Applications of Lyusternik–Schnirelmann theory to Hamiltonian systems

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§1 Introduction

Let \( x = (x^1, x^2, \ldots, x^n) \) and \( p = (p_1, p_2, \ldots, p_n) \) be points of \( \mathbb{R}^n \) and consider a Hamiltonian system

\[
\begin{align*}
\dot{x}^i &= \frac{\partial H}{\partial p_i}, \\
\dot{p}_i &= -\frac{\partial H}{\partial x^i}; \quad i = 1, 2, \ldots, n,
\end{align*}
\]

where \( H = H(x, p) : \mathbb{R}^n \to \mathbb{R} \) is a \( C^\infty \) function (Hamiltonian function) and \( \dot{\cdot} \) means \( \frac{d}{dt} \).

Along a solution \( (x(t), p(t)) \) of (1.1), \( H(x(t), p(t)) \) is a constant, so, for given \( e \), the energy surface \( H^{-1}(e) \equiv \{(x, p); H(x, p) = e\} \) is an invariant set. If \( H^{-1}(e) \) is not compact, then there is not necessarily a periodic solution on it.

On the existence of periodic solutions of Hamiltonian systems on energy surface, P. Rabinowitz [6] obtained a remarkable

**Theorem 1.** If \( H^{-1}(e) \) is star shaped, then there exist at least one periodic solution of (1.1) on it.

For this theorem, the Hamiltonian function \( H(x, p) \) is an arbitrary function. But originally in classical mechanics, the Hamiltonian function had a special form, namely "kinetic energy + potential". This means \( H(x, p) \) is of the form

\[
H(x, p) = \frac{1}{2}a^{ij}(x)p_ip_j + U(x),
\]

where \( (a^{ij}) \) is symmetric and positive definite. We call the Hamiltonian system (1.1) with Hamiltonian function of the form (1.2) a classical Hamiltonian system. Then we have [1] [2]

**Theorem 2.** For classical Hamiltonian systems, if \( H^{-1}(e) \) is compact, then there exists at least one periodic solution on it.

In order to obtain more than one periodic solutions on compact energy surfaces of classical Hamiltonian systems, we have an eye to the following point. We put \( T = \frac{1}{2}a^{ij}(x)p_ip_j \), then we have \( T \geq 0 \). Hence, if a point \( (x, p) \) satisfies \( T + U = e \), then \( U(x) \leq e \). Thus we
consider, for a fixed $e$, the set

$$W \equiv \{x; U(x) \leq e\}.$$  

Remark that "$H^{-1}(e)$ is compact" if and only if "$W$ is compact".

From now on, we assume that $e$ is a regular value of $H$ (equivalently of $U$). Then $W$ is a compact manifold with boundary $[U = e]$. In this note, we propose a conjecture "there may be at least $\nu(W)$ periodic solutions on the energy surface of the classical Hamiltonian system", and give some circumstantial evidence of this conjecture. The number $\nu(W)$ is a topological invariant of $W$ given below.

§2 Geodesics as solutions of (1.1)

For a classical Hamiltonian (1.2), the Hamiltonian system (1.1) is equivalent to the Lagrangian system

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{x}^i} = \frac{\partial L}{\partial x^i}, \quad i = 1, 2, \ldots, n,$$

where $L = T - U$ is the Lagrangian with

$$T = T(x, \dot{x}) \equiv \frac{1}{2} a_{ij}(x) \dot{x}^i \dot{x}^j, \quad (a_{ij}) = (a^{ij})^{-1}.$$  

If $(x(t), p(t))$ is a solution of (1.1), (1.2) on $H^{-1}(e)$, then $x(t)$ is a solution of (2.1) with $T + U = e$. Conversely, if $x(t)$ is a solution of (2.1), then $T(x, \dot{x}) + U(x)$ is a constant $e$ and $(x(t), p(t))$ is a solution of (1.1), (1.2) on $H^{-1}(e)$, where $p(t)$ is properly determined by $x(t)$. Also, it is known [Maupertuis–Jacobi’s variational principle] that the above $x(t)$ is, after a time change, a geodesic for a Riemannian metric

$$ds^2 = (e - U(x)) \frac{1}{2} a_{ij}(x) dx^i dx^j.$$  

This metric is called Jacobi metric for $e$. This Jacobi metric is positive on Int $W = [U < e]$ and degenerate on $\partial W = [U = e]$. Maupertuis–Jacobi’s principle gives

**Lemma 1** If $\gamma: [0, 1] \rightarrow W$ is a $C^\infty$ curve with

- $\gamma(s)$ is a geodesic for the Jacobi metric in Int $W$,
- $\gamma(0), \gamma(1) \in \partial W$,
then \((x(t), p(t))\), where \(x(t)\) is obtained by \(\gamma(s)\) after proper time change \(t \rightarrow s\) and \(p(t)\) is determined from \(x(t)\) as above, is a periodic solution of (1.1) on \(H^{-1}(e)\).

In fact, let \(x(t)\) be the solution of (2.1)

- \(x(t)\) in \(\text{Int} W\), \(t_0 < t < t_1\),
- \(x(t_0), x(t_1) \in \partial W\),

for some \(t_0 < t_1\). Then the solution \(x(t)\) stops at the times \(t = t_0\) and \(t_1\), because on the boundary \([U = e]\), we have \(T = e - U = 0\) at the times, hence \(\dot{x} = 0\). By the reversibility of the system (2.1), the inverse curve \(x(t_1 - t)\) is also a solution of (2.1) with same total energy \(T + U\). This stops again at \(t = t_1 + (t_1 - t_0)\). Connecting these solutions

- \(x(t), \ t_0 \leq t \leq t_1\),
- \(x(t_1 - t), \ t_1 \leq t \leq t_1 + (t_1 - t_0)\),
- \(x(t), \ t_1 + (t_1 - t_0) \leq t \leq t_1 + 2(t_1 - t_0)\),
- \(\cdots\),

we have a desired periodic solution.

As pointed out above, the Jacobi metric is degenerate on \(\partial W = [U = e]\). To avoid this, we consider a compact manifold \(W_\delta\), which is contained in \(\text{Int} W\) and diffeomorphic to \(W\), as follows.

Fix a small \(\delta > 0\). For \(b \in B = \partial W\), let \(x_b(t)\) be the solution of (2.1) with \(x_b(0) = b\), \(\dot{x}(0) = 0\), and \(t(b, \delta)\) the first time for which the length of the curve \(x_b(t), 0 \leq t \leq t(b, \delta)\), with respect to the Jacobi metric equals to \(\delta\). We put

\[ b_\delta = x_b(t(b, \delta)) \quad \text{and} \quad B_\delta = \bigcup_{b \in B} b_\delta. \]

Finally let \(W_\delta\) be the compact set consisting of the points "inside" \(B_\delta\). For sufficiently small \(\delta\), \(W_\delta \approx W\) and it is known that if a geodesic with respect to Jacobi metric intersect with \(B_\delta\) orthogonally, then the geodesic can be extended so as to reach the boundary \(B\). We call a geodesic of a compact manifold with boundary an orthogonal geodesic chord, if it starts and ends at points of the boundary orthogonally. The above consideration and Lemma 1 give
Lemma 2  Orthogonal geodesic chords of $W_{\delta}$ with respect to the Jacobi metric give periodic solutions of the original Hamiltonian system (1.1) with (1.2) on $H^{-1}(e)$.

§3  Lyusternik–Schnirelmann theory for orthogonal geodesic chords

For the existence and the number of orthogonal geodesic chords of compact Riemannian manifolds with boundary, the following is known.

Theorem 3  Let $Y$ be a compact Riemannian manifold with geodesically convex boundary. Then we have at least $\nu(Y)$ orthogonal geodesic chords.

The topological invariant $\nu(Y)$ is defined as follows. We put $B = \partial Y \neq \emptyset$ and

\begin{equation}
\Omega_{Y} \equiv \{ \omega : [0,1] \to Y; \text{continuous and } \omega(0), \omega(1) \in B \}
\end{equation}

with compact open topology. In the following, the coefficients of the (co)homology shall be understood as $Z_2 = Z/2Z$. We define

1. $\nu_{\pi}(Y) = \begin{cases} 1 & \text{if } \pi_k(\Omega_Y, B) \neq 0 \text{ for some } k \geq 1, \\ 0 & \text{otherwise} \end{cases}$

2. if $H_*(\Omega_Y, B) = 0$, then $\nu_H(Y) = 0$ and otherwise

$\nu_H(Y) = \text{Max} \{ k \geq 1 ; \exists \alpha_1, \alpha_2, \ldots, \alpha_{k-1} \in H^*(\Omega_Y) \text{ with } \deg \alpha_j > 0 \\
\text{and } \exists a \in H_*(\Omega_Y, B) \text{ such that } (\alpha_1 \cup \cdots \cup \alpha_{k-1}) \cap a \neq 0 \}$

3. $\nu_{\Pi}(Y)$ is obtained as $\nu_H(Y)$, exchanging $H^*(\Omega_Y)$ and $H_*(\Omega_Y, B)$ to $H^\Pi_*(\Omega_Y)$ and $H^\Pi_*(\Omega_Y, B)$. Here, $H^\Pi_*$ and $H_*^\Pi$ are equivariant (co)homology with respect to the involution $\omega \mapsto \omega^{-1} \equiv \omega(1-\cdot)$.

4. $\nu(Y) \equiv \text{Max}\{\nu_{\pi}(Y), \nu_H(Y), \nu_{\Pi}(Y)\}$.

The proof is given by Lyusternik–Schnirelmann theory applied to the following variational problem. Let $\Lambda$ be the path space consisting of all piecewise $C^\infty$ pathes $\lambda : [0,1] \to Y$ with $\lambda(0), \lambda(1) \in B$. Also define $E : \Lambda \to \mathbb{R}$ by

\begin{equation}
E(\lambda) = \frac{1}{2} \int_0^1 dt|\dot{\lambda}(t)|^2.
\end{equation}

Nontrivial ($E > 0$) "critical points" of $E$ correspond to nonconstant orthogonal geodesic chords. The assumption of the geodesical convexity corresponds to the condition (C) of
Palais-Smale. For example, let $a \in H_k(\Lambda, B)$ be a nonzero element (remark that $\Lambda$ is homotopically equivalent to $\Omega_Y$). For a representative of $a$

$$z = \sum_i \sigma_i, \quad \sigma_i : \Delta^k \rightarrow \Lambda,$$

we put

$$|z| = \bigcup_i \text{Im} \sigma_i \subset \Lambda$$

and define

$$\kappa_a \equiv \inf_{z \in a} \text{Max} E(|z|).$$

Then $\kappa_a$ is a nontrivial critical value.

If there is an $\alpha \in H^*(\Lambda)$ with $\deg \alpha > 0$ satisfying $b \equiv \alpha \cap a \neq 0$, then, in general, $\kappa_b \leq \kappa_a$ and when $\kappa_b = \kappa_a$, there exist infinitely many critical points on the level. This means, in that case, there exist at least two critical points, giving the meaning of the definition of $\nu_H(Y)$.

The topological invariant $\nu(Y)$ has the following properties.

1. for any $Y$, we have $\nu(Y) \geq 1$.

2. if $Y$ is contractible, then $\nu(Y) = \dim Y$, in particular $\nu(D^n) = n$.

3. for solid torus $S^1 \times D^2$, we have $\nu(S^1 \times D^2) \geq 3$.

Corresponding to these properties, we have the following results on the classical Hamiltonian systems.

1. there is always at least one periodic orbit [1] [2].

2. when $W \approx D^n$, there exist at least $n$ periodic solutions for systems near a rotationally symmetric one [3].

3. when $W \approx S^1 \times D^2$, there exist at least 3 periodic solutions for systems near one with some symmetry [5].

Thus it is plausible that the following may be valid: on a compact energy surface of a classical Hamiltonian system, there may be at least $\nu(W)$ periodic solutions on it.
References


