'[2,n])1 \text{ (by induction hypothesis)}$$

$$= U n D'1 2 [3,n]$$

$$= U n D'2 1 [3,n]$$

$$= U n D[1,n].$$

Case 2. $U = 2 U'$

$$[1,n]D 1 U = [1,n]D 1 2 U'$$
Lemma 4.6 Suppose that $1 \leq n \leq m$, $Y = y_1 \ldots y_{m-1}$ decreasing. Define $U_0 := [1,n-1]$, $D_0 := [n,2]$, and for $1 \leq j \leq m-1$, define

$$U_j := \text{the word obtained by deleting } y_j \text{ from } U_{j-1},$$
$$D_j := \text{the word obtained by deleting } x_j \text{ from } D_{j-1}.$$

Then, it follows that $U_j D_j X n Y = X Y U_{j-1} D_{j-1}$.

Proof. Define $[i,j] := \{i,j\} \cap \mathbb{Z}$.

$$U_j D_j X n Y$$
$$= U_j [n,x_j+1] [x_j-1,2] D_j - 1 [2,x_j-1] X [x_j,n-1] X n [n-1,x_j+1] Y [x_j-1,2] Y$$
$$= U_j [x_j-1,2] D_j - 1 [2,x_j-1] X [n,x_j] [x_j+1,n-1] X n [n-1,x_j+1] Y [x_j-1,2] Y$$
$$= U_j [x_j-1,2] D_j - 1 [2,x_j-1] X [n-1,x_j+1] Y [x_j] [x_j+1,n-1] X [n,x_j] [x_j-1,2] Y$$

(by Lemma 4.5)
$$= U_j [n-1,x_j+1] Y [x_j-1,2] D_j - 1 [2,x_j-1] X [x_j,n-1] X D_j$$
$$= U_j Y [x_j-1,2] D_j - 1$$
$$= [1,y_j-1] [y_j+1,n-1] U_j [n-1,y_j+1] Y [y_j,2] Y [y_j-1,2] X [y_j+1,n-1] X D_j$$
$$= [y_j+1,n-1] X [n-1,y_j+1] Y [1, y_j] [y_j-1,2] Y [2, y_j-1] X [y_j+1,n-1] X D_j$$
$$= [y_j+1,n-1] X [n-1,y_j+1] Y [2, y_j-1] X y_j [y_j-1,2] Y [1, y_j] [y_j+1,n-1] X D_j$$

(by Lemma 4.5)
$$= [2, y_j-1] X [y_j+1,n-1] X [n-1,y_j+1] Y [y_j,2] Y U_{j-1} D_{j-1}$$
$$= X Y U_{j-1} D_{j-1}.$$

Lemma 4.7 Under the same assumption in the previous lemma, it follows that $(1 X n Y)^m = (X Y)^{m-1} [1,n,2]$.

Proof. Using Lemma 4.6, we have

$$(X Y)^{m-1} [1,n,2] = (X Y)^{m-1} U_0 D_0$$
$$= (X Y)^{m-2} U_1 D_1 1 X n Y$$
$$= \ldots$$
$$= U_{m-1} D_{m-1} (1 X n Y)^{m-1}$$
\[(1 \times n \times Y)^m.\] 

Proposition 4.8  \(A \in \mathcal{S}_n, n = 2m,\) then \((A \overline{A})^m A = \Delta_n.\)

Proof. Induction on \(m.\)

Case 1. \(A = 1B.\)

By Lemma 4.4, \(A \overline{A} = 1XnY\) holds for some \(X\) increasing, \(Y\) decreasing, with \(|X| = |Y|.
\[
(A \overline{A})^m A = (1XnY)^m A \\
= (X Y)^m_{[1,n,2]} B \text{ (by Lemma 4.7)} \\
= (B \overline{B})^m_{[1,n,1]} B \\
= (B \overline{B})^m_{[1,n,1]} B[1,n,1] \\
= \Delta[2, \ldots, n-1]_{[1,n,1]} \text{ (by induction hypothesis)} \\
= [1,n][n-1][2,n-1][2,n-2][2,2] \\
= \Delta_n.
\]

Case 2. \(A = B 1.\)

\[
(A \overline{A})^m A \cdot \overline{A} = (B 1 \overline{B} n)_{m+1} \\
= B \cdot (1 \overline{B} n B)^m \overline{B} \cdot n \\
= B \Delta_n n \text{ (by case 1)} \\
= \Delta_n \overline{B} n = \Delta_n \overline{A}.
\]

By Lemma 3.2, we have \((A \overline{A})^m A = \Delta_n.\)

Case 3. \(1 \notin A.\)

By cases 1, 2, we have \((A \overline{A})^m A = \Delta_n.\) Hence,
\[
(A \overline{A})^m A = \overline{A} \Delta_n = \Delta_n.\]  

Lemma 4.9 The following two conditions are equivalent.

1. \(A = \overline{A} \in \mathcal{S}_n, n = 2m + 1.\)

2. There exist \(P \equiv p_1 \ldots p_m\) increasing, \(Q \equiv q_1 \ldots q_m\) decreasing such that \(A = P \times n \times Q\) and \(Q = \overline{P^{(n-1)}},\) i.e., \(p_j + q_j = n\) for \(1 \leq j \leq m.\)

Proof. (1) implies (2):

Let \(A \equiv p_1 \overline{p_2} \ldots \overline{p_k} n q_1 q_2 \ldots q_l,\) \(\{p_j\}\) increasing, \(\{q_j\}\) decreasing. Suppose that \(p_1 = 1.\) Since
\[
\overline{A} \equiv \overline{p_1} \ldots \overline{p_k} 1 q_1 \ldots q_l \equiv A,
\]
we have \(q_1 = 2,\) i.e., \(q_1 = n - 1.\) Next, \(q_2 = 2\) or \(q_l = 2,\) and say \(q_l = 2.\) Since
\[
A \equiv 1 p_2 \ldots p_k n q_1 \ldots p_{l-1} 2
\]
we have $q_k = 3$, i.e., $q_k = n - 2$. Continuing this way, we have (2).

(2) implies (1):

Suppose that $A \equiv p_1 \ldots p_m n q_1 \ldots q_m$ satisfies the condition of (2). Assume that $p_1 = 1$, then $q_1 = 2$ and so

$$A \equiv p_1 \ldots p_m 1 2 q_2 \ldots q_m = 1 B$$

holds for some $B$. Next, $p_2 = 2$ or $q_m = 2$, and say $p_2 = 2$. Then, we have $q_2 = 3$ and so,

$$A \equiv p_1 \ldots p_m 1 2 3 q_3 \ldots q_m = 1 2 C$$

holds for some $C$. Continuing this way, we have (1).

Lemma 4.10 Suppose that $n X \in S_n$, $X \equiv x_2 \ldots x_n$, $x_2 > \ldots > x_i < \ldots < x_n$ for some $i$, $2 \leq i \leq n$. Define $x_1 := n$, $V_0 := [n,1][1,n]$, and for $V_{j-1} := P x_j Q x_j R$ with $1 \leq j \leq n$, define $V_j := P x_j Q R$ if $1 \leq j \leq i$, $V_j := P Q x_j R$ if $i < j \leq n$. Then, $X V_{j-1} = V_j n X$ holds.

Proof. Induction on $n$. Define $Z(k) := Z \cap \{1, \ldots, k\}$. If $j = 1$, then

$$X V_0 = X[n,1][1,n] = [n,1][1,n] X = [n,1][1,n] n X = V_1 n X.$$ 

So, we may assume $j > 1$.

**Case 1.** $x_n = n - 1$, $V_{j-1} \equiv n \cdot n - 1 \cdot V_{j-1}^{(n-2)} \cdot n - 1$, $X^{(n-2)} \neq V_{j-1}^{(n-2)}$.

$$X V_{j-1} = X^{(n-2)} \cdot n - 1 \cdot n \cdot n - 1 \cdot V_{j-1}^{(n-2)} \cdot n - 1$$

$$= X^{(n-2)} \cdot n \cdot n - 1 \cdot n \cdot V_{j-1}^{(n-2)} \cdot n - 1$$

$$= n \cdot X^{(n-2)} \cdot (n - 1 \cdot V_{j-1}^{(n-2)}) \cdot n \cdot n - 1$$

$$= n(n - 1 \cdot X^{(n-2)})(n - 1 \cdot X^{(n-2)}) \cdot n \cdot n - 1$$

(by induction hypothesis)

$$= V_j X^{(n-2)} \cdot n \cdot n - 1$$

$$= V_j \cdot n \cdot X^{(n-2)} \cdot n \cdot n - 1 = V_j n X.$$

**Case 2.** $x_n = n - 1$, $V_{j-1} \equiv n \cdot n - 1 \cdot V_{j-1}^{(n-2)} \cdot n - 1$, $X^{(n-2)} = V_{j-1}^{(n-2)}$.

$$X V_{j-1} = X^{(n-2)} \cdot n - 1 \cdot n \cdot n - 1 \cdot V_{j-1}^{(n-2)} \cdot n - 1$$

$$= X^{(n-2)} \cdot n - 1 \cdot n \cdot X$$

$$= n \cdot X^{(n-2)} \cdot n - 1 \cdot n \cdot X = V_j n X.$$

**Case 3.** $x_n = n - 1$, $V_{j-1} \equiv n \cdot n - 1 \cdot V_{j-1}^{(n-2)} \cdot n - 1$.

$$X V_{j-1} = n - 1 \cdot X^{(n-2)}[n,1][1,n - 1]$$
= n - 1 \cdot [n,1][1,n - 1] X^{(n-2)} \\
= n \cdot n - 1 \cdot n[n - 2,1][1,n - 2] n - 1 \cdot X^{(n-2)} \\
= [n,1][1,n - 2]n X = V_j n X.

Case 4. \( x_2 = n - 1 \), \( V_{j-1} = n \cdot n - 1 \cdot V_j^{(n-2)} \), \( X^{(n-2)} \neq V_j^{(n-2)} \).

\[
X V_{j-1} = n - 1 \cdot X^{(n-2)} n \cdot n - 1 \cdot V_j^{(n-2)} \\
= n - 1 \cdot n \cdot X^{(n-2)} (n - 1 \cdot V_j^{(n-2)}) \\
= n - 1 \cdot n (n - 1 \cdot V_j^{(n-2)}) (n - 1 \cdot X^{(n-2)})
\]
(by induction hypothesis)

= n \cdot n - 1 \cdot n \cdot V_j^{(n-2)} X = V_j n X.

Case 5. \( x_2 = n - 1 \), \( V_{j-1} = n \cdot n - 1 \cdot V_j^{(n-2)} \), \( X^{(n-2)} = V_j^{(n-2)} \).

\[
X V_{j-1} = (n - 1 \cdot X^{(n-2)}) n (n - 1 \cdot V_j^{(n-2)}) = V_j n X.
\]

**Proposition 4.11** If \( A = \overline{A} \in S_n \), \( n = 2m + 1 \), then \( A^{m+1} = \Delta_n \).

**Proof.** We distinguish two cases.

Case 1. \( A = n X \).

By Lemma 4.10, \( X^{mV_0} = V_m(n X)^m \), where \( V_i \) is defined in Lemma 4.10. By Lemma 4.9, we have \( n X = P n Q \), where \( P, Q \) satisfy the conditions in Lemma 4.9 (2). So, we have

\[
(P Q)^m[n,1][1,n] = [n,1] \overline{Q} \overline{A}^m.
\]

Since \( Q = \overline{P}^{(n-1)} \), using Proposition 4.8, we have

\[
Q (P Q)^m[n,1][1,n] = \Delta_{n-1}[n,1][1,n] = [n,1] \Delta_n.
\]

On the other hand,

\[
Q[n,1] \overline{Q} \overline{A}^m = [n,1] \overline{P} \overline{Q} \overline{A}^m = [n,1] \overline{A} \overline{A}^m = [n,1] A^{m+1}.
\]

Combining these identities, we have \( A^{m+1} = \Delta_n \).

Case 2. \( A = X n \).

By case 1, we have \( t(A)^{m+1} = \Delta_n \). So, \( A^{m+1} = t \Delta_n = \Delta_n \). ■

**Proposition 4.12** In \( A_n \), the following hold.

(1) If \( A B \in A_n \), \( n = 2m \), and \((A B)^m A = \Delta_n \), then \( B = \overline{A} \), \( A B \in S_n \).

(2) If \( A \in A_n \), \( n = 2m + 1 \), and \((A B)^{m+1} = \Delta_n \), then \( A = \overline{A} \in S_n \).

**Proof.** (1) Suppose that \((A B)^m A = \Delta_n \), then \( A B \in S_n \). Suppose \( Z \overline{Z} \in S_n \), then by Proposition 4.8, \( (Z \overline{Z})^m Z = \Delta_n \). By Lemma 4.2, \( G A B = Z \overline{Z} G \) holds for some \( G \in A_n \). Hence,
\[ \overline{G \overline{B}} \Delta_n = G \Delta_n B = G (A B)^{m+1} = (Z \overline{Z})^{m+1} G = Z \Delta_n G = Z \overline{G} \Delta_n, \]
which implies \( G \overline{B} = Z \overline{G} \), or \( \overline{G} B = \overline{Z} G \). So,
\[ GA = Z \overline{Z} G = Z \overline{G} B, \]
which implies \( G A = Z \overline{G} \) or \( \overline{G} A = Z \overline{G} \). This gives \( B = \overline{A} \).

(2) Suppose that \( A^{m+1} = \Delta_n \), then \( A \in S_n \), Suppose \( Z \in S_n \), then by Proposition 4.11, \( Z^{m+1} = \Delta_n \). By Lemma 4.2, \( GA = Z G \) holds for some \( G \in A_n \). Hence,
\[ G \Delta_n = GA^{m+1} = Z^{m+1} G = \Delta_n G = \overline{G} \Delta_n, \]
which implies \( G = \overline{G} \). Therefore,
\[ G \overline{A} = \overline{G} A = Z \overline{G} = Z G = G A. \]
This gives \( A = \overline{A} \). ■

Proof of Theorem 4.1 Suppose that \( A^{m+1} = \Delta_n \), then \( A \in S_n \). This and Formulae 4.3 (iv) give (1), since \( \Delta_n = [1,n]^{n+1} \). (2) and (3) follow from Propositions 4.8, 4.11, and 4.12. ■

Remark. Since \( k(A_n) = n + 1 \), Theorem 4.1 gives \( k(A_n) = n + 1 \). ■

5. Types \( B_n \) and \( D_n \)

In this section, we deal with \( B_n \) and \( D_n \).

Lemma 5.1 For all \( A \in S_n \), \( GA = [1,n] G \) holds for some \( G \in S_{n-1} \) in \( B_n \) and \( D_n \).

Proof. Similar to the proof of Lemma 4.2. ■

Formulae 5.2 In \( B_n \), the following hold.

(i) \( k[1,n] = [1,n] k - 1 \) for \( 3 \leq k \leq n \),
(ii) \( 2[1,n]^2 = [1,n]^2 n \),
(iii) \( 1 \geq [1,n] = 2[1,n] \),
(iv) \( k[1,n] = [1,n] k \) for \( 1 \leq k \leq n \).

Proof. It is not difficult to check these formulae using the defining relations of \( B_n \). ■

Theorem 5.3 For \( A \in B_n \), \( A^n = [1,n]^n \) holds iff \( A \in S_n \).

Proof. The proof of this theorem is similar to the proofs of Formulae
4.3 (iv) and Proposition 4.12 (2).

In $D_n$, we define $\bar{i} := i$ if $3 \leq i \leq n$, $\bar{a} := \beta$ if $\{\alpha, \beta\} = \{1, 2\}$. Note that this is a graph isomorphism of $D_n$.

**Formulae 5.4** In $D_n$, the following hold.

(i) $k[1,n] = [1,n]k - 1$ for $4 \leq k \leq n$,
(ii) $3[1,n]^2 = [1,n]^2n$,
(iii) $\alpha[1,n]^2 = [1,n]^2\overline{\alpha}$ for $\alpha = 1, 2$,
(iv) $k[1,n]^{n-1} = [1,n]^{n-1}\overline{k}$ for $1 \leq k \leq n$, $n$ odd,
(v) $k[1,n]^{n-1} = [1,n]^{n-1}k$ for $1 \leq k \leq n$, $n$ even.

**Proof.** It is not difficult to check these formulae using the defining relations of $D_n$.

**Theorem 5.5** In $D_n$, the following hold.

(1) For $A \in D_n$, $A^{2n-2} = [1,n]^{2n-2}$ iff $A \in S_n$.
(2) Suppose that $A \in D_n$, $n$ odd. Then, $A^{n-1} = [1,n]^{n-1}$ holds iff $A \in S_n$ and $\overline{A} = A$.
(3) Suppose that $A \in D_n$, $n$ even. Then, $A^{n-1} = [1,n]^{n-1}$ holds iff $A \in S_n$.

**Proof.** Using Formulae 5.4 (iv) and (v), one can prove (1) and (3) in the similar way of the proofs of Formulae 4.3 (iv) and Proposition 4.12 (2). We prove (2). The proof that $A^{n-1} = [1,n]^{n-1}$ implies $\overline{A} = A \in S_n$ is also similar to the proof of Proposition 4.12 (2). So, suppose that $\overline{A} = A \in S_n$.

Since $\overline{A} = A$, $A$ is represented as $A = pxQ$ where $x = \alpha \beta$, $(\alpha, \beta) = \{1, 2\}$. Note that $x, 3, 4, \ldots, n$ satisfy the relations of $B_{n-1}$:

\[
\begin{array}{cccc}
\_ & \_ & \_ & \_ \\
3 & 4 & \_ & \_ \\
\_ & \_ & \_ & \_ \\
\end{array}
\]

Therefore, by Theorem 5.3,

$A^{n-1} = (x \cdot 3 \cdot 4 \cdot \ldots \cdot n)^{n-1} = [1,n]^{n-1}$.

**Remark.** Since $h(\overline{D}_n) = 2n$ and $h(D_n) = 2n - 2$, Theorems 5.3, 5.5 give $k(B_n) = n$, $k(D_{2m}) = 2m - 1$, and $k(D_{2m+1}) = 4m$.

**6. Other types**

Results in this section are obtained by calculations using a computer.
Theorem 6.1  In the exceptional types, we have
\[ k(E_6) = 12, \quad k(E_7) = 9, \quad k(E_8) = 15, \quad k(F_4) = 6, \quad k(H_3) = 5, \quad k(H_4) = 15. \]

In these six types, \( 2k(X_n) = h(\overset{\circ}{X}_n) \) holds except \( E_6 \) which satisfies \( k(E_6) = h(\overset{\circ}{E}_6) \). In \( E_6 \), define \( \alpha := \beta \) if \( \{\alpha, \beta\} = \{3, 4\}, \{5, 6\} \), and \( i := i \) otherwise. Note that this is a graph isomorphism of \( E_6 \).

Theorem 6.2  For \( A \in E_6 \), \( A^6 = [1,6]^6 \) holds iff \( A \in S_6 \) and \( \overline{A} = A \).

References