Discontinuous Group in a Non-Riemannian Homogeneous Space

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Table of contents

§1. Notations and preliminaries
§2. Homogeneous spaces of semidirect product groups
§3. Homogeneous spaces of solvable groups
§4. R-rank one semisimple group manifolds
§5. A necessary condition for the existence of a uniform lattice

1. Notations and preliminaries

1.1. proper action
First of all, let us recall the definition of the properness of a continuous map.

DEFINITION 1.1.1. Let $f : X \rightarrow Y$ be a continuous map between locally compact Hausdorff spaces. $f$ is called proper iff one of the following equivalent conditions holds.

1. $f$ is a closed map, and $f^{-1}(y)$ is compact for any $y \in Y$.
2. For any topological space $Z$, $f : X \times Z \rightarrow Y \times Z$ is a closed map.
3. $f^{-1}(S)$ is compact for any compact subset $S$ of $Y$.

If $f$ is a proper map, then it follows easily that a closed subset $Z$ of $X$ is compact iff $f(Z)$ is contained in some compact set of $Y$.

DEFINITION 1.1.2. The action of a locally compact topological (Hausdorff) group $G$ acting continuously on a locally compact Hausdorff space $X$ is called proper iff the map $G \times X \ni (g, x) \mapsto (x, gx) \in X \times X$ is proper. Equivalently, $\{g \in G : f(g, S) \cap S \neq \emptyset\}$ is compact for every compact subset $S$ in $X$. We call the action is properly discontinuous iff $G$ is discrete and acts properly on $X$.

Suppose that $H$ is a closed subgroup of $G$. $\Gamma$ is called a discontinuous group in $G/H$ iff $\Gamma$ is a discrete subgroup of $G$ and $\Gamma$ acts properly on $G/H$.

LEMMA 1.1.3. Let $G_i (i = 1, 2)$ be locally compact groups and $L_i, H_i \subset G_i$ be closed subgroups. Suppose that $f : G_1 \rightarrow G_2$ is a (continuous) homomorphism such that $f(L_1) \subset L_2$, $f(H_1) \subset H_2$. Assume that $f(L_1)$ is closed in $G_2$.

1) Assume that $L_1 \cap \text{Ker } f$ is compact. If the $L_2$ action on $G_2/H_2$ is proper, then the $L_1$ action on $G_1/H_1$ is also proper.

2) Assume that $f(G_1)H_2 = G_2$, that $G_1 \rightarrow G_2/H_2$ is an open map, and that the
quotients $L_2/f(L_1)$, $f^{-1}(H_2)/H_1$ are compact. If the $L_1$ action on $G_1/H_1$ is proper, then the $L_2$ action on $G_2/H_2$ is also proper.

Remark 1.1.4. If $G_i$ are (separable) Lie groups, then it is automatically satisfied from the assumption $f(G_1)H_2 = G_2$ that the map $G_1 \rightarrow G_2/H_2$ is open.

Proof of Lemma (1.1.3): 1) Fix any compact subset $S$ of $G_1$. We have

$$f(L_1 \cap SH_1S^{-1}) \subset L_2 \cap f(S)H_2f(S)^{-1}.$$  

If $L_2$ acts on $G_2/H_2$ properly, then $f(L_1 \cap SH_1S^{-1})$ is contained in a compact set because $f(S)$ is compact. Then $L_1 \cap SH_1S^{-1}$ is compact, since $f|_{L_1}: L_1 \rightarrow L_2$ is a proper map as it is a composition of proper maps: $L_1 \rightarrow L_1/L_1 \cap \text{Ker} f \hookrightarrow L_2$. That is, $L_1$ acts on $G_1/H_1$ properly.

2) As $f(L_1)$ is a closed and cocompact subgroup of $L_2$, $L_2$ acts properly iff $f(L_1)$ acts properly. So we may and do assume $f(L_1) = L_2$. Take a compact set $S_1$ of $G_1$ such that $f^{-1}(H_2) = S_1H_1$ and that $S_1$ contains the unit of $G_1$. Fix any compact subset $S$ of $G_2$. We can find a compact subset $\tilde{S}$ of $G_1$ such that $f(\tilde{S})H_2 \supset S$ as it follows from the assumption that $G_1 \rightarrow G_2/H_2$ is an open map and $f(G_1)H_2 = G_2$. Then we have

$$f^{-1}(L_2 \cap SH_2S^{-1}) \subset f^{-1}(L_2) \cap \tilde{S}f^{-1}(H_2)\tilde{S}^{-1}.$$  

In particular, $f|_{L_1}^{-1}(L_2 \cap SH_2S^{-1})$ is compact if $L_1$ acts properly on $G_1/H_1$, because

$$f|_{L_1}^{-1}(L_2 \cap SH_2S^{-1}) \subset L_1 \cap f^{-1}(L_2) \cap \tilde{S}f^{-1}(H_2)\tilde{S}^{-1} \subset L_1 \cap \tilde{S}S_1H_1S_1^{-1}\tilde{S}^{-1}.$$  

Under our assumption $f(L_1) = L_2$, we have $L_2 \cap SH_2S^{-1} = f|_{L_1} \circ f|_{L_1}^{-1}(L_2 \cap SH_2S^{-1})$ is compact. Thus $L_2$ acts on $G_2/H_2$ properly. □
1.2. **property** (CI)

Let $H, L$ be closed subgroups of a locally compact topological group $G$. If $L$ acts properly on $G/H$, then any $L$-orbit is closed with a compact isotropy group. In general, this is not a sufficient condition for the properness of the $L$-action. Anyway, the second condition about compact isotropy group is easier to check. We call that the triplet $(L, G, H)$ has **property (CI)** iff $L \cap gHg^{-1}$ is compact for any $g \in G$. We call that the triplet $(L, G, H)$ is **proper** iff $L$ acts properly on $G/H$. Then the following is easily checked from the definition (see [Bou] for the first part.)

**Lemma 1.2.1.** With notation as above, the following conditions are equivalent:

1) $(L, G, H)$ is proper,

1)' $(H, G, L)$ is proper,

1)" $(\text{diag} G, G \times G, H \times L)$ is proper,

which imply the following equivalent conditions.

2) $(L, G, H)$ has property (CI)

2)' $(H, G, L)$ has property (CI)

2)" $(\text{diag} G, G \times G, H \times L)$ has property (CI)

As we mentioned above, it is easier to check property (CI) than properness. So we are interested in how property (CI) approximates properness.

**Example 1.2.2.**

1) Suppose that $G$ is a linear reductive Lie group, and that $H, L$ are closed subgroups reductive in $G$ (see §1.3 for definition). Then property (CI) $\Leftrightarrow$ properness. This is a
restatement of one of our main results in [Ko], Theorem 4.1.

2) Suppose that \( G \) is a linear reductive noncompact Lie group. Let \( G = KAN \) be an Iwasawa decomposition and let \( H := A, L := N \). Then property (CI) is always satisfied for \( (L, G, H) \), while \( L \) never acts properly on \( G/H \).

3) If \( L \) is normal in \( G \) and if \( HL \) is closed, then property (CI) \( \iff \) properness.

4) Suppose that \( G = GL(2, \mathbb{R}) \ltimes \mathbb{R}^2, H = GL(2, \mathbb{R}) \). Then for any connected closed Lie subgroup \( L \) of \( G \), property (CI) \( \iff \) properness.

The proof of (2), (3) is easy. As for (4), we shall classify the maximal connected Lie groups \( L \) of \( G \) such that \( (L, G, H) \) has property (CI) in the proof in §2.2, which we also see is in fact proper.

### 1.3. notations for reductive groups

In this subsection we set up notation.

Let \( G \) be a real linear reductive Lie group, with real Lie algebra \( \mathfrak{g} \). Given a Cartan involution \( \theta \) of \( G \), we always write a Cartan decomposition of its Lie algebra as \( \mathfrak{g} = k + p \).

Fix a maximally abelian subspace \( a \subset p \). \( a \) is called a maximally split abelian subspace for \( G \). We write \( W(\mathfrak{g}, a) \) for the Weyl group associated to the root system of \( \Sigma(\mathfrak{g}, a) \), \( \mathbb{R}\text{-}\text{rank } G := \dim a \) (\( \leq \text{rank } G \geq \)) \( c\text{-}\text{rank } G := \text{rank } K \), and \( d(G) := \dim G/K = \dim p \).

Let \( H \) be a closed subgroup in \( G \). If there exists a Cartan involution of \( G \) which stables \( H \), then \( H \) is called reductive in \( G \) and \( G/H \) is called a homogeneous space of reductive type. In this case, \( H \) is of finite connected components, \( H \) has a Cartan decomposition \( H = (H \cap K) \exp(\mathfrak{h} \cap p) \), and \( \mathfrak{h} \) is reductive in \( \mathfrak{g} \), namely, the adjoint representation \( \mathfrak{h} \to \mathfrak{gl}(\mathfrak{g}) \) is completely reducible. Let \( a_H \) be a maximally split abelian subspace for \( H \).
Then there exists an element $g$ of $G$ such that $\text{Ad}(g)a_H \subseteq a$. Put $a(H) := \text{Ad}(g)a_H$, which is uniquely defined up to conjugacy of $W(g,a)$.

Remark 1.3.1. Definition-Lemma (2.6) in [Ko] is not accurate if $H$ is not an algebraic group defined over $\mathbb{R}$ (cf. [M]). Our definition here is equivalent to (2.6.1), and implies (2.6.2) there. Any statement there is valid for a homogeneous space of reductive type in this sense.

We will use the standard notation $\mathbb{N}, \mathbb{Z}, \mathbb{R}, \mathbb{C}$ and $\mathbb{H}$. Here $\mathbb{N}$ means the set of non-negative integers and $\mathbb{H}$ means the $\mathbb{R}$-algebra of quarternionic numbers.

2. Homogeneous spaces of semidirect product groups

2.1. semidirect product

Proposition 2.1.1 Let $G$ be a Lie group and $H$ be a closed subgroup. Assume that $\mathfrak{h}$ contains a maximal semisimple algebra of $\mathfrak{g}$. Then any connected closed subgroup $L$ such that $(L, G, H)$ has property (CI) is amenable.

Proof: Let $\mathfrak{l} = \mathfrak{l}_s + \mathfrak{l}_n$ be a Levi decomposition of $\mathfrak{l}$, where $\mathfrak{l}_s$ is a maximal semisimple algebra and $\mathfrak{l}_n$ is the radical. It follows from the assumption that there exists $g \in G$ such that $\mathfrak{l}_s \subseteq \text{Ad}(g)\mathfrak{h}$. Thus, $L \cap gHg^{-1} \supset L_s$, where $L_s$ is a connected semisimple
Lie subgroup with Lie algebra $\mathfrak{l}_s$. Therefore $L_s$ must be compact. $L$ is thus a compact extension of a solvable group, namely, an amenable group.

2.2. affine transformation group of $\mathbb{R}^2$

Let $G = GL(2, \mathbb{R}) \ltimes \mathbb{R}^2$, the affine transformation group of $\mathbb{R}^2$. The multiplicative structure is given by $(g_1, v_1) \cdot (g_2, v_2) := (g_1 g_2, g_1 v_2 + v_1)$, where $g_i \in GL(2, \mathbb{R})$, $v_i \in \mathbb{R}^2$. The Lie algebra $\mathfrak{g}$ is identified with $M(3, 2; \mathbb{R}) = \{(A, u) : A \in \mathfrak{g}l(2, \mathbb{R}), u \in \mathbb{R}^2\}$ equipped with $[(A_1, u_1), (A_2, u_2)] = ([A_1, A_2], A_1 u_2 - A_2 u_1)$. The adjoint action is given by $Ad((g, v))(A, u) = (g A g^{-1}, g u - g A g^{-1} v)$. Let $H = GL(2, \mathbb{R})$, the isotropy subgroup of $G$ at $0 \in \mathbb{R}^2$. Here is a classification of maximal connected Lie groups acting properly on $G/H \simeq \mathbb{R}^2$.

**PROPOSITION 2.2.1.** Up to conjugacy the maximal connected Lie subgroups of $G$ acting properly on $G/H$ are of the following forms:

$$L_1 = \left\{ \begin{pmatrix} e^b & 0 & a \\ 0 & 1 & b \end{pmatrix} : a, b \in \mathbb{R} \right\},$$

$$L_2 = \left\{ \begin{pmatrix} 1 & b & a \\ 0 & 1 & b \end{pmatrix} : a, b \in \mathbb{R} \right\},$$

$$L_3 = \left\{ \begin{pmatrix} \cos \theta & -\sin \theta & a \\ \sin \theta & \cos \theta & b \end{pmatrix} : a, b, \theta \in \mathbb{R} \right\}.$$

It can be checked directly that $L_i$ acts properly on $G/H$ ($i = 1, 2, 3$). Conversely, if a connected group $L$ acts properly on $G/H$, then $(L, G, H)$ has property (CI). We shall classify $L$ such that $(L, G, H)$ has property (CI) in the following way. First, $L$ is a compact extension of a solvable group from Proposition (2.1.1). In our case, a maximal compact sub group of $G$ is of one dimension, and thus $L$ itself is a solvable Lie group.
So we can take a sequence $0 = I^{(0)} \triangleleft I^{(1)} \triangleleft \cdots \triangleleft I^{(n)} = I$ such that $I^{(i)}$ is a codimension one ideal in $I^{(i+1)}$. (It is easy to see that $n \leq 3$.) Now checking property (CI) is reduced to the calculation of the normalizer $N_{\mathfrak{g}}(I^{(i)})$ and to the case of $\dim L = 1$ (Lemma (2.2.3)).

The rest of this section is devoted to complete the proof of Proposition (2.2.1) by this procedure.

**Lemma 2.2.2.** A complete representative of the adjoint orbit in $\mathfrak{g}$ is given by

\[
X(a, b) := \begin{pmatrix}
  a & 0 & 0 \\
  0 & b & 0
\end{pmatrix} \quad (a, b \in \mathbb{R}, a \leq b), \quad
W(a) := \begin{pmatrix}
  0 & 0 & 1 \\
  0 & a & 0
\end{pmatrix} \quad (a \in \mathbb{R}),
\]

\[
Y(a) := \begin{pmatrix}
  a & 1 & 0 \\
  0 & a & 0
\end{pmatrix} \quad (a \in \mathbb{R}), \quad
V := \begin{pmatrix}
  0 & 1 & 0 \\
  0 & 0 & 1
\end{pmatrix},
\]

\[
Z(a, b) := \begin{pmatrix}
  a & -b & 0 \\
  b & a & 0
\end{pmatrix} \quad (a, b \in \mathbb{R}, b > 0).
\]

**Lemma 2.2.3.** Up to conjugacy, the one dimensional connected Lie subgroups of $G$ which act properly on $G/H$ have one of the following Lie algebras: $\mathbb{R}Z(0,1), \mathbb{R}W(1), \mathbb{R}W(0), \mathbb{R}V$.

**Proof:** We notice that if $a \neq 0$ then there exists $g \in G$ such that $\text{Ad}(g)\mathbb{R}W(a) = \mathbb{R}W(1)$. So the necessity is shown by checking the property (CI). We have already seen the sufficiency before. \[\blacksquare\]

The proof of the following two lemmas is straightforward and so omitted.
**Lemma 2.2.4.** The normalizers of the Lie algebras in Lemma (2.2.3) are given by,

\[
\begin{align*}
N_g(\mathfrak{r}Z(0,1)) &= \mathfrak{r}Z(0,1) + \mathfrak{r}Z(1,0), \\
N_g(\mathfrak{r}W(0)) &= \{X \in M(3,2;\mathbb{R}) : X_{21} = 0\}, \\
N_g(\mathfrak{r}W(1)) &= \mathfrak{r}W(1) + \mathfrak{r}W(0), \\
N_g(\mathfrak{r}V) &= \mathfrak{r}V + \mathfrak{r}X(2,1) + \mathfrak{r}W(0).
\end{align*}
\]

Set \( W'(a) := \left( \begin{array}{ccc} a & 0 & 0 \\ 0 & 0 & 1 \end{array} \right) \) (\( a \in \mathbb{R} \)), \( V' := \left( \begin{array}{ccc} 0 & 0 & 1 \\ 0 & 0 & 0 \end{array} \right) \), which are conjugate to \( W(a) \), \( V \) respectively. Put \( P := N_G(\mathfrak{r}W(0)) \), \( Q := N_G(\mathfrak{r}V) \subset G \).

**Lemma 2.2.5.**

1. \( \text{Ad}(G)Z(a,b) \cap \mathfrak{p} = \emptyset \) if \( b \neq 0 \).
2. \( \text{Ad}(G)W(a) \cap \mathfrak{p} = \text{Ad}(P)W(a) \coprod \text{Ad}(P)W'(a) \) (\( a \in \mathbb{R} \)).
3. \( \text{Ad}(G)V \cap \mathfrak{p} = \text{Ad}(P)V \).
4. \( \text{Ad}(G)Z(a,b) \cap \mathfrak{q} = \emptyset \) if \( b \neq 0 \).
5. \( \text{Ad}(G)W(a) \cap \mathfrak{q} = \begin{cases} \\
\emptyset & \text{if } a \neq 0 \\
\mathbb{R}^XW(0) = \text{Ad}(Q)W(0) & \text{if } a = 0
\end{cases} \)
6. \( \text{Ad}(G)V \cap \mathfrak{q} = \bigsqcup_{c \in \mathbb{R}} \text{Ad}(Q)(V + cW(0)) \).

**Lemma 2.2.6.** Up to conjugacy the two dimensional connected Lie subgroups \( L \) of \( G \) which act properly on \( G/H \) are of the following Lie algebras:

\( \mathfrak{r}W'(0) + \mathfrak{r}W(0), \mathfrak{r}W'(1) + \mathfrak{r}W(0), \mathfrak{r}V + \mathfrak{r}W(0) \).

**Proof:** We have seen already that the corresponding Lie subgroups in Lemma (2.2.3) act properly on \( G/H \). Let us verify the necessity part by the property (CI). As \( I \) is a solvable Lie algebra, we can assume that one of the Lie algebras in Lemma (2.2.3) is an
ideal of $I$. First consider the case where $RW(0) \triangleleft I$. Then $I \subset N_{\mathfrak{g}}(RW(0)) = \mathfrak{p}$. Up to conjugacy by $P$, we may assume that a complimentary subspace of $RW(0)$ in $I$ is one of $RW(0), RW(1), RW'(0), RW'(1), RV$ from Lemma (2.2.5). Then $RW(0)$ is excluded because of linear dependency. $RW(1)$ is also excluded because $RW(0) + RW(1)$ contains a subspace $R(W(1) - W(0))$, whose corresponding connected Lie subgroup cannot act properly on $G/H$. The remaining is properly discontinuous cases. Similarly, we can treat the cases where $RZ(0, 1) \triangleleft I$, $RW(1) \triangleleft I$, $RV \triangleleft I$, yielding Lemma. □

The final step is done similarly by using the following lemma.

**Lemma 2.2.7.** The normalizers of the Lie algebras in Lemma (2.2.6) are given by,

\[
N_{\mathfrak{g}}(RW'(0) + RW(0)) = \mathfrak{g},
\]

\[
N_{\mathfrak{g}}(RW'(1) + RW(0)) = RX(1, 0) + RW'(1) + RW(0),
\]

\[
N_{\mathfrak{g}}(RV + RW(0)) = RX(2, 1) + RY(0) + RV + RW(0).
\]

3. Homogeneous spaces of solvable groups

First we recall a nice topological property of a subgroup of a solvable Lie group due to Chevalley.

**Fact 3.1, [Ch].** Let $G$ be a 1-connected (real) solvable group and $H$ be a connected subgroup of $G$. Then $H$ is closed and 1-connected.
Our main theorem in this section is,

**Theorem 3.2.** Let $G$ be a connected (real) solvable group and $H$ be a closed proper subgroup of $G$. If the fundamental group $\pi_1(G/H)$ is finite, then there exists a discontinuous group in $G/H$ which is isomorphic to $\mathbb{Z}$.

This result should be in sharp contrast to the case of homogeneous spaces of reductive type, which is a phenomenon first observed in [C-M] and is settled in general in [Ko].

**Fact 3.3, [C-M; Wo1; Wo2; Ku; Ko].** Let $G/H$ be a homogeneous space of reductive type. Then the followings are equivalent:

1. Any discontinuous group in $G/H$ is finite.
2. $\mathbb{R}$-rank $G = \mathbb{R}$-rank $H$.

A stupid observation is when $G$ is solvable and reductive, namely, $G$ is isomorphic to $\mathbb{R}^m \times T^n$. Suppose that the first Betti number of $H$ is $n'$. Then obviously,

$$|\pi_1(G/H)| < \infty \iff n = n' \iff G = H \text{ or } \text{R-rank } G > \text{R-rank } H.$$

This means a compatibility of Theorem (3.2) and Fact (3.3).

Thanks to Lemma (1.1.3)(2) with $G_1$ a universal covering group of $G_2 := G$ and with $H_1$ a connected subgroup of $G_1$ having the same Lie algebra of $\mathfrak{h}_2 := \mathfrak{h}$, Theorem (3.2) is reduced to the following Theorem (3.2)'.

**Theorem 3.2'.** Let $G$ be a 1-connected (real) solvable group and $H$ be a connected proper subgroup of $G$. Then there exists a discontinuous group in $G/H$ which is isomorphic to $\mathbb{Z}$. 
PROOF: We proceed by the induction on the dimension of $G$. Theorem (3.2)' is clear when $\dim G = 1$, namely, when $G \simeq \mathbb{R} \supset H \simeq \{0\}$. Suppose that $\dim G \geq 2$. Then there exists a connected normal subgroup $N$ of $G$ with $0 < \dim N < \dim G$. We will divide into two cases according as $HN \subsetneqq G$ or $HN = G$.

I) Assume that $HN \subsetneqq G$. A subgroup $HN$ is connected and so closed. So $\overline{H} := H/H \cap N = HN/N$ is a proper closed subgroup of $\overline{G} := G/N$. We write the canonical projection $\pi: G \to \overline{G} = G/N$. From the inductive assumption, we can find a discrete group $\Gamma$ of $\overline{G}$ such that $\Gamma$ is isomorphic to $\mathbb{Z}$ and acts on $\overline{G}/\overline{H}$ properly. Fix an element $\gamma \in G$ such that $\pi(\gamma)$ is a generator of $\overline{\Gamma}$. Put $\Gamma := \langle \gamma \rangle$. We have $\pi(\Gamma) = \overline{\Gamma}$, and therefore $\Gamma \simeq \mathbb{Z}$ and $\Gamma \cap N = \{e\}$. On the other hand, $\overline{\Gamma}$ is discrete and so does $\Gamma$. Applying Lemma (1.1.3)(1), we have now shown that $\Gamma$ acts on $G/H$ properly discontinuously.

II) Assume that $HN = G$. We have $G/H \simeq N/N \cap H$ and $N \cap H \subsetneqq N$. Since $\pi_1(N/N \cap H) = \pi_1(G/H) = \{e\}$, $N \cap H$ is connected. Thus $(N, N \cap H)$ satisfies the assumption of Theorem (3.2)' and $\dim N < \dim G$. Therefore we can find a discrete group $\Gamma \simeq \mathbb{Z}$ of $N$ which acts on $N/N \cap H$ from the inductive assumption. Clearly, $\Gamma$ is a subgroup of $G$ acting properly discontinuously on $G/H$. 

4. R-rank one semisimple group manifolds

Throughout this section, we assume that $G$ is a connected real reductive linear Lie group. See §1.3 for notations. We shall find some property of a discontinuous group in
a group manifold $G \times G / \text{diag } G$ when $\text{R-rank } G = 1$.

**Lemma 4.1.1.** If $\text{R-rank } G = 1$ and $x \in G$ is a semisimple and non-elliptic element, then $Z_G(x)$ is a direct product of a compact group and $\mathbb{R}$.

**Proof:** Choose a Cartan subgroup $J$ of $G$ containing $x$ and a Cartan involution $\theta$ such that $\theta J = J$. Put $L := Z_G(x)$, then we have $\theta 1 = 1$ (see [War] Proposition 1.4.3.2).

As $L$ is of maximal rank reductive Lie subalgebra of $\mathfrak{g}$, we have $N_G(1) \supset L \supset L_0$ have the same Lie algebra ([War] Proposition 1.4.2.4). Since $\theta N_G(1) = N_G(1)$, we have $\theta L = L$. So we can write the center of $L$ as $C = (C \cap K) \exp(\mathfrak{c} \cap \mathfrak{p})$. As $L$ is a reductive linear Lie group with finitely many connected components, if follows from $(x) \simeq \mathbb{Z} \subset C$ that $\dim \mathfrak{c} \cap \mathfrak{p} \geq 1$. Then $\mathfrak{c} \cap \mathfrak{p} = \mathfrak{p}$ because $1 = \text{R-rank } G \geq \text{R-rank } L = \text{R-rank}[L, L] + \dim \mathfrak{c} \cap \mathfrak{p}$. Hence $L = (L \cap K) \exp(\mathfrak{c} \cap \mathfrak{p})$. \[\Box\]

**Lemma 4.1.2.** If $\text{R-rank } G = 1$ and $\Gamma$ is an infinite discrete subgroup of $G$, then there exists a compact set $S$ of $G$ such that $S \Gamma S^{-1} = G$.

**Proof:** An infinite discrete subgroup $\Gamma$ in a linear Lie group contains necessarily an element of infinite order because $\Gamma$ has a torsion-free subgroup $\Gamma'$ such that $[\Gamma : \Gamma'] < \infty$ ([Se]). Thus it suffices to show Lemma (4.1.2) when $\Gamma$ is isomorphic to $\mathbb{Z}$. Let $\gamma$ be a generator of $\Gamma$ and $\gamma = \gamma_s \gamma_u$ be a Jordan decomposition (see [War] Proposition 1.4.3.3).

We divide into two cases according as $\langle \gamma_s \rangle$ is discrete in $G$ or not.

1) Assume that $\langle \gamma_s \rangle$ is discrete in $G$. It follows from Lemma (4.1.1) that $\gamma_u \in Z_G(\gamma_s)$ is the identity. Thus $\gamma = \gamma_s$ is contained in a maximally split Cartan subgroup $J$. Choose a Cartan involution $\theta$ which stables $J$ and we write $J = TA$ as usual. We can write $\gamma = t \exp(Y)$ where $t \in T, Y \in \mathfrak{a}$. Define a compact subset of $G$ by $S := \ldots$
$K \{\exp sY : 0 \leq s \leq 1\}$. Then $S(\gamma)S^{-1} \supset KAK = G$.

II) Assume that $\langle \gamma_s \rangle$ is not discrete in $G$. Then $\gamma_u \neq 1$ since $\langle \gamma \rangle = \{\gamma_s \gamma_u^n : n \in \mathbb{Z}\}$ is discrete in $G$. By the theorem of Jacobson-Morozov, there is a homomorphism $\psi : SL(2, \mathbb{R}) \rightarrow G$ such that $
abla (\begin{array}{ll}1 & 1 \\ 0 & 1 \end{array}) = \gamma_u$. Since $\langle \gamma \rangle = \{\gamma_s^{n} \gamma_u^{m} : n, m \in \mathbb{Z}\}$ is discrete in $G$, there is a Cartan involution $\theta$ of $G$ such that $\theta \psi(SL(2, \mathbb{R})) = \psi(SL(2, \mathbb{R}))$ (see [He], p.277). In particular, $A := \psi \left( \left\{ \begin{array}{ll}a & 0 \\ 0 & a^{-1} \end{array} : a > 0 \right\} \right)$ is a maximally split abelian subgroup of $G$. Define a compact subset of $G$ by $S := K \psi \left( \left\{ \begin{array}{ll}1 & x \\ 0 & 1 \end{array} : 0 \leq x \leq 1 \right\} \right)$

$L_0 \{\gamma_s \rangle$. Then

$$S(\gamma)S^{-1} \supset K \psi \left( \left\{ \begin{array}{ll}1 & x \\ 0 & 1 \end{array} : x \in \mathbb{R} \right\} \right) K \supset K \psi(SL(2, \mathbb{R})) K \supset KAK = G.$$

**Lemma 4.1.3.** Let $G$ be a connected reductive Lie group. Then the following conditions are equivalent.

1. **R-rank** $G \geq 2$
2. There exists infinite discrete subgroups $\Gamma_i$ of $G$ ($i = 1, 2$) such that $\Gamma := \Gamma_1 \times \Gamma_2$ acts properly discontinuously on a group manifold $G \times G/\text{diag} G$.

**Proof:** We may restrict ourselves to the case where $\text{R-rank} G \geq 1$.

Suppose that $\text{R-rank} G \geq 2$. We find abelian subspaces $a_1$, $a_2 \subset a$ such that $\dim a_i \geq 1$ and that $W(g, a) \cdot a_1 \cap a_2 = \{0\}$. Put $A_i := \exp a_i$, then $A_1$ acts properly on $G/A_2$.

Take any lattices $\Gamma_i$ in abelian Lie groups $A_i$ ($i = 1, 2$). Then the discrete group $\Gamma_1 \times \Gamma_2$ acts properly discontinuously on $G \times G/\text{diag} G$ (see Lemma (1.2.1)).

Suppose that $\text{R-rank} G = 1$. Let $\Gamma_i$ ($i = 1, 2$) be both infinite discrete subgroups of $G$. Then there exists a compact set $S$ of $G$ such that $ST_iS^{-1} = G$ by Lemma (4.1.2).
In particular, \((S \times S)(\Gamma_1 \times \Gamma_2)(S^{-1} \times S^{-1}) = G \times G\), which implies that any subgroup \(H\) of \(G \times G\) acting properly on \(G \times G/\Gamma_1 \times \Gamma_2\) must be compact. Thus, \(\Gamma_1 \times \Gamma_2\) cannot act properly discontinuously on \(G \times G/\text{diag}\ G\). 

**Theorem 4.1.4.** Let \(G\) be a connected noncompact reductive Lie group. Then the following conditions are equivalent.

(1) \(\text{R-rank } G = 1\)

(2) Any torsionless discontinuous group \(\Gamma\) in \(G \times G/\text{diag}\ G\) is of the following form up to switch of factor: \(\Gamma = \{(\gamma, \rho(\gamma)) : \gamma \in \Phi\}\) with a subgroup \(\Phi \subset G\) and with a homomorphism \(\rho : \Phi \to G\).

**Proof:** 2) \(\Rightarrow\) 1) If \(\text{R-rank } G \geq 2\), then there exists a discrete group \(\Gamma_i \simeq \mathbb{Z}^{n_i} (n_i \geq 1)\) of \(G\) such that \(\Gamma_1 \times \Gamma_2\) acts properly discontinuously on \(G \times G/\text{diag}\ G\) as we saw it in the previous lemma.

1) \(\Rightarrow\) 2) Suppose that \(\Gamma\) is a discontinuous group in \(G \times G/\text{diag}\ G\). Let \(p_j : G \times G \to G\) \((j = 1, 2)\) be natural projections to the \(j\)-th factor. Let \(\Gamma_j := \text{Ker}\ p_j \cap \Gamma\) for \(j = 1, 2\). Then \(\Gamma_1 \times \Gamma_2\) is regarded as a subgroup of \(\Gamma \subset G \times G\), and so is also a discontinuous group in \(G \times G/\text{diag}\ G\). It follows from the previous Lemma that at least one of \(\Gamma_j\) must be finite if \(\text{R-rank } G = 1\). We can assume \(\Gamma_1\) is a finite group after changing factor if necessary. As \(\Gamma\) is torsion-free, a finite subgroup \(\Gamma_1\) must be trivial, namely, \(p_1|\Gamma : \Gamma \to G\) is injective. Now \(\Gamma\) is of the desired form if we define \(\Phi := p_1(\Gamma)\) and \(\rho := p_2 \circ p_1|\Gamma^{-1}\).

**Remark.** R.Kulkarni and F.Raymond first proved \((1) \Rightarrow (2)\) when \(G = SL(2, \mathbb{R})\) (see Theorem 5.2 and Introduction in [Ku-R]). They also show that \(\Psi\) can be chosen to be discrete. The proof there depends on the fact that no discontinuous group in \(G \times\)
$G/\text{diag} G$ contains a subgroup $\simeq \mathbb{Z}^2$ if $G = SL(2, \mathbb{R})$. However, this is not always true even if $G$ is of $\mathbb{R}$-rank one. For example, there exists a discontinuous group $\simeq \mathbb{Z}^{n-1}$ in $G \times G/\text{diag} G$ if $G = SO(n, 1)$.

5. A necessary condition for the existence of a uniform lattice

5.1. theorem

A homogeneous space of reductive type $G/H$ does not always admit a uniform lattice. There are known two necessary conditions for the existence of a uniform lattice. One is the requirement that there should exist a discontinuous group $\simeq \mathbb{Z}$ in $G/H$ (see Fact(3.3)), and the other is a requirement from Euler characteristic ([Ko] Proposition (4.10), see also [Ku] Corollary 2.10, [Ko-O] Corollary 5 for partial results):

FACT 5.1.1. For the existence of a uniform lattice, $(G, H)$ must satisfy that

1) $\mathbb{R}$-rank $G > \mathbb{R}$-rank $H$ unless $G/H$ itself is compact.

2) If $\text{rank} G = \text{rank} H$ then $c$-rank $G = c$-rank $H$.

By a comparison with various reductive subgroups in $G$, we can exclude the possibility of the existence of uniform lattice in some of homogeneous spaces of reductive type. The following simple theorem is based on this idea.
**Theorem 5.1.2.** Let $G/H$ be a homogeneous space of reductive type. If there exists a closed subgroup $G'$ reductive in $G$ such that

\begin{align*}
(5.1.3)(a) & \quad a(G') \subseteq W(g, a) \cdot a(H) \\
(5.1.3)(b) & \quad d(G') > d(H)
\end{align*}

then $G/H$ does not admit a uniform lattice (see §1.3 for notations).

**Proof:** Suppose that there were $\Gamma \subset G$, a uniform lattice in $G/H$. Then $\Gamma$ is virtually torsionless and the cohomological dimension $\text{cd}_R(\Gamma) = d(G) - d(H)$ from Corollary 5.5 (1) in [Ko]. On the other hand, the condition (5.1.3)(a) implies that $\Gamma$ acts on $G/G'$ properly discontinuously. Using Corollary 5.5 in [Ko] again, we have $\text{cd}_R(\Gamma) \leq d(G') - d(G'')$. Thus $d(G') \leq d(H)$, which contradicts (5.1.3)(b).

**Remark 5.1.4.** One of the simplest applications is a comparison of $G/H$ with $G/G$ by taking $G' = G$, yielding Fact (5.1.1)(1). Indeed, if $R$-rank $G = R$-rank $H$, then Theorem (5.1.2) implies

\[ G/H \text{ has a uniform lattice} \iff d(G) = d(H) \iff G/H \text{ is compact.} \]

Here, the second equivalence is derived immediately from a fiber bundle structure

\[ G/H \simeq K/H \cap K \times_{H \cap K} p/\mathfrak{h} \cap p. \]

The proof of Theorem (5.1.2) is almost obvious as we saw above. Throughout the rest of this section we will clarify its typical applications and limitations.

**5.2. Example**
EXAMPLE 5.2.1. Let $G/H = U(i+j, k+l; F)/U(i, k; F) \times U(j, l; F)$, where $F = R, C, H$, and $i \leq j, k, l$. Here, we use a notation: $U(p, q; R) = SO(p, q)$, $U(p, q; C) = U(p, q)$, and $U(p, q; H) = Sp(p, q)$ (of rank $p + q$). If $G/H$ admits a uniform lattice, then $G/H$ is compact ($i = j = 0$ or $i = k = 0$), or $H$ is compact ($i = l = 0$), or $0 = i < l \leq j - k$.

PROOF: To see the condition $0 = i < l \leq j - k$, it suffices to apply Theorem (5.1.2) with $G' = G_i$, where $G_1 := U(i+t, k+l; F)$, $G_2 = U(i+j, i+t; F)$, and $t := \min(j, l)$.

REMARK 5.2.2. Assume moreover that $F = R$ in the above Example. Then it is also necessary that $jkl \equiv 0 \mod 2$ from Fact (5.1.1)(2). Conversely, it is known that if $i = l = 0$ or if $(i, j, k, l) = (0, 2n, 1, 1), (0, 4n, 1, 3), (0, 4, 2, 1)$, then there exists a uniform lattice in $G/H$.

5.3. semisimple orbit

Let us apply Theorem (5.1.2) to a semisimple orbit of the adjoint action. First we fix notations. Let $G$ be a connected real linear reductive Lie group and $X$ be an element of its Lie algebra $\mathfrak{g}$. $G \cdot X \simeq G/Z_G(X)$ is called an (adjoint) orbit in $\mathfrak{g}$, where $G \cdot X := \{Ad(g)X : g \in G\}$, $Z_G(X) := \{g \in G : Ad(g)X = X\}$.

COROLLARY 5.3.1. In the above setting, suppose that $X$ is a semisimple element. If $G \cdot X \simeq G/Z_G(X)$ admits a uniform lattice, then there is an elliptic element $X_1 \in \mathfrak{g}$ such that $Z_G(X) = Z_G(X_1)$. In particular, the orbit $G \cdot X$ carries a $G$-invariant complex structure.

REMARK 5.3.2. We should note that $G$ itself never carries a complex Lie group structure if $G \cdot X \simeq G/Z_G(X)$ admits a uniform lattice with a nonzero semisimple element.
$X \in \mathfrak{g}$. This follows from the fact that \( \text{R-rank} \, Z_G(X) = \text{R-rank} \, G \) if \( G \) is a complex reductive Lie group.

**Proof of Corollary (5.3.1):** There exists a Cartan involution \( \theta \) which stables \( Z_G(X) \). Since \( \text{rank} \, Z_G(X) = \text{rank} \, G \) ([War] Proposition 1.4.3.2) and since \( G/Z_G(X) \) admits a uniform lattice, we have \( c\text{-rank} \, Z_G(X) = c\text{-rank} \, G \) from Fact (5.1.1)(2). Therefore \( \mathfrak{z}_g(X) \) contains a fundamental Cartan subalgebra \( j^c \) of \( \mathfrak{g} \). We may and do assume \( j^c \) is \( \theta \)-stable and we write \( X = X_1 + X_2 \) corresponding to the direct sum decomposition \( j^c = t^c + a^c := j^c \cap t + j^c \cap p \). Then the first statement of Corollary follows from Theorem (5.1.2) with \( H = Z_G(X) \subset G' = Z_G(X_1) \) combined with the following Claim (5.3.3)

The last statement is directly from a (generalized) Borel embedding. 

**Claim 5.3.3.** With notation as above, we have

1) \( Z_G(X) = Z_G(X_1) \cap Z_G(X_2) \),

2) \( \text{R-rank} \, Z_G(X) = \text{R-rank} \, Z_G(X_1) \),

3) either \( Z_G(X) = Z_G(X_1) \) or \( d(Z_G(X)) < d(Z_G(X_1)) \).

**Proof of Claim (5.3.3):** The first claim is a direct consequence of the equation \( Z_G(X) = \theta(Z_G(X)) = Z_G(\theta X) \). Take a maximally split abelian subspace \( a_1 \) of \( Z_g(X_1) \) such that \( a_1 \) contains \( X_2 \). This is possible because \( X_2 \in p \cap Z_g(X_1) \). Then \( a_1 \subset Z_g(X_1) \cap Z_g(X_2) = Z_g(X) \). This means that \( a_1 \) is also a maximally split abelian subspace of \( Z_g(X) \), whence the second part.

Let us prove the third part. Suppose \( Z_G(X) \subsetneq Z_G(X_1) \). As the centralizer of an elliptic element is necessarily connected, it follows that \( Z_g(X) \subsetneq Z_g(X_1) \). Noting \( j^c \subset Z_g(X) \subsetneq Z_g(X_1) \), we find an \( \alpha \in \Delta(Z_g(X_1), j^c) \setminus \Delta(Z_g(X), j^c) \). If we write
\[ \alpha = \alpha_1 + \alpha_2 \] corresponding to the direct sum decomposition \( j^c* = t^c* + \mathfrak{a}^c* \), we have \( \alpha_1(X_1) = 0 \), \( \alpha(X) = \alpha_1(X_1) + \alpha_2(X_2) \neq 0 \). Fix a nonzero element \( Y \in \mathfrak{g}(j^c; \alpha) \) and set \( Z := Y - \theta Y \in \mathfrak{p} \). Since \( \alpha \neq \theta \alpha \), \( Y \) and \( \theta Y \) are linearly independent. Now we have \( \alpha_1(X_1) = 0 \), \( \alpha(X) = \alpha_1(X_1) + \alpha_2(X_2) \neq 0 \). Thus \( Z \in \mathfrak{p} \cap (Z_{\mathfrak{g}}(X_1) \setminus Z_{\mathfrak{g}}(X)) \). Hence we have shown \( d(Z_G(X)) < d(Z_G(X_1)) \).

Example 5.3.4. The following homogeneous space of reductive type is an elliptic orbit which admits a uniform lattice and which does not carry any invariant Riemannian metric: \( SU(2n, 2)/U(2n, 1) \), \( SO(2n, 2)/U(n, 1) \), and \( SO(4,3)/SO(4,1) \times SO(2) \).

5.4. Semisimple symmetric space

Let us recall the notion of \( \epsilon \)-family of semisimple symmetric spaces introduced by Oshima-Sekiguchi. We also review some necessary notions of semisimple symmetric pair for the benefit of the reader. Let \( \mathfrak{g} \) be a semisimple Lie algebra, \( \sigma \) be an involution of \( \mathfrak{g} \), \( \theta \) be a Cartan involution of \( \mathfrak{g} \) commuting with \( \sigma \). Let \( \mathfrak{g} = \mathfrak{k} + \mathfrak{p} = \mathfrak{h} + \mathfrak{q} \) be direct sum decomposition corresponding \( \theta, \sigma \) respectively. Put \( \mathfrak{h}^{a} := \{X \in \mathfrak{g} : \sigma \theta(X) = X\} = \mathfrak{h} \cap \mathfrak{k} + \mathfrak{q} \cap \mathfrak{p} \). Then \( (\mathfrak{g}, \mathfrak{h}^{a}) \) is called the associated symmetric pair of \( (\mathfrak{g}, \mathfrak{h}) \). Note that \( (\mathfrak{h}^{a})^{a} = \mathfrak{h} \). Take a maximal abelian subspace \( a_{\mathfrak{p},q} \) of \( \mathfrak{p} \cap \mathfrak{q} \). Then \( \Sigma(a_{\mathfrak{p},q}) \) satisfies the axiom of root system and is called the restricted root system of \( (\mathfrak{g}, \mathfrak{h}) \).

The signature of a restricted root is a map \( (m^{+}, m^{-}) : \Sigma(a_{\mathfrak{p},q}) \rightarrow \mathbb{N} \times \mathbb{N} \) defined by \( m^{+}(\lambda) := \dim \mathfrak{h}^{a} \cap \mathfrak{g}(a_{\mathfrak{p},q}; \lambda), m^{-}(\lambda) := \dim \mathfrak{g}(a_{\mathfrak{p},q}; \lambda) - m^{+}(\lambda) \). A map \( \epsilon : \Sigma(a_{\mathfrak{p},q}) \cup \{0\} \rightarrow \{1, -1\} \) is called a signature of \( \Sigma(a_{\mathfrak{p},q}) \) if \( \epsilon \) is a semigroup homomorphism with \( \epsilon(0) = 1 \) (see [O-S2] (1.9.3.1)). Note that any map \( \Psi \rightarrow \{1, -1\} \) is uniquely extended to a signature, where \( \Psi \) is a fundamental system for \( \Sigma(a_{\mathfrak{p},q}) \). To a signature \( \epsilon \) of \( \Sigma(a_{\mathfrak{p},q}) \), we associate an involution \( \sigma_{\epsilon} \) by \( \sigma_{\epsilon}(X) := \epsilon(\lambda)\sigma(X) \) if \( X \in \mathfrak{g}(a_{\mathfrak{p},q}; \lambda) \),
$\lambda \in \Sigma(a_{p,q}) \cup \{0\}$. Then $\sigma_\epsilon$ defines a symmetric pair $(g, h_\epsilon)$. The set $F((g, h)) := \{(g, h_\epsilon) : \epsilon \text{ is a signature of } \Sigma(a_{p,q})\}$ is called an $\epsilon$-family of symmetric pairs ([O-S2] §6). Among $\epsilon$-family, there is a distinguished symmetric pair called basic characterized by,

$$m^+(\lambda) \geq m^- (\lambda) \text{ for any } \lambda \in \Sigma(a_{p,q}) \text{ such that } \frac{\lambda}{2} \notin \Sigma(a_{p,q}).$$

It is known that there exists a basic symmetric pair of $F = F((g, h))$ unique up to isomorphisms ([O-S2] Proposition 6.5). If the basic symmetric pair of $F$ is a Riemannian symmetric pair, then $m^- \equiv 0$ and $F$ is $K_\epsilon$-family in the sense of [O-S1]. Typical examples of basic symmetric pairs are $(g, t)$ (Riemannian symmetric pair), $(g, g)$ (trivial case), $(g + g, \text{diag } g)$, $(g_C, g)$, $(u(p, q; F), u(m; F) + u(p - m, q; F))$ ($F = R, C, H$), whose associated symmetric pair are $(g, g)$, $(g, t)$, $(g_C, t_C)$, $(u(p, q; F), u(m, q; F) + u(p - m; F))$, respectively. Now we are ready to state our application of Theorem (5.1.2) to semisimple symmetric spaces:

**COROLLARY 5.4.1.** If an irreducible symmetric space $G/H$ admits a uniform lattice, then the associated symmetric pair $(g, h^a)$ is basic in $\epsilon$-family $F((g, h^a))$.

The proof of Corollary (5.4.1) is derived from Theorem (5.1.2) combined with (1),(2) of the following lemma.

**LEMMA 5.4.2.** With notations as above, let $(g, h)$ be basic in the $\epsilon$-family $F = F((g, h))$ and $(g, h_\epsilon)$ be not basic in $F$. Then we have

1) $a(H^a) \sim a(H_\epsilon^a)$ by an element of $W(g, a)$.

2) $d(H^a) > d(H_\epsilon^a)$.

3) $R$-rank$(H) = R$-rank$(G/H^a) \leq R$-rank$(G/H_\epsilon^a) = R$-rank$(H_\epsilon)$.
PROOF: (1) is clear because $a_{p,q}$ is a maximally split abelian subspace of $H_{\epsilon}^a$ as well as of $H^a$. The proof of (2) and (3) is based on the classification in [O-S2](see also [Be]). They are trivial if $H$ is compact, because $H^a = G$ in this case. If $H$ is noncompact, we can check them by using Table V; Table I and (1.14-16) in [O-S2]. (h) of $D_{l,A}^1$ in Table I there should read $\mathfrak{so}(l, C).$]

Here is a list of $G/H_{\epsilon}^a$ which does not admit a uniform lattice from Corollary (5.4.1). We omitted here the cases where $R$-rank $G = R$-rank $H_{\epsilon}^a$ (see Fact (5.1.1)(1)). In particular, $H^a$ is necessarily noncompact. We also omitted the cases treated in §5.2, namely, an indefinite Grassmann manifold $G/H = U(i+j, k+l; F)/U(i, k; F) \times U(j, l; F)$, $(F = R, C, H)$. (We can find the same necessary condition with that of Example (5.2.1) if we apply Corollary (5.4.1).)

<table>
<thead>
<tr>
<th>$g$</th>
<th>$h_{\epsilon}^a$</th>
<th>$h^a$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\mathfrak{sl}(2l, R)$</td>
<td>$\mathfrak{so}(l, l)$</td>
<td>$\mathfrak{sp}(l, R)$</td>
</tr>
<tr>
<td>$\mathfrak{su}^*(4l)$</td>
<td>$\mathfrak{so}^*(4l)$</td>
<td>$\mathfrak{sp}(l, l)$</td>
</tr>
<tr>
<td>$\mathfrak{su}(2l, 2l)$</td>
<td>$\mathfrak{so}^*(4l)$</td>
<td>$\mathfrak{sp}(l, l)$</td>
</tr>
<tr>
<td>$\mathfrak{sp}(2l, R)$</td>
<td>$\mathfrak{su}(l, l)$</td>
<td>$\mathfrak{sp}(l, C)$</td>
</tr>
<tr>
<td>$\mathfrak{so}(2l, 2l)$</td>
<td>$\mathfrak{so}(2l, C)$</td>
<td>$\mathfrak{u}(l, l)$</td>
</tr>
<tr>
<td>$\mathfrak{so}^*(4l + 4)$</td>
<td>$\mathfrak{so}^*(4l + 4)$</td>
<td>$\mathfrak{so}^<em>(2) + \mathfrak{so}^</em>(4l + 4)$</td>
</tr>
<tr>
<td>$\epsilon_6(6)$</td>
<td>$\mathfrak{sp}(4, R)$</td>
<td>$\mathfrak{f}_4(4)$</td>
</tr>
<tr>
<td>$\epsilon_7(2)$</td>
<td>$\mathfrak{su}(4, 2) + \mathfrak{su}(2)$</td>
<td>$\mathfrak{so}^*(10) + \sqrt{-1}R$</td>
</tr>
<tr>
<td>$\epsilon_7(7)$</td>
<td>$\mathfrak{su}(4, 4)$</td>
<td>$\epsilon_6(2) + \sqrt{-1}R$</td>
</tr>
<tr>
<td>$\epsilon_7(25)$</td>
<td>$\mathfrak{su}(6, 2)$</td>
<td>$\epsilon_6(-14) + \sqrt{-1}R$</td>
</tr>
<tr>
<td>$\epsilon_8(8)$</td>
<td>$\mathfrak{so}^*(16)$</td>
<td>$\epsilon_7(-9) + \mathfrak{su}(2)$</td>
</tr>
<tr>
<td>$\mathfrak{so}(2l + 2, C)$</td>
<td>$\mathfrak{so}(2l + 2, C) + \mathfrak{so}(2l - 2p + 1, C)$</td>
<td>$\mathfrak{so}(2l + 1, C)$</td>
</tr>
<tr>
<td>$\epsilon_6, C$</td>
<td>$\mathfrak{sp}(4, C)$</td>
<td>$\mathfrak{f}_4, C$</td>
</tr>
</tbody>
</table>

Table 5.4.3.

There is still a room for applications of Theorem (5.1.2). Here is a list of some other typical examples of $G/H$ which does not admit a uniform lattice. For most of
parameters below, $G'$ stands for a reductive group satisfying the conditions in Theorem (5.1.2). We have no intention to make a complete list in Table 5.4.4.

<table>
<thead>
<tr>
<th>$g$</th>
<th>$h$</th>
<th>$p + q \leq n$, $pq &gt; 0$</th>
<th>$g'(p + q)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$gl(n, R)$</td>
<td>$sp(m, R)$</td>
<td></td>
<td>$so(m, n - m)$</td>
</tr>
<tr>
<td>$sl(n, C)$</td>
<td>$sp(m, C)$</td>
<td>$0 &lt; 2m \leq n - 2$</td>
<td>$u(m, n - m)$</td>
</tr>
<tr>
<td>$sl(n, C)$</td>
<td>$so(m, C)$</td>
<td>$0 &lt; 2m \leq n - 1$</td>
<td>$u(m/2, n - m/2)$</td>
</tr>
<tr>
<td>$sl(2l, C)$</td>
<td>$u(l, l)$</td>
<td></td>
<td>$sp(l, C)$</td>
</tr>
<tr>
<td>$so^*(2l, C)$</td>
<td>$u(l, n - l)$</td>
<td>$3l \leq 2n \leq 6l$, $n \geq 3$</td>
<td>$so^*(4l + 2)$</td>
</tr>
<tr>
<td>$sp(n, R)$</td>
<td>$sp(l, C)$</td>
<td>$0 &lt; 2l \leq n$</td>
<td>$u(l, n - l)$</td>
</tr>
<tr>
<td>$su^*(2n)$</td>
<td>$so^*(2l)$</td>
<td>$1 &lt; l \leq n$</td>
<td>$sp(l/2, n - l/2)$</td>
</tr>
</tbody>
</table>

Table 5.4.4.

We give here some remarks on Table (5.4.4).

1) In the first line, $g(n)$ stands for one of the following classical Lie algebras: $gl(n, R)$, $gl(n, C)$, $so^*(2n)$, $so(n, C)$, $sp(n, R)$, $sp(n, C)$. We also remark that $g' = g(p + q)$ should be modified by $g(p + q - 1)$ if $g(n) = so^*(2n)$ or $so(n, C)$ and if both $p$ and $q$ are odd integers.

2) As for $(g, h) = (so^*(2l, C), u(l, n - l))$, the choice $g' = so^*(4l + 2)$ is valid for $2l < n$.

If $3l \leq 2n < 4l$ or $n = 2l$, then we can choose $g' = so^*(4n - 4l + 2)$, $g' = so^*(2n)$, respectively.

The condition $3l \leq 2n \leq 6l$ looks strange. It is interesting to note that if $(n, l) = (4, 1)$ or $(4, 3)$, $G/H = SO^*(2n)/U(l, n - l)$ admits a uniform lattice.

3) As for $(g, h) = (sp(n, R), sp(l, C))$, we have to use Example (4.11) in [Ko] if $n = 2l$ instead of Theorem (5.1.2).

REMARK 5.4.5. It is likely that a complex irreducible semisimple symmetric space
$G_C/H_C$ admits a uniform lattice if and only if $G_C/H_C$ is locally isomorphic to a group manifold. From Fact (3.3) and Tables (5.4.3) and (5.4.4), we are left with $(g_C, \mathfrak{h}_C) = (\mathfrak{gl}(2n, \mathbb{C}), \mathfrak{sp}(n, \mathbb{C})), (\mathfrak{so}(2n + 1, \mathbb{C}), \mathfrak{so}(2n, \mathbb{C})), (\mathfrak{e}_6, \mathfrak{f}_4).$

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