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The cohomology groups of degree 3 of Siegel modular varieties of genus 2

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§0. Introduction.

We discuss two types of modular symbols on Siegel modular varieties of genus 2: One is holomorphic, another totally real. In §1, we give a simple result for the first case. In §2, on the second case, we discuss a conjecture on its Hodge type, and explain how to reduce it to another conjectures on L-functions.

Notation. \( \text{Sp}(2; \mathbb{R}) \) : the real symplectic group of rank 2.
\( \text{Sp}(2; \mathbb{Z}) \) : the integral symplectic subgroup in \( \text{Sp}(2; \mathbb{R}) \).
\( l \) : natural number.
\( \Gamma(l) \) : the principal congruence subgroup of \( \text{Sp}(2; \mathbb{Z}) \).
\( \mathfrak{h}_2 \) : the Siegel upper half space of genus 2.
\( \mathcal{V} = \Gamma(l) \mathfrak{h}_2 \) : the quotient of \( \mathfrak{h}_2 \) by \( \Gamma(l) \), which is a smooth algebraic variety over \( \mathbb{C} \).
\( \bar{\mathcal{V}} \) : a smooth toroidal compactification of \( \mathcal{V} \).
§1. Modular sub-varieties and coniveau filtration.

(4.1) Coniveau filtration.

Let $X$ be a smooth compact algebraic variety over $\mathbb{C}$ with dimension $n$. Let $Y$ be a subvariety of dimension $m$ in $X$. Then by restriction, we have a natural map of Hodge structures:

$$H^k(X, \mathcal{O}) \to H^k(Y, \mathcal{O}).$$

If $Y$ is smooth, then by Poincaré duality, we have

$$H^{2m-k}(Y, \mathcal{O})(m) \to H^{2n-k}(X, \mathcal{O})(n).$$

Or, equivalently, we have

$$H^i(Y, \mathcal{O})(-d) \to H^{i+2d}(X, \mathcal{O}).$$

Here $(m)$ etc. are Tate twists. In general, we have an exact sequence of local cohomology

$$\to H^i_Y(X, \mathcal{O}) \to H^i(X, \mathcal{O}) \to H^i(X-Y, \mathcal{O}) \to \cdots.$$ 

By Deligne, this is an exact sequence of mixed Hodge structure $\mathcal{M}$.

When $Y$ is smooth, we have $H^i_Y(X, \mathcal{O}) \cong H^{i-2d}(Y, \mathcal{O})(-d)$ by Grothendieck isomorphism.

Definition-Proposition. The rational sub-Hodge structure of $H^i(X, \mathcal{O})$ defined by

$$\mathcal{F}^d H^i(X, \mathcal{O}) := \sum_{Y \subseteq X} \text{Im} \left( H^i_Y(X, \mathcal{O}) \to H^i(X, \mathcal{O}) \right)$$

has Hodge type $\{ (a,b) \in \mathbb{Z} \times \mathbb{Z} \mid a+b=i, a \geq d, b \geq d \}$. 

The above filtration is called “coniveau” filtration by Grothendieck. It is assumed that this filtration has a crucial role in the arithmetic of cycles on $X$. 

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(1.2) The "boundary part" of Hodge structure $H^3(V, \mathbb{Q})$.

Let $\overline{V}$ be a smooth toroidal compactification of $V$ along the cusps. Put $D = \overline{V} - V$ and $D = \bigcup D_i$ the decomposition into irreducible components $D_i$. Since each $D_i$ is smooth in $\overline{V}$, the restriction map $H^3(\overline{V}, \mathbb{Q}) \to H^3(D_i, \mathbb{Q})$ defines a morphism of rational Hodge structures $\rho_{D_i} : H^1(D_i, \mathbb{Q})(-1) \to H^3(\overline{V}, \mathbb{Q})$.

**Definition.** $H^3(M_0, \mathbb{Q}) := \sum_i \text{Im} \rho_{D_i}$.

$H^3(M_0, \mathbb{Q})$ is a polarized sub-Hodge structure with Hodge type $(2,1), (1,3)$.

If we take the Igusa model for $\overline{V}$ as [Igusa], each $D_i$ is an elliptic modular surface over a modular curve $C_i$. Then $H^1(D_i, \mathbb{Q}) \cong H^1(C_i, \mathbb{Q})$.

(1.3) **Hilbert modular surfaces.**

By moduli embedding, or Satake embedding, a Hilbert modular surface $S$ is mapped to $V$; $S \to V$. We can compactify $f : \overline{S} \to \overline{V}$. Since $b_1(\overline{S}) = 0$ for any Hilbert modular surface, $H^1(\overline{S}, \mathbb{Q})(-1) \to H^3(\overline{V}, \mathbb{Q})$ is zero map.

(1.4) "Diagonal" embedding.

Let $(z_1, z_2) \in H \times H \to (\begin{smallmatrix} z_1 & * \\ 0 & z_2 \end{smallmatrix}) \in G_2$ be the holomorphic
map from the product of the upper half plane $H$ to $\mathbb{H}$. This induces
a holomorphic map from a product $E = E_1 \times E_2$ of elliptic
modular curves $E_i (i=1, 2)$ to $V : \mathit{f}_{E_i} : E_i = E_1 \times E_2 \rightarrow V$.

By Kunneth formula
\[
H^1 (\mathit{f}_{E_i} (\mathbb{C}), \mathbb{Q}) = H^1 (\mathbb{C}, \mathbb{Q}) \oplus H^1 (\mathbb{C}, \mathbb{Q}),
\]
In general, $H^1 (\mathit{f}_{E_i} (\mathbb{C}), \mathbb{Q}) \neq 0$, hence $H^1 (\mathit{f}_{E_i} (\mathbb{C}), \mathbb{Q}) \neq 0$.

(1.5) Proposition. Consider the Poincaré dual $\mathit{f}_{E_i} : H^1 (\mathit{f}_{E_i} (\mathbb{C}), \mathbb{Q}) \rightarrow H^3 (\mathbb{C}, \mathbb{Q})$ of the restriction map $H^3 (\mathbb{C}, \mathbb{Q}) \rightarrow H^3 (\mathit{f}_{E_i} (\mathbb{C}), \mathbb{Q})$. Then $\mathit{Im} \mathit{f}_{E_i}$ is contained in $H^3 (\mathit{f}_{E_i} (\mathbb{C}), \mathbb{Q})$.

(1.6) Proof of Proposition. Recall that $H^1 (\mathbb{C}, \mathbb{Q}) = 0$. Hence the intersection form
\[
\mathit{E} : H^3 (\mathbb{C}, \mathbb{Q}) \times H^3 (\mathbb{C}, \mathbb{Q}) \rightarrow \mathbb{Q}(-3)
\]
is non-degenerate, and its restriction to $H^3 (\mathit{f}_{E_i} (\mathbb{C}), \mathbb{Q})$ is also non-degenerate.

Therefore, if $\mathit{Im} \mathit{f}_{E_i} \neq H^3 (\mathit{f}_{E_i} (\mathbb{C}), \mathbb{Q})$ is true, then there exists a
cohomology class $\gamma \neq 0 \in \mathit{Im} \mathit{f}_{E_i}$ such that for any $\eta \in H^3 (\mathit{f}_{E_i} (\mathbb{C}), \mathbb{Q})$ the pairing $\mathit{E} (\gamma, \eta)$ = 0. Thus it suffices to show the following.

(1.7) Lemma. For any $\gamma \neq 0 \in \mathit{Im} \mathit{f}_{E_i}$, there exists $\eta \in H^3 (\mathit{f}_{E_i} (\mathbb{C}), \mathbb{Q})$ such that $\mathit{E} (\gamma, \eta)$ = 0.
In order to prove the above lemma, it is necessary to write
the intersection number $\Phi(x, \eta)$ in terms of
$\overline{\gamma}' \in H^1(\overline{E}_x, \mathbb{Q})(-1)$ and $\gamma' \in H^1(D_i, \mathbb{Q})(-1)$.
If we choose $\overline{\gamma}'$, $\gamma'$ by
$\rho_{\overline{E}_x}(\overline{\gamma}') = \overline{\gamma}'$, $\rho_{D_i}(\gamma') = \gamma'.

(1.9) Lemma. Let $Y$ be the intersection of $\overline{E}_x$ and $D_i$.
$Y$ is possibly empty. Let us consider the morphisms by restriction:
\[
\begin{align*}
x_1 : H^1(\overline{E}_x, \mathbb{Q})(-1) & \to H^1(Y, \mathbb{Q})(-1) \\
x_2 : H^1(D_i, \mathbb{Q})(-1) & \to H^1(Y, \mathbb{Q})(-1).
\end{align*}
\]
Then
$\Phi(x, \eta) = \Phi_Y(x_1(\gamma'), x_2(\gamma')).$

Here $\Phi_Y$ is the intersection form of the curve $Y$:
$\Phi_Y : H^1(Y, \mathbb{Q}) \times H^1(Y, \mathbb{Q}) \to \mathbb{Q}(-1)$.

The above lemma is a standard fact in the intersection theory,
and may be found in the text book of [Fulton].

Thus the proof of Lemma (1.7) is reduced to the following.

(1.9) Lemma. For any $\gamma \neq 0$ in $\rho_{\overline{E}_x}$, there exists an element
$\gamma' \in H^1(D_i, \mathbb{Q})$ of some irreducible component $D_i$ of $D$ such that
for $Y = D_i \cap \overline{E}_x$, $\Phi_Y(x_1(\gamma'), x_2(\gamma')) \neq 0.$
This lemma is immediate from the following result of [Yamazaki].

(1.10) Proposition. For \( \widetilde{E}_a = \widetilde{R}_1 \times \widetilde{R}_2 \rightarrow \widetilde{V} \), there exists an irreducible component \( \mathcal{D}_c \) of \( \mathcal{D} \) such that for \( Y = \mathcal{D}_c \cap \widetilde{E}_a \),
\[ H^k(\mathcal{D}_c, \mathbb{Q}) \cong H^k(Y, \mathbb{Q}) \]

This completes the proof of Proposition (1.5).

§2. Conjectures on totally real modular symbols.

(2.1) In the construction of \( L \)-functions associated to Siegel modular forms of genus 2, Andrianov considered a kind of totally real embedding of hyperbolic three spaces into the Siegel upper half space of degree 2.

Let \( K \) is an imaginary quadratic field, \( \text{SL}_2(\mathcal{O}_K) \) is the special linear group with entries in the ring of integers \( \mathcal{O}_K \) of \( K \), which is a discrete subgroup of \( \text{SL}_2(\mathbb{C}) \). For some congruence subgroup \( \Gamma' \) of \( \text{SL}_2(\mathcal{O}_K) \), there exists a map
\[ m : \Gamma' \backslash \text{SL}_2(\mathbb{C}) / \mathcal{K}' \rightarrow \Gamma(2) \backslash \text{Sp}(2; \mathbb{R}) / \mathbb{K} \]
induced from an injective homomorphism \( \text{SL}_2(\mathbb{C}) \rightarrow \text{Sp}(2; \mathbb{R}) \). Here \( \mathcal{K}' \) and \( \mathbb{K} \) are maximal compact subgroups of \( \text{SL}_2(\mathbb{C}) \) and \( \text{Sp}(2; \mathbb{R}) \), respectively.
Let $\overline{V}$ be the Tate duality compactification of $V = \Gamma(\mathcal{O}) \backslash G_2$. Then the canonical compactification of $\infty$ defines an element in $H_3(\overline{V}; \mathbb{Z})$. More careful investigation show the following.

(2.2) Lemma. $\overline{\infty}$ defines an element $\overline{\eta} \in IH_3(\overline{V}; \mathbb{Z})$. Here $IH_3(\overline{V}; \mathbb{Z})$ is the intersection homology group of degree 3 of $\overline{V}$ with middle perversity.

We denote by $[\overline{\infty}]^* \in IH^3(\overline{V}; \mathbb{Q})$ the Poincaré dual of the image of the fundamental class $[\overline{\infty}]$ of $\Gamma' \backslash \mathbb{H}_2(\mathcal{O}_K)/\mathbb{K}$. The cohomology group $IH^3(\overline{V}; \mathbb{Q})$ has a Hodge structure defined by Horiuchi and Inose.

(2.3) Conjecture. The cycle $[\overline{\infty}]^* \in IH^3(\overline{V}; \mathbb{Q})$ has Hodge type $(3,0); (0,3)$.

The above conjecture is reduced to the following conjectures for L-functions for harmonic forms on $\overline{V}$.

(2.4) Conjecture. Let $\omega$ be a $L^2$-harmonic form on $\overline{V}$ of type (2.1). Assume that $\omega$ is an common eigen form of all Hecke operators. Then,
(i) Let $L(s, \omega)$ be the $L$-function for Spinor representation associated to $\omega$. Then $L(s, \omega)$ has a probability simple pole at $s=2$, & 
\quad R_{s=2} L(s, \omega) = \int \omega,$

with $\omega$ is a linear combination of cycles $[v_1] \in H_2 (\tilde{V}; \mathbb{Z})$.

(ii) $L(s, \omega)$ is an entire function if it is multiplied with the $\Gamma$-factor $\Gamma(s) \Gamma(s-1)$.

[References]

