<table>
<thead>
<tr>
<th>Title</th>
<th>A modulus of uniform continuity with some order in $L^s_{loc}(\Omega; \mathbb{R}^N)$ ($2 \leq s \leq \infty$) and a sharp estimate of Lebesgue points of the first-derivatives of minimizers of a Quasi-convex functional in the calculus of variations</th>
</tr>
</thead>
<tbody>
<tr>
<td>Author(s)</td>
<td>HORIHATA, KAZUHIRO</td>
</tr>
<tr>
<td>Citation</td>
<td>数理解析研究所講究録 (1991), 738: 1-9</td>
</tr>
<tr>
<td>Issue Date</td>
<td>1991-01</td>
</tr>
<tr>
<td>URL</td>
<td><a href="http://hdl.handle.net/2433/102063">http://hdl.handle.net/2433/102063</a></td>
</tr>
<tr>
<td>Right</td>
<td>Type</td>
</tr>
<tr>
<td>Textversion</td>
<td>Departmental Bulletin Paper</td>
</tr>
<tr>
<td></td>
<td>publisher</td>
</tr>
<tr>
<td></td>
<td>Kyoto University</td>
</tr>
</tbody>
</table>
A modulus of uniform continuity with some order in $L^s_{loc}(\Omega; \mathbb{R}^N)$ ($2 \leq s < \infty$) and a sharp estimate of Lebesgue points of the first-derivatives of minimizers of a Quasi-convex functional in the calculus of variations.

KAZUHIRO HORIHATA
Department of Mathematics, Faculty of Science and Technology, Keio University

Abstract. This paper establishes that minimizers of strictly quasi-convex variational functionals, satisfy a modulus of uniform continuity with some order in the norm of $L^s_{loc}(\Omega; \mathbb{R}^N)$ with $2 \leq s \leq \infty$. This modulus of uniform continuity combined with a result in the present author's paper and one of Evans's results implies a local H"older continuity and a sharp estimate for the Hausdorff dimension of Lebesgue points of the first derivatives of minimizers.

1. INTRODUCTION

In this paper we establish that minimizers for certain functionals in the calculus of variations satisfy a modulus of uniform continuity of some order in the norm of $L^s_{loc}(\Omega; \mathbb{R}^N)$ with $2 \leq s < +\infty$. This functional is given as follows: Let $n, N$ be positive integers. We denote by $M^{n \times N}$ the space of all real $n \times N$ matrices and suppose that $\Omega \subset \mathbb{R}^n$ is a bounded with smooth boundary. Then for $v : \Omega \rightarrow \mathbb{R}^N$, we consider the functional

$$I[v] \equiv \int_{\Omega} F(\nabla v) dx,$$

where $v = (v^i), \nabla v = (\partial v^i / \partial x_\alpha)$ ($\alpha = 1, \ldots, n, i = 1, \ldots, N$) is the gradient matrix of $v$ and $F : M^{n \times N} \rightarrow \mathbb{R}$ is any given mapping, which is strictly defined later. Here we introduce another notation which will be used in this paper: $L^s(\Omega; \mathbb{R}^N)$ is $s$th-power integrable function space. We also denote by $L^s_{loc}(\Omega; \mathbb{R}^N)$ locally $s$th-power integrable function space, $H^{1,s}(\Omega; \mathbb{R}^N)$ and $\tilde{H}^{1,s}(\Omega; \mathbb{R}^N)$ are the usual Sobolev spaces. Also $|A|$ and $H^\gamma(A)$ means the Lebesgue measure and the $\gamma$-dimensional Hausdorff measure of measurable set $A$ in $\mathbb{R}^n$, respectively, (see Giaquinta [Gm1] and Giusti [Gj] for detailed definition).

We introduce a forward translation operator and also a forward difference operator of a map in $f \in L^s(\Omega; \mathbb{R}^N)$: Let $h$ be any small number and $e$ be a unit vector in $\mathbb{R}^n$. We define a forward translate operator $+:$

$$f^+(x) \equiv f(x + he)$$

and define a forward difference operator $\tau_h$ by

$$\tau_h f = f^+ - f.$$
We adopt the summation convention: For $\forall A, P, Q \in M^{n \times N}$, we define

$$DF(A) = \left( \frac{\partial F}{\partial p_{\alpha}^{i}}(A) \right),$$

$$D^2F(A) = \left( \frac{\partial^2 F}{\partial p_{\alpha}^{i} \partial p_{\beta}^{j}}(A) \right),$$

$(\alpha, \beta = 1, \cdots, n, i, j = 1, \cdots, N),$

$$DF(A) \cdot P = \sum_{\alpha=1}^{n} \sum_{i=1}^{N} \frac{\partial F}{\partial p_{\alpha}^{i}}(A) P_{\alpha}^{i},$$

and

$$D^2F(A) < P, Q >= \sum_{\alpha, \beta=1}^{n} \sum_{i,j=1}^{N} \frac{\partial^2 F}{\partial p_{\alpha}^{i} \partial p_{\beta}^{j}}(A) P_{\alpha}^{i} Q_{\beta}^{j}.$$ 

Let $F(x, z, p): \Omega \times R^{N} \times M^{n \times N} \mapsto \succ R$ be a function satisfying

(H1) $F(x, z, p) \leq K[1 + p^{s}]$

(H2) $F(x, z, p) \geq m$

(H3) $|F(x, z, p_{1}) - F(x, z, p_{2})| \leq K[1 + |p_{1}|^{s-1} + |p_{2}|^{s-2}]|p_{1} - p_{2}|$

(H4) $|F(x_{1}, z_{2}, p) - F(x_{1}, z_{2}, p)| \leq K[1 + |p|^{s}][|x_{1} - x_{2}| + |z_{1} - z_{2}|]

for $\exists m K > 0$ and $s(1 \leq s < \infty)$. The first question in the calculus of variations can be considered as the existence problem of minimizers in some function space. Under the above condition, Morrey [Mo] has isolated that a necessary and sufficient condition of certain functional $F(x, z, p)$ for the lower semicontinuity of $I[\cdot]$ on some Sobolev space is quasi-convex:

$$\int_{\Omega} F(x_{0}, z_{0}, p_{0}) dy \leq \int_{\Omega} F(x_{0}, z_{0}, p_{0} + \nabla \phi) dy \quad \text{for} \quad \forall (x_{0}, z_{0}, p_{0}) \in \Omega \times R^{N} \times M^{n \times N},$$

for an arbitrary smooth, bounded, open set $O \subset R^{n}$, $\forall A \in M^{n \times N}$ and $\forall \phi \in C_{0}^{1}(O ; R^{N})$.

Recently Acerbi and Fusco [AF] has refined Morrey's theorem, who have obtained the following for $F(p)$:

**Theorem 0 ([AF]).** Assume that $F : M^{n \times N} \mapsto R$ is continuous and for some positive numbers $C$ and $s$ the following

$$0 \leq F(p) \leq C(1 + |p|^{s})$$

holds for $\forall p \in M^{n \times N}$. Then $I[\cdot]$ is weakly sequentially lower semicontinuous on the Sobolev space $H^{1,m}(\Omega ; R^{N})$ if and only if $F$ is quasi-convex.

Also the second question can be considered as the regularity problem of such minimizers. However one often encounters that a minimizer is not necessarily regular everywhere in $\Omega$, even when $F$ is uniform convex (see [Gm1], [Gm2], [Gm3], [GG2] and [GI]). For the study of partial regularity, Evans [Ev] (see also [EG] and [GM]) has showed that minimizers has Hölder continuous first derivatives on some open subset $\Omega_{0} \subset \Omega$ satisfying $|\Omega/\Omega_{0}| = 0$, when $F \in C^{2}(M^{n \times N}; R^{N})$ and $D^2F(p)$ is uniform continuous in $M^{n \times N}$ and strictly quasi-convex: For $\exists \gamma > 0$ and $\exists s (2 \leq s < \infty)$ $F$ satisfies

$$\gamma \int_{\Omega} (1 + |\nabla \phi|^{s-2})|\nabla \phi|^{2} dy \leq \int_{\Omega} [F(A + \nabla \phi) - F(A)] dy$$

2
for $\forall A \in M^{n \times N}$ and $\forall \phi \in \mathcal{C}^1(\Omega; R^N)$.

and suppose that

\[(H5) \quad |D^2F(p)| \leq C_0(1 + |p|^{s-2})\]

for some constant $C_0$ and $\forall p \in M^{n \times N}$.

We remark that assumption $(H5)$ implies that there exist positive constants $C_1$ and $C_2$ such that

\[(H6) \quad |F(p)| \leq C_1(1 + |p|^s)\]

\[(H7) \quad |DF(p)| \leq C_2(1 + |p|^{s-1})\]

for all $p \in M^{n \times N}$. Under the above condition, Evans has proved

**Theorem 1** ([Ev]). Assume that $2 \leq s < +\infty$, the function $F$ satisfies (1.5) and $(H5)$. Let $u \in H^{1,s}(\Omega; R^N)$ be a minimizer of $I[\cdot]$. Then there exists an open subset $\Omega_0$ of $\Omega$ such that

\[(1.5) \quad |(\Omega/\Omega_0)| = 0\]

and the first derivatives of a minimizer $u$ are locally Hölder continuous on $\Omega_0$:

\[
\nabla u \in C^\alpha(\Omega_0; M^{n \times N})
\]

for each $0 < \alpha < 1$.

This proof is performed by combining a blow-up argument with the following Caccioppoli inequality:

**Theorem 2** ([Ev]). There exists a constant $C_3$ independent of $r$ such that a minimizer $u$ satisfies

\[
(1.6) \quad \int_{B_r(x)} (1 + |\nabla u|^{s-2})|\nabla u|^2 dx \leq C_3 \left(\frac{1}{r^2} \int_{B_r(x)} |u - a|^2 dx + \frac{1}{r^s} \int_{B_r(x)} |u - a|^s dx\right)
\]

for $\forall B_r(x) \subset \subset \Omega$ and $\forall a \in R^N$.

From Theorem 2 and a Gehring inequality [Gm], it follows that

**Theorem 3.** When $\nabla u$ satisfies the inequality (1.7) of Theorem 2, there exist positive numbers $t (t > s)$ depending only on $C_3, s, \Omega$ and $C_4$ depending only on $C_3, s, \Omega$ and $\tilde{\Omega}$ such that $\nabla u \in L_{loc}^t(\Omega; R^N)$ and moreover the following holds:

\[
(1.7) \quad \left[\frac{1}{|\tilde{\Omega}|} \int_{\tilde{\Omega}} (1 + |\nabla u|)^t dx\right]^{1/t} \leq C_4 \left[\frac{1}{|\Omega|} \int_{\Omega} (1 + |\nabla u|)^s dx\right]^{1/s}
\]

for $\forall \tilde{\Omega} \subset \subset \Omega$.

2. Main result

Now we can state the main theorem
THEOREM 4 (Main Theorem). Assume that $2 \leq s < +\infty$, the function $F$ satisfies (1.4) and (H5). Let $u$ be minimizer of $I[\cdot]$ in $H^{1,s}(\Omega; R^{N})$. Then for an arbitrary open set $\tilde{\Omega}$ compactly contained in $\Omega$, the following holds:

\[(2.1) \quad \int_{\tilde{\Omega}} |\tau_{h} \nabla u|^{2} dx \leq C_5 \cdot h \quad \text{for} \quad 0 < h < \frac{1}{8} \text{dist}(\tilde{\Omega}, \partial \Omega),\]

where $C_5$ is a constant depending only on $n$, $N$, $\gamma$, $C_0$, $\|\nabla u\|_{L^{s}}$, $\tilde{\Omega}$ and $\Omega$.

Here we notice that as in the same way of author’s previous result, one finds

THEOREM 5([H0]). Let $f$ be a function belonging to $L_{loc}^{p}(\Omega; R^{N})(1 \leq p < \infty)$ with the following condition: Let $\tilde{\Omega}$ be an arbitrary open set compactly contained in $\Omega$ and suppose that there exist positive numbers $C_6$ and $\alpha (0 < \alpha < n/p)$ independent of $h$ such that $f$ satisfies

\[(2.2) \quad \int_{\tilde{\Omega}} |\tau_{h} f|^{p} dx \leq C_6 \cdot h^{p\alpha}\]

for any number $h$ with $0 < h < \frac{1}{4} \text{dist}(\tilde{\Omega}, \partial \Omega)$. Then for the singular set $S_f$ of the map $f$ defined by

\[(2.3) \quad S_f = \{x \in \Omega : \lim_{\rho \rightarrow 0} f(x, \rho) = +\infty \} \cup \{x \in \Omega : \lim_{\rho \rightarrow 0} \int_{B_{\rho}(x)} |f(x, \rho)|^{p} dy > 0 \}\]

where $f(x, \rho) = 1/|B_{\rho}| \int_{B_{\rho}(x)} f(y) dy$, the following holds:

\[(2.4) \quad H^{(\beta)}(S_f) = 0\]

for any positive number $\beta$ with $n - p\alpha < \beta$.

From Theorem 4 and Theorem 5, we obtain

THEOREM 6. A singular set $S_{\nabla u}$ of the first derivatives of such minimizers, have at most

\[(2.5) \quad H^{n-1+\epsilon}(S_{\nabla u}) = 0 \quad \text{for} \quad \forall \epsilon > 0.\]

In addition, noting [Ev] and [EG], one finds that (2.5) shows the first derivatives of minimizers satisfy local Hölder continuity on $\Omega/S$:

\[\nabla u \in C^{\alpha}(\Omega/S; M^{n \times N}) \quad \text{for} \quad 0 < \alpha < 1.\]

3. PROOF OF THEOREM 4

Since $u$ is a minimizer of $I[\cdot]$ in $H^{1,s}(\Omega; R^{N})$, $u$ satisfies the following first-variational formula:

\[(3.1) \quad \int_{\Omega} DF(\nabla u) \cdot \nabla \phi dx = 0 \quad \text{for} \quad \forall \phi \in \overset{\circ}{H}^{1,s}(\Omega; R^{N}).\]
Transferring $x$ to $x + he$ along the direction of a unit vector $e$, we have

\[(3.2) \quad \int_{\Omega} DF(\nabla u^+) \cdot \nabla \phi dx = 0 \quad \text{for} \quad \forall \phi \in \dot{H}^{1,s}(\Omega_1; R^N).\]

where $\Omega_0 = \tilde{\Omega}, \Omega_k = \{ x \in \Omega : \text{dist}(x, \tilde{\Omega}) < \frac{k}{4} \text{dist}(\tilde{\Omega}, \partial \Omega) \}$ $(k = 0, 1, \cdots, 4)$. (3.1) subtracted after (3.2) gives

\[(3.3) \quad \int_{\Omega} [DF(\nabla u^+) - DF(\nabla u)] \cdot \nabla \phi dx = 0 \quad \text{for} \quad \forall \phi \in \dot{H}^{1,s}(\Omega_1; R^N).\]

Thus we have

\[(3.4) \quad \int_{\Omega} \int_{0}^{1} D^2F(\nabla u + t \nabla(\tau_hu)) < \nabla(\tau_hu), \nabla \phi > dt dx = 0 \quad \text{for} \quad \forall \phi \in \dot{H}^{1,m}(\Omega_1; R^N).\]

Substituting $\tau_hu\eta^2$ for $\phi$, where a cut-off function $\eta \in C_0^\infty(\Omega)$ satisfies

\[
\eta = \begin{cases} 
1 \quad \text{in} \ \Omega_0, \\
0 \quad \text{outside} \ \Omega_1
\end{cases}
\]

with

\[
|\nabla \eta| \leq \frac{2}{\text{dist}(\tilde{\Omega}, \partial \Omega)},
\]

\[0 \leq |\eta| \leq 1.
\]

We can proceed the calculation of (3.3) as follows:

\[
\int_{\Omega} \tau_h[DF(\nabla u), \nabla(\tau_hu)\eta^2] > dx \\
= \int_{\Omega} \int_{0}^{1} D^2F(\nabla u + t \nabla(\tau_hu)) \\
[\nabla(\tau_hu), \nabla(\tau_hu)\eta^2] + 2 \nabla(\tau_hu), \tau_hu\eta \nabla \eta >] dt dx
\]

Consequently, the following

\[
\int_{\Omega} D^2F(A) < \nabla(\tau_hu), \nabla(\tau_hu) > \eta^2 dx \\
= \int_{\Omega} [D^2F(A) - \int_{0}^{1} D^2F(\nabla u + t \nabla(\tau_hu))] \\
< \nabla(\tau_hu), \nabla(\tau_hu) > \eta^2 dt dx \\
- 2 \int_{\Omega} \int_{0}^{1} D^2F(\nabla u + t \nabla(\tau_hu)) dt < \nabla(\tau_hu)\eta, \tau_hu \nabla \eta > dt dx
\]

holds for $\forall A \in M^{n \times N}$. Now let $\Omega_1$ be approximated by a union of hypercubes $D_{k,i}$ with each edge length $1/k$ sufficiently large $k > 0$:

\[
\Omega_1 \subset \bigcup_{i=1}^{I} D_{k,i} \quad \text{with} \quad \Omega_1 \subset H_k \subset \Omega_2,
\]

\[\hat{D}_{k,i} \cap \hat{D}_{k,j} = \emptyset \quad \text{in} \quad i \neq j,
\]

\[|H_k - \Omega_2| \rightarrow 0 \quad \text{as} \quad k \rightarrow +\infty,
\]

\[(3.7) \quad |D_{k,i}| = (1/k)^n.
\]
Moreover we remark that there exists subsequence of $I$ which we call $I(k)$ such that $H_k = \bigcup_{i=1}^{I(k)} D_{k,i}$ satisfies $\Omega_1 \subset H_k \subset \Omega_2$ and $|\Omega_2 - H_k| \to 0$ as $k \to +\infty$. For $x \in H_k$, we define
\[
\overline{\nabla u}(x) \equiv \frac{1}{|D_{k,i}|} \int_{D_{k,i}} \nabla u(y) dy \text{ for } x \in D_{k,i} \text{ and } i = 1, \cdots, I(k).
\]

When we adopt $\nabla u(x) + s \nabla (\tau_h u)(x)$ ($0 \leq s \leq 1$), $\nabla u^+(x) \equiv \overline{\nabla u}(x) - \overline{\nabla u}(x)$ as $A$, then it follows from (3.5, 3.6) and (3.7) that
\[
\int_{\Omega} D^{2}F(\overline{\nabla u} + s \overline{\nabla (\tau_h u)}) < \nabla (\tau_h u), \nabla (\tau_h u) > \eta^2 ds dx
\]
\[
= \int_{0}^{1} \int_{H_k} D^{2}F(\nabla u + t \nabla (\tau_h u)) < \nabla (\tau_h u), \nabla (\tau_h u) > \eta^2 dt dx
\]
\[
+ 2 \int_{0}^{1} \int_{\Omega_{2}/H_k} D^{2}F(\nabla u + t \nabla (\tau_h u)) < \nabla (\tau_h u), \nabla (\tau_h u) > \eta^2 dt dx.
\]

By integrating (3.8) over $[0, 1]$ for $s$, we obtain
\[
\int_{\Omega} \int_{0}^{1} D^{2}F(\overline{\nabla u} + s \overline{\nabla (\tau_h u)}) < \nabla (\tau_h u), \nabla (\tau_h u) > \eta^2 ds dx
\]
\[
= \int_{0}^{1} \int_{\Omega_{2}/H_k} D^{2}F(\overline{\nabla u} + t \overline{\nabla (\tau_h u)}) - D^{2}F(\nabla u + t \nabla (\tau_h u)) < \nabla (\tau_h u), \nabla (\tau_h u) > \eta^2 dt dx
\]
\[
- 2 \int_{0}^{1} D^{2}F(\nabla u + t \nabla (\tau_h u)) < \nabla (\tau_h u), \nabla (\tau_h u) > \eta^2 dt dx.
\]

The above (3.9) is a starting point to our proof. The original technique used here is seen in [Da] and [Mo]. At first, we estimate the left-hand side in (3.9) from below:
\[
\int_{\Omega} \int_{0}^{1} D^{2}F(\overline{\nabla u} + s \overline{\nabla (\tau_h u)}) < \nabla (\tau_h u), \nabla (\tau_h u) > \eta^2 ds dx
\]
\[
\geq \int_{0}^{1} \int_{H_k} D^{2}F(\overline{\nabla u} + s \overline{\nabla (\tau_h u)}) < \nabla (\tau_h u), \nabla (\tau_h u) > \eta^2 ds dx
\]
\[
+ \int_{0}^{1} \int_{\Omega_{2}/H_k} D^{2}F(\overline{\nabla u} + s \overline{\nabla (\tau_h u)}) < \nabla (\tau_h u), \nabla (\tau_h u) > \eta^2 ds dx
\]
\[
= \sum_{i=1}^{I(k)} \int_{0}^{1} \int_{D_{k,i}} D^{2}F(\overline{\nabla u} + s \overline{\nabla (\tau_h u)}) < \nabla (\tau_h u), \nabla (\tau_h u) > \eta^2 ds dx
\]
\[
+ \int_{0}^{1} \int_{\Omega_{2}/H_k} D^{2}F(\overline{\nabla u} + s \overline{\nabla (\tau_h u)}) < \nabla (\tau_h u), \nabla (\tau_h u) > \eta^2 ds dx.
\]

If we use the mean value theorem for $s$, then there exist positive numbers $s_{0,i}$ ($i = 1, \cdots, I(k)$) such that
\[
= \sum_{i=1}^{I(k)} \int_{D_{k,i}} D^{2}F(\overline{\nabla u} + s_{0,i} \overline{\nabla (\tau_h u)}) < \nabla (\tau_h u), \nabla (\tau_h u) > \eta^2 ds dx
\]
\[
+ \int_{0}^{1} \int_{\Omega_{2}/H_k} D^{2}F(\overline{\nabla u} + s \overline{\nabla (\tau_h u)}) < \nabla (\tau_h u), \nabla (\tau_h u) > \eta^2 ds dx.
\]
Here we remark that from Morrey ([Mo], Th 4.4.3) and Federer ([Fe], Th 5.1.10) assumption (1.4) implies the strong Legendre - Hadamard condition:

\[
\sum_{\alpha, \beta} \sum_{i,j} \frac{\partial^2 F}{\partial p^\alpha_i \partial p^\beta} (A) \xi_\alpha \xi_\beta \eta^i \eta^j \geq \gamma |\xi|^2 |\eta|^2
\]

for \( A \in M^{n \times N} \), \( \xi \in \mathbb{R}^n \) and \( \eta \in \mathbb{R}^N \).

Thus by noting that \( \nabla u \) is a constant on each hypercube \( D_{k,i} \) \( (i = 1, \cdots, I) \), we have

\[
\begin{align*}
\int_{\Omega} \int_{0}^{1} \left[ D^2 F(\nabla u + s \nabla (\tau_h u)) - D^2 F(\nabla u + s \nabla u) \right] < \nabla \nabla (\tau_h u), \nabla (\tau_h u) > \eta^2 dtdx \\
\leq C_7 \int_{\Omega_1} \int_{0}^{1} \left[ 1 + |\nabla u + s \nabla (\tau_h u)|^{s-2} + |\nabla u + s \nabla u|^{s-2} \right] \\
\cdot w(|\nabla u - \nabla u|^2 + |\nabla u^+ - \nabla u^+|^2) |\nabla \nabla (\tau_h u)|^2 dx \\
\leq 2C_7 2^{s-1} \int_{\Omega_1} \left[ 1 + |\nabla u|^{s-2} + |\nabla u^+|^{s-2} + |\nabla u|^{s-2} + |\nabla u^+|^{s-2} \right] \\
\cdot w(|\nabla u|^2 + |\nabla u^+|^2) dx.
\end{align*}
\]

(3.13)

Since \( \nabla u \in L^t_{loc} (\Omega; \mathbb{R}^N) \) \( (t > s) \) from (1.8) of Theorem 3, we can apply Hölder inequality to (3.13) as follows: For \( s_1 = t/(s-2) \), \( s_2 = t/2 \) and \( s_3 = t/(t-s) \), we estimate the right-hand in (3.13)

\[
\begin{align*}
\int_{\Omega_1} \left[ 1 + |\nabla u|^t + |\nabla u^+|^t + |\nabla u|^t + |\nabla u^+|^t \right] dx \\
\cdot w(|\nabla u|^2 + |\nabla u^+|^2) dx.
\end{align*}
\]

Since \( \nabla u \in L^t_{loc} (\Omega; \mathbb{R}^N) \) \( (t > s) \) from (1.8) of Theorem 3, we can apply Hölder inequality to (3.13) as follows: For \( s_1 = t/(s-2) \), \( s_2 = t/2 \) and \( s_3 = t/(t-s) \), we estimate the right-hand in (3.13)

\[
\begin{align*}
\leq 2^s C_7 2 \left\{ \int_{\Omega_1} \left[ 1 + |\nabla u|^t + |\nabla u^+|^t + |\nabla u|^t + |\nabla u^+|^t \right] dx \right\}^{(s-2)/t} \\
\cdot \left\{ \int_{\Omega_1} |\nabla u|^t + |\nabla u^+|^t \right\}^{2/t} \left\{ \int_{\Omega_1} w^{t/(t-s)} |\nabla u|^2 + |\nabla u^+|^2 \right\}^{(t-s)/t}.
\end{align*}
\]

(3.14)
Successively by using bounded and concave properties of $w(t)$, we have

$$\begin{align*}
&\leq 2^s10C_7\{\int_{\Omega_2}[1 + |\overline{\nabla u}|^t + |\nabla u|^t]dx\}^{(s-2)/t} \\
&\{\int_{\Omega_2} |\overline{\nabla u}|^tdx\}^{2/t}\{\int_{\Omega_1} w(\overline{\nabla u} - \nabla u)^2 + |\overline{\nabla u^+} - \nabla u^+|2) dx\}^{(t-s)/t} \\
&\leq 2^s10C_7|\Omega_1|^{(t-s)/t}\{\int_{\Omega_2} [1 + |\overline{\nabla u}| + |\nabla u|]dx\}^{s/t} \\
&\{\frac{1}{|\Omega_1|}\int_{\Omega_1} w(\overline{\nabla u} - \nabla u| + |\overline{\nabla u^+} - \nabla u^+|) dx\}^{(t-s)/t} \\
&\leq 2^sC_7|\Omega|^{1-s/t}\{\int_{\Omega_2} [1 + |\overline{\nabla u}| + |\overline{\nabla u^+}|]dx\}^{(t-s)/t}.
\end{align*}$$

(3.14)

From $L_1$-norm continuity of integrable function, for $\forall \epsilon > 0$, there exists $k = k(\epsilon)$ such that

$$\begin{align*}
(3.14) &\leq 2^s10C_7|\Omega|^{1-s/t} \cdot \epsilon \cdot \{\int_{\Omega_2} [1 + |\overline{\nabla u}| + |\overline{\nabla u^+}|]dx\}^{s/t}.
\end{align*}$$

Finally we shall estimate the second term on the right-hand side in (3.9): From assumption (H5) and using Newton-Leibnitz formula we obtain

$$\begin{align*}
-2\int_{\Omega_1} \int_0^1 (D^2F)(\nabla u + t\nabla(\tau\nu u))dt &< \nabla(\tau\nu u)\eta, \tau\nu u\nabla\eta > dtdx \\
&\leq 2C_0 \int_{\Omega_1} \int_0^1 (1 + |\nabla u + t\nabla(\tau\nu u)|^{s-2})|\nabla(\tau\nu u)| \cdot |\tau\nu u| \cdot |\nabla\eta|dx \\
&\leq 2^sC_0 \int_{\Omega_1} \int_0^1 (1 + |\overline{\nabla u}|^{s-2} + |\nabla u|^{s-2})|\nabla(\tau\nu u)| \cdot |\tau\nu u| \cdot |\nabla\eta|dx \\
&\leq 2^sC_0 \frac{2}{\text{dist}(\Omega_0, \Omega_1)} \int_{\Omega_2} [1 + |\overline{\nabla u}|^{s-2} + |\nabla u|^{s-2}]^{s/(s-2)}dx \{\int_{\Omega_1} [||\nabla u^+| + |\nabla u|]dx\}^{1/s} \{\int_{\Omega_1} |\tau\nu u|^{s}dx\}^{1/s} \\
&\leq 2^s12C_0 \frac{h}{\text{dist}(\Omega_0, \Omega_1)} \int_{\Omega_2} [1 + |\nabla u|^s]dx \{\int_{\Omega_2} |\nabla u|^{s}dx\}^{1/s}.
\end{align*}$$

(3.16)

Consequently it follows from (3.12), (3.15) and (3.16) that

$$\begin{align*}
\gamma \int_{H_k} |\nabla(\tau\nu u)|^2 dx &\leq 2^s10C_7|\Omega|^{1-s/t} \cdot \epsilon \cdot \{\int_{\Omega_2} [1 + |\overline{\nabla u}| + |\overline{\nabla u^+}|]dx\}^{s/t} \\
&+ \frac{122^sC_0h}{\text{dist}(\Omega_0, \Omega_1)} \int_{\Omega_2} [1 + |\nabla u|^s]dx \{\int_{\Omega_2} |\nabla u|^sdx\}^{1/s}.
\end{align*}$$

(3.17)
Now letting pass to the limit $k \to \infty$, we deduce the desired estimates:

$$
\int_{\Omega} |\nabla(\tau_h u)|^2 \, dx
$$

(3.18)

$$
\leq \gamma^{-1} \frac{2^8 50 C_0 h}{\text{dist}(\tilde{\Omega}, \partial \Omega)} \{ \int_{\Omega} (1 + |\nabla u|)^s \, dx \}.
$$

This completes our proof.

Acknowledgement

The author is very grateful to Prof. N.KIKUCHI for especially drawing my attention to the problem and his constant encouragement.

References


[AF]. E.Acerbi and N.Fusco, Semicontinuity problem in the calculus of variations, Arch Rat Mech Anal.

[Da]. B.Dacorgna, LNM No 922, "Weak Continuity and Weak Lower semi Continuity of Non-Linear Functionals," Springer.


[Gm3]. ---------, Quasi - Convexity , growth conditions and partial regularity, preprint.


