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ENTROPY for CANONICAL SHIFTS

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§1. Introduction.

The notion of the entropy for *-automorphisms of finite von Neumann algebras is introduced by Connes and Størmer ([3]). In the previous paper [2], we defined the entropy for *-endomorphisms of finite von Neumann algebras as an extended version of it. It is possible to define the entropy for a general completely positive linear map $\alpha$ using results in [4] by a similar method to one for *-endomorphisms. However, the formula of the definition of the entropy for $\alpha$ implies that the entropy is apt to be zero if $\alpha^k$ converges to $\alpha$ when $k$ tends to infinity. The conditional expectation is a trivial example of such a map $\alpha$. For that reason, the interesting completely positive map $\alpha$ for us to discuss the entropy are those which have the property that $\alpha^k$ goes away from $\alpha$ as $k$ tends to infinity.

In this paper, we shall study such a class of *-endomorphisms of injective finite von Neumann algebras.

In §3, we shall introduce, for a *-endomorphism $\sigma$ of an injective finite von Neumann algebra $A$, the notion of an $n$-shift on the tower $(A_j)_j$ of finite dimensional von Neumann subalgebras of $A$ which generates $A$ and we obtain the formula of the entropy $H(\sigma)$ for an $n$-shift $\sigma$.

In the work [8] on the classification for subfactors of the hyperfinite type $\text{II}_1$-factor, Ocneanu introduced a special kind of *-endomorphism which is called the canonical shift.
on the tower of relative commutants. The $^*$-endomorphism $\Gamma$ is a generalization of the comultiplication for Hopf algebras and also considered as the canonical shift on the string algebras. In a part, $\Gamma$ has similar properties to the canonical endomorphism of an inclusion of infinite von Neumann algebras due to Longo [7].

The canonical shift $\Gamma$ naturally induces a 2-shift for the injective finite von Neumann algebra $A$ which generated by the tower $(A_j)_j$ of relative commutants and the entropy $H(\Gamma)$ is determined by the following:

$$H(\Gamma) = \lim_{k \to \infty} \frac{H(A_{2k})}{k}.$$  

For a $^*$-endomorphism $\sigma$ of a von Neumann algebra $A$, the entropy $H(\sigma)$ is a conjugacy invariant, that is, if there is an isomorphism $\theta$ of $A$ onto a von Neumann algebra $B$ such that $\theta \sigma = \phi \theta$ for a $^*$-endomorphism $\phi$ of $B$, then $H(\sigma) = H(\phi)$. On the other hand, two conjugate $^*$-endomorphisms $\sigma$ and $\phi$ of $A$ give two conjugate von Neumann subalgebras $\sigma(A)$ and $\phi(A)$ under automorphisms of $A$.

In [9], Pimsner and Popa introduced two conjugacy invariants for von Neumann subalgebras. One is the relative entropy $H(A | B)$ for a von Neumann subalgebra $B$ of a finite von Neumann algebra $A$, which is defined an extended version of one for finite dimensional algebras due to Connes-Størmer [3]. The other is the constant $\lambda(A, B)$, which plays a role like the index for subfactors due to Jones [6]. In fact in the case of factors $B \subset A$, $\lambda(A, B)^{-1}$ is Jones index $[A : B]$.

We shall investigate relations among those invariants.

In §4, we restrict our attention to finite dimensional von Neumann algebras. We need those results later. The Jones index for a subfactor $N$ of a finite factor $M$ is given as $1/\tau(e)$ for the projection $e$ of $L^2(M)$ onto $L^2(N)$ where $\tau$ is the trace on the basic extension algebra of $N \subset M$. In the case of finite dimensional von Neumann algebras, we shall show that the constant $\lambda(\ , \ )^{-1}$ coincides with Jones index in such a sense.
In §5, it is obtained that in general the following relation holds for an $n$-shift $\sigma$:

$$H(A | \sigma(A)) \leq 2H(\sigma).$$

A condition that the equality holds is also given.

In §6, we shall obtain the relation between $H(\sigma)$ and the constant $\lambda(A, \sigma(A))$. We shall define a locally standard tower for an increasing sequence $(A_j)\rangle$ of finite dimensional von Neumann algebras. The tower $(A_j)\rangle$ of relative commutants for the inclusion of finite factors $N \subset M$ satisfies this condition. If a $\ast$-endomorphism $\sigma$ of $A$ is an $n$-shift on a locally standard tower which generates $A$, then we have the following:

$$H(A | \sigma(A)) \leq 2H(\sigma) \leq \log \lambda(A, \sigma(A))^{-1}.$$

In §7, we shall apply the above results to the canonical shift $\Gamma$ for the tower of relative commutants. Let $N \subset M$ be type $\Pi_1$-factors with the finite index. Considering the tower $(M_j)\rangle$ of factors obtained by iterating Jones basic construction from $N \subset M$, it is obtained the increasing sequence $(A_j)\rangle$ of finite dimensional von Neumann algebras, where $A_j = M' \cap M_j$. The $\ast$-endomorphism $\Gamma$ is defined on the algebra $\bigcup_j A_j$ as a mapping such that $\Gamma(M'_k \cap M_j) = M'_{k+2} \cap M_{j+2}$ for all $k \leq j$. First, we remark that $\Gamma$ is extended to the trace preserving $\ast$-endomorphism of a finite von Neumann algebra $A = \bigcup_j (A_j)'$. Then $\Gamma$ has an ergodic property that

$$\bigcap_k \Gamma^k(A) = C1$$

and satisfies the conditions of Definition for a 2-shift, except only one. In order that $\Gamma$ satisfies all conditions for 2-shifts, a condition for the inclusion $N \subset M$ is necessary. For example, in the case where $N' \cap M = C1$, $\Gamma$ is a 2-shift and the following relation holds:

$$H(A | \Gamma(A)) \leq 2H(\Gamma) \leq 2\log[M : N].$$

Furthermore, if the inclusion $N \subset M$ has finite depth ([8], [12]), then we have:

$$H(M | N) = H(\Gamma) = \log[M : N]^{-1}.$$
In §8, we shall discuss conditions for a $^*$-endomorphism $\sigma$ of a factor $M$ to be extended to an automorphism $\theta$ of a factor containing $M$ so that $H(\sigma) = H(\theta)$. If the inclusion $N \subset M$ has finite depth, then $\Gamma$ is extended to an ergodic $^*$-automorphism $\Theta$ which satisfies the following:

$$H(M \mid N) = H(\Theta) = H(\Gamma) = \log[M : N]^{-1}.$$ 

§2. Preliminaries.

In this section, we shall fix the notations and terminologies frequently used in this paper.

Throughout this section, $M$ will be a finite von Neumann algebra with a fixed normal faithful trace $\tau$, $\tau(1) = 1$. The inner product $\langle x, y \rangle = \tau(xy^*)$ gives $M$ as a vector space the structure of a pre-Hilbert space. Let $||x|| = \tau(x^*x)^{1/2}$ and $L^2(M, \tau)$ the Hilbert space completion of $M$. Then $M$ acts on $L^2(M, \tau)$ by the left multiplication. The canonical conjugation on $L^2(M, \tau)$ is denoted by $J = J_M$. It is the conjugate unitary map induced by the involution $^*$ on $M$. For a von Neumann subalgebra $N$ of $M$, let $e_N$ be the orthogonal projection of $L^2(M, \tau)$ onto $L^2(N, \tau)$. Then the restriction $E_N$ of $e_N$ to $M$ is the faithful normal conditional expectation of $M$ onto $N$.

The letter $\eta$ designates the function on $[0, \infty)$ defined by $\eta(t) = -t \log t$. For each $k$, we let $S_k$ be the set of all families $(x_{i_1, i_2, \ldots, i_k})_{i_j \in N}$ of positive elements of $M$, zero except for a finite number of indices and satisfying

$$\sum_{i_1, \ldots, i_k} x_{i_1, i_2, \ldots, i_k} = 1.$$ 

For $x \in S_k$, $j \in 1, 2, \ldots, k$ and $i_j \in N$, put

$$x_{i_1}^j = \sum_{i_{1-j}, i_{j+1}, \ldots, i_k} x_{i_1, i_2, \ldots, i_k}.$$ 

Let $N_1, N_2, \ldots, N_k$ be finite dimensional von Neumann subalgebras of $M$. Then

$$H(N_1, \ldots, N_k) = \sup_{x \in S_k} \left[ \sum_{i_1, \ldots, i_k} \eta \tau(x_{i_1, \ldots, i_k}) - \sum_j \sum_{i_j} \tau E_{N_j}(x_{i_j}^j) \right].$$
Let $\sigma$ be a $\tau$-preserving $*$-endomorphism of $M$ and $N$ a finite dimensional von Neumann subalgebra of $M$, then

$$H(N,\sigma) = \lim_{k \to \infty} \frac{1}{k} H(N, \sigma(N), \ldots, \sigma^{k-1}(N))$$

exists by [2]. The entropy $H(\sigma)$ for $\sigma$ is defined as the supremum of $H(N, \sigma)$ for all finite dimensional subalgebras $N$ of $M$.

If there exists an increasing sequence $(N_j)_j$ of finite dimensional subalgebras which generates $M$, then by [2]

$$H(\sigma) = \lim_{j \to \infty} H(N_j, \sigma).$$

The relative entropy $H(M \mid N)$ for a von Neumann subalgebra $N$ of $M$ is defined ([10]) as an extension form of one ([3]) by

$$H(M \mid N) = \sup_{x \in S_1} \sum_j [\tau_1(x_1) - \tau E_N(x)]$$

This $H(M \mid N)$ is a conjugacy invariant for subalgebras of $M$. Another conjugacy invariant $\lambda(M, N)$ is introduced in [10] as a generalization of Jones index defined by

$$\lambda(M, N) = \max\{\lambda \geq 0; E_N(x) \geq \lambda x, x \in M_+\}.$$

For an inclusion $N \subset M$ of finite von Neumann algebras, the von Neumann algebra on $L^2(M, \tau)$ generated by $M$ and $e = e_N$ is called the standard basic extension (or basic construction) for $N \subset M$ and denoted by $M_1 = \ll M, e \gg$. Then by the properties of $J = J_M$ and $e = e_N$, we have $M_1 = \ll M, e \gg = JN'J([6])$. If $M_1$ is finite and if there is a trace $\tau_1$ on $M_1$ such that $\tau_1(xe) = \lambda \tau(x)$ for all $x \in M$, then the trace $\tau_1$ is called the $\lambda$-Markov trace for $N \subset M$. If $M \supset N$ are factors and there is the $\lambda$-Markov trace of $M_1$ for $N \subset M$, then Jones index $[M : N] = \lambda^{-1}$ ([6]).

We shall call an increasing sequence $(M_j)_{j \in N}$ of von Neumann algebras a standard tower (cf. [5], [9], [13]) if $M_{j-1} \subset M_j \subset M_{j+1}$ is the basic construction obtained from $M_{j-1} \subset M_j$ for each $j$. 

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Let $L$ be a finite factor containing $M$. We shall call $L$ the algebraic basic construction for the factors $N \subset M$ if there is a nonzero projection $e \in M$ satisfying:

(i) $exe = E_N(x)e$ for $x \in M$

and

(ii) $L$ is generated by $e$ and $M$ as a von Neumann algebra.

In this case, there is an isomorphism $\phi$ of $M_1$ onto $L$ such that $\phi(e_N) = e$ and $\phi(x) = x$ for all $x \in M$ ([11]).

We shall call such a projection $e$ the basic projection for $N \subset M$ and a decreasing sequence $(N_j)_{j \in \mathbb{N}}$ of finite factors a standard tunnel (cf. [5], [9], [13]) if $N_{j-1} \supset N_j \supset N_{j+1}$ is the algebraic basic construction for $N_j \supset N_{j+1}$ for each $j$.

§3. Entropy of $n$-shift

In this section, we shall give the definition of $n$-shifts and a formula of the entropy for $n$-shifts. Let $A$ be an injective finite von Neumann algebra with a fixed faithful normal trace $\tau$, with $\tau(1) = 1$. Let $(A_j)_{j=1,2,\ldots}$ be an increasing sequence of finite dimensional von Neumann algebras such that $A = \text{the weak closure of } \bigcup_j A_j = \{A_j : j\}''$. Assume that $\sigma$ is a $\tau$-preserving *-endomorphism of $A$. Then $\sigma$ is a ultra-weakly continuous, one to one mapping with $\sigma(1) = 1$.

**Definition 1.** Let $n$ be a natural number. A $\tau$-preserving *-endomorphism $\sigma$ of $A$ is called an $n$-shift on the tower $(A_j)_j$ for $A$ if the following conditions are satisfied:

1. For all $j$ and $m$, the von Neumann algebra $\{A_j, \sigma(A_j), \ldots, \sigma^m(A_j)\}''$ generated by $\{\sigma^j(A_j) : j = 0, \ldots, m\}$ is contained in $A_{j+n'm}$.

2. There exists a sequence $(k_j)_{j \in \mathbb{N}}$ of integers with the properties:

$$\lim_{j \to \infty} \frac{nk_j - j}{j} = 0$$

and
$x\sigma^m(y) = \sigma^m(y)x, \quad \tau(z\sigma^l(x)) = \tau(z)\tau(x)$

for all $l \in \mathbb{N}$, $x, y \in A_j$, $m \in k_j\mathbb{N}$ and $z \in \{A_j, \sigma^k(A_j), \ldots, \sigma^{(l-1)k}(A_j)\}''$.

(3) Let $E_B$ be the conditional expectation of $A$ onto a von Neumann subalgebra $B$ of $A$. Then for an $j \geq n$

$$E_{A_j}E_{\sigma(A_j)} = E_{\sigma(A_{j-n})}$$

(4) For each $j$, there exists a $\tau$-preserving $*$-automorphism or antiautomorphism $\beta$ of $A_{nj+n}$ such that $\sigma(A_{nj}) = \beta(A_{nj})$.

**Remark 1.** The number $n$ of an $n$-shift depends on the choice of the sequence $(A_j)_j$. Every given $n$-shift can be 1-shift on a suitable tower for the same von Neumann algebra.

**Example 1.** Let $S$ be the $*$-endomorphism corresponding to the translation of 1 in the infinite tensor product $R = \bigotimes_{i \in \mathbb{N}} (M_i, tr_i)$ of the algebra $M_i$ of $m \times m$ matrices with the normalized trace $tr_i$ on $M_i$ for each $i \in \mathbb{N}$. For each $j$, let $A_j = \bigotimes_{i=1}^j (M_i, tr_i)$. Then for all $n$, $S^n$ is an $n$-shift on the tower $(A_j)_j$ for $R$.

In fact, for an $n \in \mathbb{N}$, let $k_j = \lfloor \frac{j}{n} \rfloor + 1$. Then $(k_j)_j$ satisfies the following properties $(2')$ which are stronger than $(2)$:

$$\lim_{j \to \infty} \frac{nk_j - j}{j} = 0$$

and

$$x\sigma^m(y) = \sigma^m(y)x, \quad \tau(z\sigma^{lk}(x)) = \tau(z)\tau(x)$$

for all $l \in \mathbb{N}$, $x, y \in A_j$, $k_j \leq k$, $m \in \mathbb{N}$ and $z \in \{A_j, \sigma^k(A_j), \ldots, \sigma^{(l-1)k}(A_j)\}''$. It is obvious that another conditions are satisfied by $S^n$. 

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Example 2. Let \((e_j)\) be the sequence of projections with the following properties for some natural number \(k\) and \(\lambda \in (0, 1/4] \cup \{1/(4\cos^2(\pi/n)); n \geq 3\}:

(a) \(e_i e_j e_i = \lambda e_i\) if \(|i - j| = k\)

(b) \(e_i e_j = e_j e_i\) if \(|i - j| \neq k\)

(c) \((e_j)\) generates the hyperfinite type \(\Pi_1\)-factor \(R\)

(d) \(\tau(we_i) = \lambda \tau(u;)\) for the trace \(\tau\) of \(R\) and a reduced word \(w\) on \(\{1, e_1, ..., e_{i-1}\}\).

Let \(A_j\) be the von Neumann algebra generated by \(\{e_1, ..., e_j\}\). Then, by [6], \(A_j\) is finite dimensional. Let \(\sigma\) be the \(*\)-endomorphism of \(R\) such that \(\sigma(e_i) = e_{i+1}\). Then \(\sigma^n\) is an \(n\)-shift on the tower \((A_j)\) of \(R\) for all \(n\). In fact, for an \(n \in N\), let \(k_j = \left\lfloor \frac{j+k}{n} \right\rfloor + 1\). Then \((k_j)\) satisfies properties \((2')\) in Example 1. The condition \((3)\) and \((4)\) are satisfied by using results by [6] and [1].

In §7, we shall show that the canonical shift due to Ocneanu is a 2-shift on the tower of relative commutant algebras.

Theorem 1. If a \(\tau\)-preserving \(*\)-endomorphism \(\sigma\) of \(A\) satisfies the condition \((1)\) and \((2)\) in Definition 1 for the tower \((A_j)\) of \(A\), then

\[ H(\sigma) = \lim_{k \to \infty} \frac{H(A_{nk})}{k}. \]

§4. Finite dimensional algebras.

In this section, \(M\) will be a finite dimensional von Neumann algebra and \(\tau\) a fixed faithful normal trace of \(M\) with \(\tau(1) = 1\). Then \(M\) is decomposed into the direct summand:

\[ M = \bigoplus_{l \in K} M_l \]

where \(M_l\) is the algebra of \(d(l) \times d(l)\) matrices and \(K = K_M\) is a finite set. Then the vector \(d_M = d = (d(l))_{l \in K}\) is called the dimension vector of \(M\). The column vector
$t_M = t = (t(l))_{l \in K}$ has $t(l)$ as the value of the trace for the minimal projections in $M_l$, and is called the trace vector of $\tau$. Let $N$ be a von Neumann subalgebra of $M$ with $N = \sum_{k \in K_N} \oplus N_k$. The inclusion matrix $[N \hookrightarrow M] = (m(k, l))_{k \in K_N, l \in K_M}$ is given by the number $m(k, l)$ of simple components of a simple $M_l$ module viewed as an $N_k$ module. Then

$$d_N[N \hookrightarrow M] = d_M \quad \text{and} \quad [N \hookrightarrow M]t_M = t_N.$$ 

Here we shall give a simple formula for $\lambda(M, N)$.

By the definition of the basic construction of $N \subset M$, there is a natural isomorphism between the centers of $N$ and $<M, e>$ via $x \rightarrow JxJ$. Hence there is a natural identification between the sets of simple summands of $N$ and $<M, e>$. We put $K = K_N = K_{<M,e>}$. The following theorem assures that in the case of finite dimensional von Neumann algebras, the constant $\lambda(\cdot, \cdot)$ plays the same role as the index for finite factors.

**Theorem 2.** (1) Assume that there is a trace of $<M, e>$ which is an extension of $\tau$. Then

$$\lambda(<M, e>, M)^{-1} = \max_{k \in K} \frac{t_N(k)}{t_{<M,e>} (k)}.$$

(2) If the trace $\tau$ of $<M, e>$ has the $\tau(e)$-Markov property, then

$$\lambda(<M, e>, M)^{-1} = 1/\tau(e) = \| [N \hookrightarrow M] \|^2.$$ 

**Definition 2.** Let $N \subset M \subset L$ be an inclusion of finite dimensional von Neumann algebras. Then $L$ is said to be the algebraic basic construction for $N \subset M$ if there is a projection $e$ in $L$ satisfying

(a) $L$ is generated by $M$ and $e$,

(b) $xe = ex$ for an $x \in N$, 

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(c) If $x \in N$ satisfies $xe = 0$, then $x = 0$.

(d) $exe = E_N(x)e$ for all $x \in M$.

In this case, there is a $^*$-isomorphism of the basic construction $M_1 = JN^*J$ onto $L$.

We shall call $N \subset M \subset L$ a \textit{locally algebraic basic extension} of $N \subset M$ if there is a projection $p \in L \cap L'$ which satisfies that the inclusion $M \subset Lp$ is the algebraic basic construction for $N \subset M$.

If $L \supset M \supset N$ is a locally standard extension of the inclusion $M \supset N$, we can identify the set $K_N$ with a subset of $K_L$ via the equality $Ne = eLe$. Under such an identification, we have the following:

**Proposition 3.** Let $L \supset M \supset N$ be a locally standard extension of $M \supset N$. Then

$$\lambda(L, M)^{-1} \geq \max_{k \in K_N} \min_{l \in K_L} \frac{t_N(k)}{t_L(l)}.$$ 

Let

$$l(M) = \sum_{l \in K} d(l)t(l)\log\frac{d(l)}{t(l)},$$

where $K = K_M$, $d = d_M$ and $t = t_M$.

**Proposition 4.**

(i) $H(M \mid N) \leq I(M) - I(N)$

(ii) $H(<M, e\mid M) = I(<M, e\mid M) - I(M)$

(iii) $I(M) \leq 2H(M)$ and the equality holds if and only if $M$ is a factor.

§5. $H(\sigma)$ and $H(A \mid \sigma(A))$

In this section we shall investigate a relation between $H(\sigma)$ and $H(A \mid \sigma(A))$ for an $n$-shift $\sigma$ on the tower $(A_i)_i$ for a finite von Neumann algebra $A$.  

Let \((A_j)\) be an increasing sequence of finite dimensional von Neumann algebras. Let \(A_j = \sum_{k \in K_j} \oplus A_j(k)\) be such a decomposition as in §4, and \(d_j\) the dimension vector of \(A_j\). Then we shall say \((A_j)\) satisfies the bounded growth conditions ([2]) if the following two conditions are satisfied:

(i) \[\sup_j \frac{|(K_j)|}{j} < +\infty\]

and

(ii) For some \(m\), \(A_{j+1}(l)\) contains at most \(d_j(k)\) \(A_j(k)\)-components for all \(j \geq m\), where \(|(K_j)|\) is the cardinal number of \(K_j\).

For examples, let us consider two towers which are treated in Examples 1 and 2. Both of them satisfies the bounded growth conditions ([2]). We shall discuss another example in §7.

**Theorem 5.** Let \(\sigma\) be a \(\tau\)-preserving *-endomorphism of an injective finite von Neumann algebra \(A\) with a faithful normal trace \(\tau\), \(\tau(1) = 1\). If \(\sigma\) is an \(n\)-shift on the tower \((A_j)\) for \(A\), then

\[H(A | \sigma(A)) \leq 2H(\sigma)\]

Furthermore, if the bounded growth conditions are satisfied, for the tower \((A_{nj})\)

\[H(A | \sigma(A)) = 2H(\sigma)\]

In order to prove Theorem 5, we need the following:

**Lemma 6.** Let \(\sigma\) be the same as in Theorem 5. If \(\sigma\) satisfies the conditions (1), (3) and (4) in Definition 1 for \(n\), then

\[H(A | \sigma(A)) = \lim_{j \to \infty} H(A_{nj}+n | A_{nj})\]
By considering the standard tower

\[ N \subset M \subset M_1 \subset M_2 \subset \ldots \subset M_n = \langle M_{n-1}, e_{n-1} \rangle \subset \ldots \]

obtained from the pair \( N \subset M \) of \( II_1 \)-factors with \([M : N] < \infty \) by iterating the basic construction, it is proved in [11] that \( H(M_n \mid N) = \log[M_n : N] \) if \( H(M \mid N) = \log[M : N] \).

Since the index has the multiplicative property ([6]), it implies that \( H(M_n \mid N) = nH(M \mid N) \) if \( H(M \mid N) = \log[M : N] \). Next corollary shows a similar result holds for the pair \( \sigma(M) \subset M \).

**Corollary 7.** Let a \(*\)-endomorphism \( \sigma \) satisfy the same condition as in Theorem 5. Then for all \( n \)

\[ H(A \mid \sigma^n(A)) = nH(A \mid \sigma(A)). \]

**§6.** \( H(\sigma) \) and \( \lambda(A, \sigma(A)) \) for \( n \)-shift \( \sigma \).

In this section, we shall investigate relations between the entropy \( H(\sigma) \) and the constant \( \lambda(A, \sigma(A)) \) for an \( n \)-shift \( \sigma \) of the tower \((A_j)_{j \in \mathbb{N}}\) for a finite von Neumann algebra \( A \) with a fixed faithful normal trace \( \tau \), \( \tau(1) = 1 \).

**Definition 3.** We shall call an increasing sequence \((A_j)_{j} \) of finite dimensional von Neumann subalgebras of a finite von Neumann algebra \( A \) with a faithful normal trace \( \tau \) a \textit{locally standard tower} for \( \alpha \) if there exists a natural number \( k \) which satisfies the following conditions:

1) For a certain central projection \( p_{k(j+1)} \) of \( A_{k(j+1)} \), the inclusion matrix \( [A_{jk} \hookrightarrow A_{k(j+1)}p_{j+1}] \) is the transpose of \( [A_{k(j-1)} \hookrightarrow A_{kj}] \), for each \( j \).
2) If $\left(t_{k(j-1)}(i)\right)_i$ is the trace vector for the restriction of $\tau$ to $A_{k(j-1)}$, then the value of $\tau$ of the minimal projections for $A_{k(j+1)}p_{k(j+1)}$ are given by $(\alpha t_{k(j-1)}(i))_i$ for each $j$.

3) There is an $c > 0$ such that $H(A_{2kj}) \leq c - j \log \alpha$ for each $j$.

We call the number $2k$ a period of the locally standard tower.

As the examples of locally standard towers, we have followings:

(i). The tower $(A_j)_j$ in Example 1 is obviously a locally standard tower for $1/m$, because the inclusion matrix in each step are all same.

(ii). The standard tower is a locally standard tower for $\|T^*T\|^{-1}$, because the inclusion matrix in the $j$-th step is the transpose of one in the $(j-1)$-th step for all $j$ ([6]). Hence the tower $(A_j)_j$ is also locally standard if $A_{j+1}$ is a locally algebraic basic extension of $A_{j-1} \subseteq A_j$.

(iii). The tower $(A_j)_j$ in Example 2 is a locally standard tower for $\lambda$, because the central support of $e_j$ in $A_j$ satisfies the condition (1) and (2) in Definition 3 and the condition (3) are proved by results in § 4.2 and § 5.1 in [6].

We shall treat another locally standard tower in the next section.

**Theorem 8.** Let $A$ be a finite von Neumann algebra with a fixed faithful normal trace $\tau$, $\tau(1) = 1$. Let $\sigma$ be an $n$-shift on the locally standard tower $(A_j)_j$ for $\alpha$ with a period $2n$, then

$$H(A | \sigma(A)) \leq 2H(\sigma) \leq -\log \alpha \leq \log \lambda(A, \sigma(A))^{-1}$$

The author would like to her hearty thanks to F.Hiai for pointing out a mistake in the proof of Theorem 8 in the preliminary version.

**Corollary 9.** Let $A$ be an injective finite factor with the canonical trace $\tau$ and $\sigma$ an $n$-shift of a locally standard tower for $A$ with a period $2n$, then
\[ H(A \mid \sigma(A)) \leq 2H(\sigma) \leq \log[A : \sigma(A)]. \]

In the case of a factor \( A \), it is obtained in [10] equivalent conditions that \( H(A \mid \sigma(A)) = \log[A : \sigma(A)] \). In such a case, we have

\[ H(A \mid \sigma(A)) = 2H(\sigma) = \log[A : \sigma(A)]. \]

For example, the shifts \( S \) in Example 1 and \( \sigma \) for \( \lambda > (1/4) \) in Example 2 satisfy the equality ([2]). However, the shifts \( \sigma \) in Example 2 have the following relation ([2]):

\[ H(R \mid \sigma(R)) = 2H(\sigma) < \log[R : \sigma(R)] \]

if \( \lambda \leq (1/4) \).

§7 Canonical shift.

In [9], Ocneanu defined a very nice *-endomorphism for the tower of the relative commutant algebras for the inclusion \( N \subset M \) of type \( II_1 \)-factors with the finite index.

At first, we shall recall from [9] the definition and main properties of the canonical shift on the tower of relative commutants.

Let \( M \) be a finite factor with the canonical trace \( \tau \) and \( N \) a subfactor of \( M \) such that \([M : N] < +\infty\). Then the basic extension \( M_1 = \langle M, e \rangle \) is a \( II_1 \)-factor with the \( \lambda = [M : N]^{-1} \) Markov trace ([6]) and there are the family \( \{m_i\} \subset M \) which forms an "orthonormal basis" in \( M \) with respect to the \( N \) valued inner product \( E_N(xy^*)(x, y \in M) \), that is, each \( x \in M \) is decomposed in the unique form as the following ([9], [10]):

\[ x = \sum_i E_N(m_i^*x)m_i. \]
Iterating the basic construction from $N \subset M$, we have the standard tower

$$M_{-1} = N \subset M_0 = M \subset M_1 \subset M_2 \subset \ldots$$

in which, $e_j$ is the projection of $L^2(M_j, \tau_j)$ onto $L^2(M_{j-1}, \tau_{j-1})$, where $\tau_j$ is the $\lambda$-Markov trace for $M_j$. Then from the family $(e_j)$, the projection $e(n,k)$ is obtained and

$$M_{n-k} \subset M_n \subset M_{n+k} = \langle M_n, e(n,k) \rangle$$

is an algebraic basic extension ([9], [11]). Furthermore it is obtained in [9] that the "orthonormal basis" in $M_n$ with respect to $M_{n-k}$ valued inner product from the family of the basis in $(M_j)_j$.

Let $A_j = M' \cap M_j$ for all $j$. The antiautomorphism $\gamma_j$ of $A_{2j} = M' \cap M_{2j}$ defined by

$$\gamma_j(x) = J_j x^* J_j, \quad x \in A_{2j}$$

is called the mirroring, where $J_j$ is the conjugate unitary on $L^2(M_j, \tau_j)$. Then for all $x \in M' \cap M_{2j}$, the following expression of the mirrorings is given:

$$\gamma_j(x) = [M_j : M] \sum_i E(em_i^* x) em_i,$$

where $E$ is the conditional expectation of $M_j$ onto $M$, $e$ is the projection of $L^2(M_j)$ onto $L^2(M)$ and $(m_i)$, a module basis of $M_j$ over $M$. The expression implies that the mirrorings satisfies the following relation:

$$\gamma_{j+1} \cdot \gamma_j = \gamma_j \cdot \gamma_{j-1}$$

for all $j \geq 1$ on $A_{2j-1}$. In the view of this relation, the endomorphism $\Gamma$ of $\bigcup A_n$ can be defined by

$$\Gamma(x) = \gamma_{j+1}(\gamma_j(x)),$$

for $x \in A_{2j}$. Ocneanu called the endomorphism $\Gamma$ the canonical shift on the tower of the relative commutants. In the case of inclusions of infinite factors, similar $^*$-endomorphisms
are investigated by Longo [8]. The mapping $\Gamma$ has the following properties; for any $k, n \geq 0$ with $n \geq k$,
\[ \Gamma(M_k^l \cap M_n) = M_{k+2}^l \cap M_{n+2}. \]

Now, we shall consider the finite von Neumann algebra $A$ generated by the tower $(A_j)_j$ and extend $\Gamma$ to a trace preserving $^*$-endomorphism of $A$ as follows.

Since $N \subset M$ are $II_1$-factors with $[M : N] < +\infty$, there is a faithful normal trace on $\bigcup_j M_j$ which extends the canonical trace $\tau$ on $M$. We denote the trace by the same notation $\tau$.

Although $M_{j+1}$ is defined as a von Neumann algebra on $L^2(M_j, \tau_j)$, each $M_j$ can be considered as von Neumann algebras on the Hilbert space $L^2(M, \tau)$. Hence $\bigcup A_j$ and $\bigcup M_j$ can be considered as von Neumann algebras acting on $L^2(M, \tau)$. Let
\[ M_\infty = \{\bigcup M_j\}'', \quad A = \{\bigcup A_j\}''. \]

Then $M_\infty$ is a finite factor with the canonical trace which is the extension of $\tau$. We denote it by the same notation $\tau$. Then $A$ is a von Neumann subalgebra of $M_\infty$. Since $\Gamma$ is a ultra-weakly continuous endomorphism of $\bigcup A_j$, $\Gamma$ is extended to a $^*$-endomorphism of $A$.

Although, in the case where discussed by Ocneanu, for all $k$, the mirroring $\gamma_k$ is a trace preserving map thanks to the assumption $N' \cap M = C1$, in general, the mirrorings are not always trace preserving. However the canonical shift is always trace preserving:

**Lemma 10.** For every $k$, $\gamma_{k+1} \cdot \gamma_k$ is a $\tau$- preserving isomorphism of $M_k^l \cap M_{2k}$ onto $M_{2k}^l \cap M_{2k+2}$.

Furthermore, if $E_{A_j}(e_1) = \lambda$ (for example $N' \cap M = C1$), then $\gamma_j$ is a trace preserving antiautomorphism of $A_{2j}$ for all $j$.

By Lemma 10, the canonical shift $\Gamma$ on the tower of the relative commutants $(A_j)_j$ of
$M$ is extended to a $\tau$-preserving $*$-endomorphism of $A$. We shall call the $*$-endomorphism of $A$ the canonical shift for the inclusion $M \supset N$ and denote it by the same notation $\Gamma$.

We shall show the canonical shift $\Gamma$ is a 2-shift on the tower $(A_j)_j$ for $A$.

**Lemma 11.** Let $L$ be a finite von Neumann algebra with a faithful normal trace $\tau$, $\tau(1) = 1$. If $M$ is a subfactor of $L$, then

$$\tau(xy) = \tau(x)\tau(y) \quad (x \in M, y \in M' \cap L).$$

**Proposition 12.** The canonical shift $\Gamma$ for the inclusion $N \subset M$ satisfies the conditions (1), (2) and (3) for 2-shifts.

If $E_{A_1}(e_1) = [M : N]^{-1}$, then $\Gamma$ is a 2-shift on the tower $(A_j)_j$ for $A$.

Next, we shall show the entropy $H(\Gamma)$ of the $*$-endomorphism $\Gamma$ of $A$ is always dominated by $\log[M : N]$.

**Lemma 13.** Let $B = A \cap N$ for von Neumann subalgebras $A$ and $N$ of a finite von Neumann algebra $M$ satisfying the commuting square condition: $E_A E_N = E_N E_A = E_B$. Then,

$$H(M \mid N) \geq H(A \mid B), \quad \lambda(M, N) \leq \lambda(A, B).$$

Let $B$ and $C$ be the von Neumann subalgebras of $A$ defined by

$$B = (\bigcup_j (M_1^j \cap M_j))'' \quad C = (\bigcup_j (M_2^j \cap M_j))''$$

**Theorem 14.** Let $\Gamma$ be the canonical shift for the inclusion $N \subset M$ of type $II_1$-factors with $[M : N] < \infty$. Then

$$H(\Gamma) = \lim_{k \to \infty} \frac{H(M' \cap M_{2k})}{k}.$$
If $E_{A_{1}}(e_{1}) = [M : N]^{-1}$, then

$$H(A | C) \leq 2H(\Gamma) \leq \log \lambda(A, C)^{-1} = 2H(M | N) = 2\log[M : N].$$

**Corollary 15.** Under the same conditions as in Theorem 14, let $A$ be a factor. Then

$$H(A | C) \leq 2H(\Gamma) \leq 2\log[A : B] = 2\log[M : N].$$

**Corollary 16.** Let $\Gamma$ be the canonical shift for the inclusion $N \subseteq M$ of type $II_{1}$-factors with $[M : N] < \infty$. If $N' \cap M = C1$, then

$$H(\Gamma) \leq H(M | N) = \log[M : N].$$

For a pair $N \subseteq M$ of hyperfinite type $II_{1}$-factors with $[M : N] < \infty$, Popa says that $N \subseteq M$ has the *generating property* if there exists a choice of the standard tunnel of subfactors $(N_{j})_{j}$ such that $M$ is generated by the increasing sequence $(N'_{j} \cap M)_{j}$.

**Corollary 17.** Assume that $N \subseteq M$ has the generating property. If $E_{N' \cap M}(e_{0}) = [M : N]^{-1}$, then

$$H(M | N) = H(\Gamma) = \log[M : N]^{-1}.$$ 

As a sufficient condition that satisfies two assumptions in Corollary 17, Ocneanu [9] introduced the following notion for a pair $N \subseteq M$ with $N' \cap M = C1$, and Popa [13] extended it to general cases. The inclusion $N \subseteq M$ of type $II_{1}$-factors with $[M : N] < +\infty$ is said to have the *finite depth* if

$$\sup_{j}(k_{j}) < +\infty$$

where $k_{j}$ is the cardinal number of simple summands of $M' \cap M_{j}$. 

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Remark 18. If the inclusion $N \subseteq M$ of type $II_1$-factors with the finite index and finite depth, then the tower $(A_j)_j$ of relative commutants satisfies the bounded growth conditions.

If an inclusion $N \subseteq M$ has the finite depth, then $E_{N \cap M}(e_0) = [M : N]^{-1}$ and $N \subseteq M$ has the generating property ([13]). Hence we have:

Corollary 19. Let $N \subseteq M$ be type $II_1$-factors with the finite index and the finite depth. Let $\Gamma$ be the canonical shift for $N \subseteq M$. Then

$$H(M \mid N) = H(\Gamma) = \log[M : N]^{-1}.$$

Remark 20. In Corollary 18, the shift $\Gamma$ is considered as an *-endomorphism of the algebra $A$ generated by the tower $(A_j)_j$ of the relative commutants of $M$. Since $N \subseteq M$ has the finite depth, the shift $\Gamma$ induces a trace preserving *-endomorphism of $M$ which transpose $M$ onto such the subfactor $P$ that $P \subseteq N \subseteq M$ is the algebraic basis extension for $P \subseteq N$. Then the *-endomorphism of $M$ has the same property as $\Gamma$.

In the last of this section, we shall show that the canonical shift has an ergodic property, which is similar to the canonical endomorphism in [7]. So that the canonical shift is a shift in the sense due to Powers [14].

Proposition 21. Let $N \subseteq M$ be type $II_1$-factors with the finite index. Then the canonical shift $\Gamma$ for $N \subseteq M$ satisfies that

$$\bigcap_k \Gamma^k(A) = C1.$$
In this section, we shall show that the canonical shift $\Gamma$ is extended to an ergodic *-automorphism $\Theta$ of a certain big von Neumann algebra such that $H(\Gamma) = H(\Theta)$.

Let $N \subseteq M$ be type $II_1$-factors with $[M : N] < \infty$. Let

$$M_{-1} = N \subseteq M = M_0 \subseteq M_1 = \langle M, e \rangle \subseteq \ldots \subseteq M_j = \langle M_{j-1}, e_{j-1} \rangle \subseteq \ldots$$

be the standard tower obtained from $N \subseteq M$. Let $M_\infty$ be the finite factor generated by the tower $(M_j)_j$.

**Proposition 22.** Let $N \subseteq M$ be type $II_1$-factors with the finite index and $\tau$ the canonical trace of $M$. Let $\sigma$ be a *-isomorphism of $M$ onto $N$. Then the following statements are equivalent:

1. There exists a *-isomorphism $\sigma_1$ of $M_1$ onto $M$ such that for all $x \in M$,

   $$\sigma_1(x) = \sigma(x).$$

2. There exists a projection $e \in M$ such that

   $$\sigma(N) = \{e\}' \cap N \quad \text{and} \quad E_N(e) = \lambda 1 = [M : N]^{-1}.$$  

3. There exists a projection $e \in M$ such that for all $y \in N$,

   $$e ye = E_{\sigma(N)}(y)e, \quad \tau(e y) = \lambda \tau(y)$$

and

$M$ is generated by $N$ and $e$ as a von Neumann algebra.

4. There exists an automorphism $\Theta$ on $M_\infty$ such that for all $x \in M$ and all $j$,

   $$\Theta(x) = \sigma(x) \quad \text{and} \quad \Theta(e_j) \in M_j.$$  

5. The decreasing sequence

   $$M \supset N \supset \sigma(N) \supset \ldots \supset \sigma^j(N) \supset \ldots$$

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Definition 4. Let $\sigma$ be a *-isomorphism of a type II$_1$-factor $M$ onto a subfactor $N$ with the finite index. If $\sigma$ satisfies the equivalent conditions in Proposition 22, then we call $\sigma$ basic *-endomorphism for the inclusion $N \subset M$.

Let $\sigma$ be the basic *-endomorphism of the inclusion $N \subset M$ of type II$_1$-factors with the finite index. Let $P_j = M \cap \sigma^j(M)'$. Then $(P_j)_j$ is an increasing sequence of finite dimensional von Neumann algebras. Let $P$ be the von Neumann algebra generated by $(P_j)_j$. Then $P$ is a von Neumann subalgebra of $M$ and we have the following:

**Proposition 23.** Let $\sigma$ be the basic *-endomorphism for the inclusion $N \subset M$ of the type II$_1$-factors with the finite index. Then,

$$H(\sigma) = \lim_{{k \to \infty}} \frac{H(M \cap \sigma^k(M)')}{k}$$

Assume that $E_{N' \cap M}(e) = [M : N]^{-1}$ for a basic projection of $\sigma(N) \subset N$. Then $\sigma^m$ is an $m$-shift on the tower $(P_j)_j$ for $P$ for all even number $m$ and satisfies the following relations. For all even $m$,

$$H(P | \sigma^m(P)) \leq 2mH(\sigma) \leq \log \lambda(P, \sigma^m(P))^{-1} = m\log[M : N]$$

**Corollary 24.** Let $\sigma$ be the same as in Proposition 23. Then

$$2H(\sigma) \leq \log[M : N].$$

Furthermore, if the inclusion $N \subset M$ has finite depth, then

$$H(M | N) = 2H_M(\sigma) = 2H(\sigma) = \log[M : N],$$

where $H_M(\sigma)$ is the entropy of $\sigma$ as a *-endomorphism of $M$. 

As an example of a basic *-endomorphism, we have the *-endomorphism $\sigma$ in Example 2.

We shall show that another good example of a basic *-endomorphisms is the canonical shift on the tower of relative commutants in §7.

**Proposition 25.** Let $M \supset N$ be type $II_1$-factors with the finite index and finite depth. Then the canonical shift $\Gamma$ for the inclusion $M \supset N$ is a basic *-endomorphism of $A = (\bigcup_j (M' \cap M_j))''$.

In [2], we proved that some kinds of *-endomorphisms are extended to ergodic *-automorphisms of big algebras with same values as entropies. Here we shall show it also holds for the canonical shifts.

Let $R$ be the von Neumann algebra which is generated by the standard tower obtained from $A \supset \Gamma(A)$ . Since $\Gamma$ is a basic *-endomorphism of $A$, there exists a *-automorphism of $R$ which is an extension of $\Gamma$. We denote it by $\Theta$.

**Theorem 26.** Let $N \subset M$ be type $II_1$-factor with finite index. Then the automorphism $\Theta$ induced by the canonical shift $\Gamma$ for the inclusion $N \subset M$ is ergodic. If $N \subset M$ has finite depth

$$H(M \mid N) = H(\Theta) = H(\Gamma) = \log[M : N]^{-1}.$$ 

**References.**


[3] A. Connes and E. Størmer : Entropy for automorphism of $II_1$ von Neumann algebras,


