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ENTROPY for CANONICAL SHIFTS

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§1. Introduction.

The notion of the entropy for *-automorphisms of finite von Neumann algebras is introduced by Connes and Størmer ([3]). In the previous paper [2], we defined the entropy for *-endomorphisms of finite von Neumann algebras as an extended version of it. It is possible to define the entropy for a general completely positive linear map α using results in [4] by a similar method to one for *-endomorphisms. However, the formula of the definition of the entropy for α implies that the entropy is apt to be zero if α^k converges to α when k tends to infinity. The conditional expectation is a trivial example of such a map α. For that reason, the interesting completely positive map α for us to discuss the entropy are those which have the property that α^k goes away from α as k tends to infinity.

In this paper, we shall study such a class of *-endomorphisms of injective finite von Neumann algebras.

In §3, we shall introduce, for a *-endomorphism σ of an injective finite von Neumann algebra A, the notion of an n-shift on the tower $(A_j)_j$ of finite dimensional von Neumann subalgebras of A which generates A and we obtain the formula of the entropy $H(\sigma)$ for an n-shift σ.

In the work [8] on the classification for subfectors of the hyperfinite type II_1-factor, Ocneanu introduced a special kind of *- endomorphism which is called the canonical shift on the tower of relative commutants. The *-endomorphism Γ is a generalization of the comultiplication for Hopf algebras and also considered as the canonical shift on the string algebras. In a part, Γ has similar properties to the canonical endomorphism of an inclusion of infinite von Neumann algebras due to Longo [7].

The canonical shift Γ naturally induces a 2-shift for the injective finite von Neumann algebra A which generated by the tower $(A_j)_j$ of relative commutants and the entropy $H(\Gamma)$ is determined by the following :

$$H(\Gamma) = \lim_{k\to\infty} \frac{H(A_{2k})}{k}.$$

For a *-endomorphism σ of a von Neumann algebra A, the entropy $H(\sigma)$ is a conjugacy invariant, that is, if there is an isomorphism θ of A onto a von Neumann algebra B such that $\theta\sigma = \phi\theta$ for a *-endomorphism ϕ of B, then $H(\sigma) = H(\phi)$. On the other hand, two conjugate *-endomorphisms σ and ϕ of A give two conjugate von Neumann subalgebras $\sigma(A)$ and $\phi(A)$ under automorphisms of A.

In [9], Pimsner and Popa introduced two conjugacy invariants for von Neumann subalgebras. One is the relative entropy $H(A \mid B)$ for a von Neumann subalgebra B of a finite von Neumann algebra A, which is defined an extended version of one for finite dimensional algebras due to Connes-Størmer [3]. The other is the constant $\lambda(A, B)$, which plays a role like the index for subfactors due to Jones [6]. In fact in the case of factors $B \subset A$, $\lambda(A, B)^{-1}$ is Jones index $[A : B]$.

We shall investigate relations among those invariants.

In §4, we restrict our attension to finite dimensional von Neumann algebras. We need those results later . The Jones index for a subfactor N of a finite factor M is given as $1/\tau(e)$ for the projection e of $L^2(M)$ onto $L^2(N)$ where τ is the trace on the basic extension algebra of $N \subset M$. In the case of finite dimensional von Neumann algebras, we shall show that the constant $\lambda(\ ,\ )^{-1}$ coinsides with Jones index in such a sense.

In §5, it is obtained that in general the following relation holds for an n-shift σ :

$$H(A \mid \sigma(A)) \leq 2H(\sigma).$$

A condition that the equality holds is also given.

In §6, we shall obtain the relation between $H(\sigma)$ and the constant $\lambda(A, \sigma(A))$. We shall define a locally standard tower for an increasing sequence $(A_j)_j$ of finite dimensional von Neumann algebras. The tower $(A_j)_j$ of relative commutants for the inclusion of finite factors $N \subset M$ satisfies this condition. If a *-endomorphism σ of A is an n-shift on a locally standard tower which generates A, then we have the following :

$$H(A \mid \sigma(A)) \leq 2H(\sigma) \leq \log \lambda(A, \sigma(A))^{-1}.$$

In §7, we shall apply the above results to the canonical shift Γ for the tower of relative commutants. Let $N \subset M$ be type II_1-factors with the finite index. Considering the tower $(M_j)_j$ of factors obtained by iterating Jones basic construction from $N \subset M$, it is obtained the increasing sequence$(A_j)_j$ of finite dimensional von Neumann algebras, where $A_j = M' \cap M_j$. The *-endomorphism Γ is defined on the algebra $\bigcup_j A_j$ as a mapping such that $\Gamma(M_k' \cap M_j) = M_{k+2}' \cap M_{j+2}$ for all $k \leq j$. First, we remark that Γ is extended to the trace preserving *-endomorphism of a finite von Neumann algebra $A = \bigcup_j (A_j)''$. Then Γ has an ergodic property that

$$\bigcap_k \Gamma^k(A) = C1$$

and satisfies the conditions of Definition for a 2-shift, except only one. In order that Γ satisfies all conditions for 2-shifts, a condition for the inclusion $N \subset M$ is necessary. For example, in the case where $N' \cap M = C1$, Γ is a 2-shift and the following relation holds :

$$H(A \mid \Gamma(A)) \leq 2H(\Gamma) \leq 2\log[M : N].$$

Furthermore, if the inclusion $N \subset M$ has finite depth ([8], [12]), then we have :

$$H(M \mid N) = H(\Gamma) = \log[M : N]^{-1}.$$

In §8, we shall discuss conditions for a *-endomorphism σ of a factor M to be extended to an automorphism θ of a factor containing M so that $H(\sigma) = H(\theta)$. If the inclusion $N \subset M$ has finite depth, then Γ is extended to an ergodic *-automorphism Θ which satisfies the following :

$$H(M \mid N) = H(\Theta) = H(\Gamma) = \log[M : N]^{-1}.$$

§2. **Preliminaries.**

In this section, we shall fix the notations and terminologies frequently used in this paper.

Throughout this section, M will be a finite von Neumann algebra with a fixed normal faithful trace $\tau, \tau(1) = 1$. The inner product $< x, y > = \tau(xy^*)$ gives M as a vector space the structure of a pre-Hilbert space. Let $\|x\| = \tau(x^*x)^{1/2}$ and $L^2(M,\tau)$ the Hilbert space completion of M. Then M acts on $L^2(M,\tau)$ by the left multiplication. The canonical conjugation on $L^2(M,\tau)$ is denoted by $J = J_M$. It is the conjugate unitary map induced by the involution * on M. For a von Neumann subalgebra N of M, let e_N be the orthogonal projection of $L^2(M,\tau)$ onto $L^2(N,\tau)$. Then the restriction E_N of e_N to M is the faithful normal conditional expectation of M onto N.

The letter η designates the function on $[0,\infty)$ defined by $\eta(t) = -t\log t$. For each k, we let S_k be the set of all families $(x_{i_1,i_2,\ldots,i_k})_{i_j\in N}$ of positive elements of M, zero except for a finite number of indices and satisfying

$$\sum_{i_1,\ldots,i_j,\ldots i_k} x_{i_1,\ldots,i_k} = 1.$$

For $x \in S_k$, $j \in 1,2,\ldots,k$ and $i_j \in N$, put

$$x^j_{i_1} = \sum_{i_1,\ldots,i_{j-1},i_{j+1},\ldots,i_k} x_{i_1,i_2,\ldots,i_k}.$$

Let $N_1, N_2, \ldots, N_k$ be finite dimensional von Neumann subalgebras of M. Then

$$H(N_1,\ldots,N_k) = Sup_{x\in S_k}[\sum_{i_1,\ldots,i_k} \eta\tau(x_{i_1,\ldots i_k}) - \sum_j\sum_{i_j} \tau\eta E_{N_j}(x^j_{i_j})].$$

Let σ be a τ-preserving *-endomorphism of M and N a finite dimensional von Neumann subalgebra of M, then

$$H(N,\sigma) = \lim_{k\to\infty} \frac{1}{k} H(N, \sigma(N), ..., \sigma^{k-1}(N))$$

exists by [2]. The entropy $H(\sigma)$ for σ is defined as the supremum of $H(N,\sigma)$ for all finite dimensional subalgebras N of M.

If there exists an increasing sequence $(N_j)_j$ of finite dimensional subalgebras which generates M, then by [2]

$$H(\sigma) = \lim_{j\to\infty} H(N_j, \sigma).$$

The relative entropy $H(M \mid N)$ for a von Neumann subalgebra N of M is defined ([10]) as an extension form of one ([3]) by

$$H(M \mid N) = Sup_{x\in S_1} \sum_i [\tau\eta(x_i) - \tau\eta E_N(x_i)].$$

This H($M \mid N$) is a conjugacy invariant for subalgebras of M. Another conjugacy invariant $\lambda(M,N)$ is introduced in [10] as a generalization of Jones index defined by

$$\lambda(M,N) = \max\{\lambda \geq 0; E_N(x) \geq \lambda x, x \in M_+\}.$$

For an inclusion $N \subset M$ of finite von Neumann algebras, the von Neumann algebra on $L^2(M,\tau)$ generated by M and $e = e_N$ is called the standard basic extension (or basic construction) for $N \subset M$ and denoted by $M_1 =< M, e >$. Then by the properties of $J = J_M$ and $e = e_N$, we have $M_1 =< M, e >= JN'J$([6]). If M_1 is finite and if there is a trace τ_1 on M_1 such that $\tau_1(xe) = \lambda\tau(x)$ far all $x \in M$, then the trace τ_1 is called the λ-Markov trace for $N \subset M$. If $M \supset N$ are factors and there is the λ -Markov trace of M_1 for $N \subset M$, then Jones index $[M : N] = \lambda^{-1}$ ([6]).

We shall call an increasing sequence $(M_j)_{j\in N}$ of von Neumann algebras a *standard tower* (cf. [5], [9], [13]) if $M_{j-1} \subset M_j \subset M_{j+1}$ is the basic construction obtained from $M_{j-1} \subset M_j$ for each j.

Let L be a finite factor containing M. We shall call L the algebraic basic construction for the factors $N \subset M$ if there is a non zero projection $e \in M$ satisfying :

(i) $exe = E_N(x)e$ for $x \in M$

and

(ii) L is generated by e and M as a von Neumann algebra.

In this case, there is an isomorphism ϕ of M_1 onto L such that $\phi(e_N) = e$ and $\phi(x) = x$ for all $x \in M$ ([11]).

We shall call such a projection e the *basic projection* for $N \subset M$ and a decreasing sequence $(N_j)_{j \in N}$ of finite factors a *standard tunnel* (cf. [5], [9], [13]) if $N_{j-1} \supset N_j \supset N_{j+1}$ is the algebraic basic construction for $N_j \supset N_{j+1}$ for each j.

§3. Entropy of n -shift

In this section, we shall give the definition of n-shifts and a formula of the entropy for n-shifts. Let A be an injective finite von Neumann algebra with a fixed faithful normal trace τ, with $\tau(1) = 1$. Let $(A_j)_{j=1,2,..}$ be an increasing sequence of finite dimensional von Neumann algebras such that $A =$ the weak closure of $\bigcup_j A_j = \{A_j : j\}''$. Assume that σ is a τ-preserving *-endomorphism of A. Then σ is a ultra-weakly continuous, one to one mapping with $\sigma(1) = 1$.

Definition 1. Let n be a natural number. A τ-preserving *-endomorphism σ of A is called an n-shift on the tower $(A_j)_j$ for A if the following conditions are satisfied:

(1) For all j and m, the von Neumann algebra $\{A_j, \sigma(A_j), ..., \sigma^m(A_j)\}''$ generated by $\{\sigma^j(A_j); j = 0, ..., m\}$ is contained in A_{j+nm}.

(2) There exists a sequence $(k_j)_{j \in N}$ of integers with the properties:

$$\lim_{j \to \infty} \frac{nk_j - j}{j} = 0$$

and

$$x\sigma^m(y) = \sigma^m(y)x, \quad \tau(z\sigma^{lk_j}(x)) = \tau(z)\tau(x)$$

for all $l \in \mathbf{N}$, $x, y \in A_j$, $m \in k_jN$ and $z \in \{A_j, \sigma^{k_j}(A_j), ..., \sigma^{(l-1)k_j}(A_j)\}''$.

(3) Let E_B be the conditional expectation of A onto a von Neumann subalgebra B of A. Then for an $j \geq n$

$$E_{A_j}E_{\sigma(A_j)} = E_{\sigma(A_{j-n})}$$

(4) For each j, there exists a τ-preserving *-automorphism or antiautomorphism β of A_{nj+n} such that $\sigma(A_{nj}) = \beta(A_{nj})$.

Remark 1. The number n of an n-shift depends on the choice of the sequence $(A_j)_j$. Every given n-shift can be 1-shift on a suitable tower for the same von Neumann algebra.

Example 1. Let S be the *-endomorphism corresponding to the translation of 1 in the infinite tensor product $R = \bigotimes_{i \in N}(M_i, tr_i)$of the algebra M_i of $m \times m$ matrices with the normalized trace tr_i on M_i for each $i \in N$. For each j, let $A_j = \bigotimes_{i=1}^{j}(M_i, tr_i)$. Then for all n, S^n is an n-shift on the tower $(A_j)_j$ for R.

In fact, for an $n \in \mathbf{N}$, let $k_j = [\frac{j}{n}] + 1$. Then $(k_j)_j$ satisfies the following properties (2') which are stronger than (2):

$$\lim_{j \to \infty} \frac{nk_j - j}{j} = 0$$

and

$$x\sigma^m(y) = \sigma^m(y)x, \quad \tau(z\sigma^{lk}(x)) = \tau(z)\tau(x)$$

for all $l \in \mathbf{N}$, $x, y \in A_j$, $k_j \leq k$, $m \in N$ and $z \in \{A_j, \sigma^k(A_j), ..., \sigma^{k(l-1)}(A_j)\}''$. It is obvious that another conditions are satisfied by S^n.

Example 2. Let $(e_j)_j$ be the sequence of projections with the following properties for some natural number k and $\lambda \in (0, 1/4] \cup \{1/(4cos^2(\pi/n); n \geq 3\}$:

(a) $e_i e_j e_i = \lambda e_i$ if $|i - j| = k$

(b) $e_i e_j = e_j e_i$ if $|i - j| \neq k$

(c) $(e_j)_j$ generates the hyperfinite type II_1-factor R

(d) $\tau(we_i) = \lambda\tau(w)$ for the trace τ of R and a reduced word w on $\{1, e_1, ..., e_{i-1}\}$.

Let A_j be the von Neumann algebra generated by $\{e_1, ..., e_j\}$. Then, by [6], A_j is finite dimensional. Let σ be the *-endomorphism of R such that $\sigma(e_i) = e_{i+1}([1])$. Then σ^n is an n-shift on the tower $(A_j)_j$ of R for all n. In fact, for an $n \in N$, let $k_j = [\frac{i+k}{n}] + 1$. Then $(k_j)_j$ satisfies properties (2') in Example 1. The condition (3) and (4) are satisfied by using results by [6] and [1].

In §7, we shall show that the canonical shift due to Ocneanu is a 2-shift on the tower of relative commutant algebras.

Theorem 1. If a τ-preserving *-endomorphism σ of A satisfies the condition (1) and (2) in Definition 1 for the tower $(A_j)_j$ of A, then

$$H(\sigma) = \lim_{k \to \infty} \frac{H(A_{nk})}{k}.$$

§4. Finite dimensional algabras.

In this section, M will be a finite dimensional von Neumann algebra and τ a fixed faithful normal trace of M with $\tau(1) = 1$. Then M is decomposed into the direct summand:

$$M = \sum_{l \in K} \bigoplus M_l$$

where M_l is the algebra of $d(l) \times d(l)$ matrices and $K = K_M$ is a finite set. Then the vector $d_M = d = (d(l))_{l \in K}$ is called the *dimension vector* of M. The column vector

$t_M = t = (t(l))_{l\in K}$ has $t(l)$ as the value of the trace for the minimal projections in M_l, and is called the *trace vector* of τ. Let N be a von Neumann subalgebra of M with $N = \sum_{k\in K_N} \bigoplus N_k$. The *inclusion matrix* $[N \hookrightarrow M] = (m(k,l))_{k\in K_N, l\in K_M}$ is given by the number $m(k,l)$ of simple components of a simple M_l module viewed as an N_k module. Then

$$d_N[N \hookrightarrow M] = d_M \quad and \quad [N \hookrightarrow M]t_M = t_N.$$

Here we shall give a simple formula for $\lambda(M,N)$.

By the definition of the basic construction of $N \subset M$, there is a natural isomorphism between the centers of N and $< M,e >$ via $x \to JxJ$. Hence there is a natural identification between the sets of simple summands of N and $< M,e >$. We put $K = K_N = K_{<M,e]>}$.

The following theorem assures that in the case of finite dimensional von Neumann algebras, the constant $\lambda(\ .\ )$ plays the same role as the index for finite factors.

Theorem 2. (1) Assume that there is a trace of $< M,e >$ which is an extension of τ. Then

$$\lambda(< M,e >, M)^{-1} = \max_{k\in K} \frac{t_N(k)}{t_{<M,e>}(k)}.$$

(2) If the trace τ of $< M,e >$ has the $\tau(e)$- Markov property, then

$$\lambda(< M,e >, M)^{-1} = 1/\tau(e) = \| [N \hookrightarrow M] \|^2 .$$

Definition 2. Let $N \subset M \subset L$ be an inclusion of finite dimensional von Neumann algebras. Then L is said to be the *algebraic basic construction* for $N \subset M$ if there is a projection e in L satisfying

(a) L is generated by M and e,

(b) $xe = ex$ for an $x \in N$,

(c) If $x \in N$ satisfies $xe = 0$, then $x = 0$,

(d) $exe = E_N(x)e$ for all $x \in M$.

In this case, there is a *-isomorphism of the basic construction $M_1 = JN'J$ onto L.

We shall call $N \subset M \subset L$ a *locally algebraic basic extension* of $N \subset M$ if there is a projection $p \in L \cap L'$ which satisfies that the inclusion $M \subset Lp$ is the algebraic basic construction for $N \subset M$.

If $L \supset M \supset N$ is a locally standard extension of the inclusion $M \supset N$, we can identity the set K_N with a subset of K_L via the equality $Ne = eLe$. Under such an identification, we have the following:

Proposition 3. Let $L \supset M \supset N$ be a locally standard extension of $M \supset N$. Then

$$\lambda(L, M)^{-1} \geq \max_{k \in K_N} \min_{l \in K_L} \frac{t_N(k)}{t_L(l)}.$$

Let

$$I(M) = \sum_{l \in K} d(l)t(l) \log \frac{d(l)}{t(l)},$$

where $K = K_M$, $d = d_M$ and $t = t_M$.

proposition 4.

(i) $H(M \mid N) \leq I(M) - I(N)$

(ii) $H(< M, e > \mid M) = I(< M, e >) - I(M)$

(iii) $I(M) \leq 2H(M)$ and the equality holds if and only if M is a factor.

§5. $H(\sigma)$ and $H(A \mid \sigma(A))$

In this section we shall investigate a relation between $H(\sigma)$ and $H(A \mid \sigma(A))$ for an n-shift σ on the tower $(A_j)_j$ for a finite von Neumann algebra A.

Let $(A_j)_j$ be an increasing sequence of finite dimensional von Neumann algebras. Let $A_j = \sum_{k \in K_j} \oplus A_j(k)$ be such a decomposition as in §4, and d_j the dimension vector of A_j. Then we shall say $(A_j)_j$ satisfies *the bounded growth conditions* ([2]) if the following two conditions are satisfied :

(i)

$$\sup_j \frac{|(K_j)|}{j} < +\infty$$

and

(ii) For some m, $A_{j+1}(l)$ contains at most $d_j(k)$ $A_j(k)$- components for all $j \geq m$, where $|(K_j)|$ is the cardinal number of K_j.

For examples, let us consider two towers which are treated in Examples 1 and 2 . Both of them satisfies the bounded growth conditions ([2]). We shall discuss another example in §7.

Theorem 5. Let σ be a τ-preserving *-endomrphism of an injective finite von Neumann algebra A with a faithful normal trace τ, $\tau(1) = 1$. If σ is an n-shift on the tower $(A_j)_j$ for A, then

$$H(A \mid \sigma(A)) \leq 2H(\sigma).$$

Furthermore, if the bounded growth conditions are satisfied, for the tower $(A_{nj})_j$

$$H(A \mid \sigma(A)) = 2H(\sigma).$$

In order to prove Theorem 5, we need the following:

Lemma 6. Let σ be the same as in Theorem 5. If σ satisfies the conditions (1), (3) and (4) in Definition 1 for n, then

$$H(A \mid \sigma(A)) = \lim_{j \to \infty} H(A_{nj+n} \mid A_{nj}).$$

By considering the standard tower

$$N \subset M \subset M_1 \subset M_2 \subset \ldots \subset M_n =< M_{n-1}, e_{n-1} > \subset \ldots$$

obtained from the pair $N \subset M$ of II_1-factors with $[M : N] < \infty$ by iterating the basic construction, it is proved in [11] that $H(M_n \mid N) = \log[M_n : N]$ if $H(M \mid N) = \log[M : N]$. Since the index has the multiplicative property ([6]), it implies that $H(M_n \mid N) = nH(M \mid N)$ if $H(M \mid N) = \log[M : N]$. Next corollary shows a similar result holds for the pair $\sigma(M) \subset M$.

Corollary 7. Let a *-endomorphism σ satisfy the same condition as in Theorem 5. Then for all n

$$H(A \mid \sigma^n(A)) = nH(A \mid \sigma(A)).$$

§6. $H(\sigma)$ and $\lambda(A, \sigma(A))$ for n-shift σ.

In this section, we shall investigate relations between the entropy $H(\sigma)$ and the constant $\lambda(A, \sigma(A))$ for an n-shift σ of the tower $(A_j)_{j \in N}$ for a finite von Neumann algebra **A** with a fixed faithful normal trace τ, $\tau(1) = 1$.

Definition 3. We shall call an increasing sequence $(A_j)_j$ of finite dimensional von Neumann subalgabras of a finite von Neumann algebra **A** with a faithful normal trace τ a *lacally standard tower* for α if there exists a natural number k which satisfies the following conditions :

1) For a certain central projection $p_{k(j+1)}$ of $A_{k(j+1)}$,the inclusion matrix $[A_{jk} \hookrightarrow A_{k(j+1)} p_{j+1}]$ is the transpose of $[A_{k(j-1)} \hookrightarrow A_{kj}]$, for each j .

2) If $(t_{k(j-1)}(i))_i$ is the trace vector for the restriction of τ to $A_{k(j-1)}$, then the value of τ of the minimal projections for $A_{k(j+1)}p_{k(j+1)}$ are given by $(\alpha t_{k(j-1)}(i))_i$ for each j .

3)There is an $c > 0$ such that $H(A_{2kj}) \leq c - j \log \alpha$ for each j.

We call the number $2k$ a *period* of the locally standard tower.

As the examples of locally standard towers , we have followings :

(i). The tower $(A_j)_j$ in Example 1 is obviously a locally standard tower for $1/m$,because the inclusion matrix in each step are all same.

(ii). The standard tower is a locally standard tower for $\| T^t T \|^{-1}$, because the inclusion matrix in the j-th step is the transpose of one in the $(j-1)$-th step for all j ([6]). Hence the tower $(A_j)_j$ is also locally standard if A_{j+1} is a locally algebraic basic extension of $A_{j-1} \subset A_j$.

(iii). The tower $(A_j)_j$ in Example 2 is a locally standard tower for λ, because the central support of e_j in A_j satisfies the condition (1) and (2) in Definition 3 and the condition (3) are proved by results in § 4.2 and § 5.1 in [6].

We shall treat another locally standard tower in the next section.

Theorem 8. Let A be a finite von Neumann algebra with a fixed faithful normal trace τ, $\tau(1) = 1$. Let σ be an n-shift on the locally standard tower $(A_j)_j$ for α with a period $2n$, then

$$H(A \mid \sigma(A)) \leq 2H(\sigma) \leq -\log \alpha \leq \log \lambda(A, \sigma(A))^{-1}$$

The author would like to her hearty thanks to F.Hiai for pointing out a mistake in the proof of Theorem 8 in the preliminary version.

Corollary 9. Let A be an injective finite factor with the canonical trace τ and σ an n-shift of a locally standard tower for A with a period $2n$, then

$$H(A \mid \sigma(A)) \leq 2H(\sigma) \leq \log[A : \sigma(A)].$$

In the case of a factor A, it is obtained in [10] equivalent conditions that $H(A \mid \sigma(A)) = log[A : \sigma(A)]$. In such a case, we have

$$H(A \mid \sigma(A)) = 2H(\sigma) = log[A : \sigma(A)].$$

For example, the shifts S in Example 1 and σ for $\lambda > (1/4)$ in Example 2 satisfy the equality ([2]). However, the shifts σ in Example 2 have the following relation([2]):

$$H(R \mid \sigma(R)) = 2H(\sigma) < \log[R : \sigma(R)]$$

if $\lambda \leq (1/4)$.

§7 Canonical shift .

In [9], Ocneanu defined a very nice *-endomorphism for the tower of the relative commutant algebras for the inclusion $N \subset M$ of type II_1-factors with the finite index.

At first, we shall recall from [9] the definition and main properties of the canonical shift on the tower of relative commutants.

Let M be a finite factor with the canonical trace τ and N a subfactor of M such that $[M : N] < +\infty$. Then the basic extension $M_1 =< M, e >$ is a II_1-factor with the $\lambda = [M : N]^{-1}$- Markov trace ([6]) and there are the family $\{m_i\} \subset M$ which forms an "orthonormal basis" in M with respect to the N valued inner product $E_N(xy^*)(x, y \in M)$, that is, each $x \in M$ is decomposed in the unique form as the following ([9], [10]):

$$x = \sum_i E_N(m_i^* x) m_i.$$

Iterating the basic construction from $N \subset M$, we have the standard tower

$$M_{-1} = N \subset M_0 = M \subset M_1 =< M_0, e_0 >\subset M_2 \subset ...$$

in which, e_j is the projection of $L^2(M_j, \tau_j)$ onto $L^2(M_{j-1}, \tau_{j-1})$, where τ_j is the λ- Markov trace for M_j. Then from the family $(e_j)_j$ the projection $e(n,k)$ is obtained and

$$M_{n-k} \subset M_n \subset M_{n+k} =< M_n, e(n,k) >$$

is an algebraic basic extension ([9], [11]). Furthermore it is obtained in [9] that the "orthonormal basis" in M_n with respect to M_{n-k} valued inner product from the family of the basis in $(M_j)_j$.

Let $A_j = M' \cap M_j$ for all j. The antiautomorphism γ_j of $A_{2j} = M' \cap M_{2j}$ defined by

$$\gamma_j(x) = J_j x^* J_j, \quad x \in A_{2j}$$

is called the *mirroring* , where J_j is the conjugate unitary on $L^2(M_j, \tau_j)$. Then for all $x \in M' \cap M_{2j}$, the following expression of the mirrorings is given :

$$\gamma_j(x) = [M_j : M] \sum_i E(em_i^* x) em_i,$$

where E is the conditional expectation of M_j onto M, e is the projection of $L^2(M_j)$ onto $L^2(M)$ and $(m_i)_i$ a module basis of M_j over M. The expression implies that the mirrorings satisfies the following relation:

$$\gamma_{j+1} \cdot \gamma_j = \gamma_j \cdot \gamma_{j-1}$$

for all $j \geq 1$ on A_{2j-1}. In the view of this relation, the endomorphism Γ of $\bigcup_n A_n$ can be defined by

$$\Gamma(x) = \gamma_{j+1}(\gamma_j(x)),$$

for $x \in A_{2j}$. Ocneanu called the endomorphism Γ the *canonical shift* on the tower of the relative commutants. In the case of inclusions of infinite factors, similar *-endomorphisms are investigated by Longo [8]. The mapping Γ has the following properties ; for any $k, n \geq 0$ with $n \geq k$,

$$\Gamma(M_k' \cap M_n) = {M_{k+2}}' \cap M_{n+2}.$$

Now, we shall consider the finite von Neumann algebra A generated by the tower $(A_j)_j$ and extend Γ to a trace preserving *-endomorphism of A as follows.

Since $N \subset M$ are II_1-factors with $[M : N] < +\infty$, there is a faithful normal trace on $\bigcup_j M_j$ which extends the canonical trace τ on M. We denote the trace by the same notation τ.

Although M_{j+1} is defined as a von Neumann algebra on $L^2(M_j, \tau_j)$, each M_j can be considered as von Neumann algebras on the Hilbert space $L^2(M, \tau)$. Hence $\bigcup A_j$ and $\bigcup M_j$ can ce considered as von Neumann algebras acting on $L^2(M, \tau)$. Let

$$M_\infty = \{\bigcup_j M_j\}'', \quad A = \{\bigcup_j A_j\}''.$$

Then M_∞ is a finite factor with the canonical trace which is the extension of τ. We denote it by the same notation τ. Then A ia a von Neumann subalgebra of M_∞. Since Γ is a ultra-weakly continuous endomorphism of $\bigcup_j A_j$, Γ is extended to a *-endomorphism of A.

Although, in the case where discussed by Ocneane, for all k, the mirroring γ_k is a trace preserving map thanks to the assumption $N' \cap M = \mathbf{C}1$, in general, the mirrorings are not always trace preserving . However the canonical shift is always trace preserving :

Lemma 10. For every k, $\gamma_{k+1} \cdot \gamma_k$ is a τ- preserving isomorphism of $M' \cap M_{2k}$ onto $M_2' \cap M_{2k+2}$.

Furthermore, if $E_{A_1}(e_1) = \lambda$ (for example $N' \cap M = \mathbf{C}1$), then γ_j is a trace preserving antiautomorphism of A_{2j} for all j.

By Lemma 10, the canonical shift Γ on the tower of the relative commutants $(A_j)_j$ of

M is extended to a τ-preserving *-endomorphism of A. We shall call the *-endomorphism of A the *canonical shift* for the inclusion $M \supset N$ and denote it by the same notation Γ.

We shall show the canonical shift Γ is a 2-shift on the tower $(A_j)_j$ for A.

Lemma 11. Let L be a finite von Neumann algebra with a faithful normal trace τ, $\tau(1) = 1$. If M is a subfactor of L , then

$$\tau(xy) = \tau(x)\tau(y) \quad (x \in M,\ y \in M' \cap L).$$

Proposition 12. The canonical shift Γ for the inclution $N \subset M$ satisfies the conditions (1), (2) and (3) for 2-shifts.

If $E_{A_1}(e_1) = [M : N]^{-1}$, then Γ is a 2-shift on the tower $(A_j)_j$ for A.

Next, we shall show the entropy $H(\Gamma)$ of the *-endomorphism Γ of A is always dominated by $\log[M : N]$.

Lemma 13. Let $B = A \cap N$ for von Neumann subalgebras A and N of a finite von Neumann algebra M satisfying the commuting square condition : $E_A E_N = E_N E_A = E_B$. Then,

$$H(M \mid N) \geq H(A \mid B)\ , \quad \lambda(M,N) \leq \lambda(A,B).$$

Let B and C be the von Neumann subalgebras of A defined by

$$B = (\bigcup_j (M_1' \cap M_j))''\ ,\ C = (\bigcup_j (M_2' \cap M_j))''$$

Theorem 14. Let Γ be the canonical shift for the inclusion $N \subset M$ of type II_1-factors with $[M : N] < \infty$. Then

$$H(\Gamma) = \lim_{k\to\infty} \frac{H(M' \cap M_{2k})}{k}.$$

If $E_{A_1}(e_1) = [M:N]^{-1}$, then

$$H(A \mid C) \leq 2H(\Gamma) \leq \log \lambda(A,C)^{-1} = 2H(M \mid N) = 2\log[M:N].$$

Corollary 15. Under the same conditions as in Theorem 14 , let A be a factor. Then

$$H(A \mid C) \leq 2H(\Gamma) \leq 2\log[A:B] = 2\log[M:N].$$

Corollary 16. Let Γ be the canonical shift for the inclusion $N \subset M$ of type II_1-factors with $[M:N] < \infty$. If $N' \cap M = \mathbf{C}1$, then

$$H(\Gamma) \leq H(M \mid N) = \log[M:N].$$

For a pair $N \subset M$ of hyperfinite type II_1-factors with $[M:N] < \infty$, Popa says that $N \subset M$ has the *generating property* if there exists a choice of the standard tunnel of subfactors $(N_j)_j$ such that M is generated by the increasing sequence $(N_j' \cap M)_j$.

Corollary 17. Assume that $N \subset M$ has the generating property. If $E_{N' \cap M}(e_0) = [M:N]^{-1}$, then

$$H(M \mid N) = H(\Gamma) = \log[M:N]^{-1}.$$

As a sufficient condition that satisfies two assumptions in Corollary 17, Ocneanu [9] introduced the following notion for a pair $N \subset M$ with $N' \cap M = \mathbf{C}1$, and Popa [13] extended it to general cases. The inclusion $N \subset M$ of type II_1-factors with $[M:N] < +\infty$ is said to have the *finite depth* if

$$\sup_j (k_j) < +\infty$$

where k_j is the cardinal number of simple summunds of $M' \cap M_j$.

Remark 18. If the inclusion $N \subset M$ of type II_1-factors with the finte index and finite depth, then the tower $(A_j)_j$ of relative commutants satisfies the bounded grouwth conditions.

If an inclusion $N \subset M$ has the finite depth, then $E_{N' \cap M}(e_0) = [M : N]^{-1}$ and $N \subset M$ has the generating property ([13]). Hence we have :

Corollary 19. Let $N \subset M$ be type II_1-factors with the finite index and the finite depth. Let Γ be the canonical shift for $N \subset M$. Then

$$H(M \mid N) = H(\Gamma) = \log[M : N]^{-1}.$$

Remark 20. In Corollary 18, the shift Γ is considered as an *-endomorphism of the algebra A generated by the tower $(A_j)_j$ of the relative commutants of M. Since $N \subset M$ has the finite depth, the shift Γ induces a trace preserving *-endomorphism of M which transpose M onto such the subfactor P that $P \subset N \subset M$ is the algebraic basis extension for $P \subset N$. Then the *-endomorphism of M has the same property as Γ.

In the last of this section, we shall show that the canonical shift has an ergodic property, which is similar to the canonocal endomorphism in [7]. So that the canonical shift is a shift in the sense due to Powers [14].

Proposition 21. Let $N \subset M$ be type II_1-factors with the finite index. Then the canonical shift Γ for $N \subset M$ satisfies that

$$\bigcap_k \Gamma^k(A) = C1.$$

§8 Extension of canonical shift.

In this section, we shall show that the canonical shift Γ is extended to an ergodic *-automorphism Θ of a certain big von Neumann algebra such that $H(\Gamma) = H(\Theta)$.

Let $N \subset M$ be type II_1-factors with $[M : N] < \infty$. Let

$$M_{-1} = N \subset M = M_0 \subset M_1 =< M, e >\subset ... \subset M_j =< M_{j-1}, e_{j-1} >\subset ...$$

be the standard tower obtained from $N \subset M$. Let M_∞ be the finite factor generated by the tower $(M_j)_j$.

Proposition 22. Let $N \subset M$ be type II_1-factors with the finite index and τ the canonical trace of M. Let σ be a *-isomorphism of M onto N. Then the following statements are equivalent :

(1) There exists a *-isomorphism σ_1 of M_1 onto M such that for all $x \in M$,

$$\sigma_1(x) = \sigma(x).$$

(2) There exists a projection $e \in M$ such that

$$\sigma(N) = \{e\}' \cap N \quad and \quad E_N(e) = \lambda 1 = [M : N]^{-1}.$$

(3) There exists a projection $e \in M$ such that for all $y \in N$,

$$eye = E_{\sigma(N)}(y)e \quad , \quad \tau(ey) = \lambda\tau(y)$$

and

M is generated by N and e as a von Neumann algebra.

(4) There exists an automorphism Θ on M_∞ such that for all $x \in M$ and all j,

$$\Theta(x) = \sigma(x) \quad and \quad \Theta(e_j) \in M_j.$$

(5) The decreasing sequence

$$M \supset N \supset \sigma(N) \supset ... \supset \sigma^j(N) \supset ...$$

is a standard tunnel.

Definition 4. Let σ be a *-isomorphism of a type II_1-factor M onto a subfactor N with the finite index. If σ satisfies the equivalent conditions in Proposition 22, then we call σ basic *-endomorphism for the inclusion $N \subset M$.

Let σ be the basic *-endomorphism of the inclusion $N \subset M$ of type II_1-factors with the finite index. Let $P_j = M \cap \sigma^j(M)'$. Then $(P_j)_j$ is an increasing sequence of finite dimensional von Neumann algebras. Let P be the von Neumann algebra generated by $(P_j)_j$. Then P is a von Neumann subalgebra of M and we have the following :

Proposition 23. Let σ be the basic *-endomorphism for the inclusion $N \subset M$ of thpe II_1-factors with the finite index. Then,

$$H(\sigma) = \lim_{k \to \infty} \frac{H(M \cap \sigma^k(M)')}{k}$$

Assume that $E_{N' \cap M}(e) = [M : N]^{-1}$ foa a basic projection of $\sigma(N) \subset N$. Then σ^m is a m-shift on the tower $(P_j)_j$ for P for all even number m and satisfies the following relations. For all even m,

$$H(P \mid \sigma^m(P)) \leq 2mH(\sigma) \leq \log \lambda(P, \sigma^m(P))^{-1} = m \log[M : N]$$

Corollary 24. Let σ be the same as in Proposition 23. Then

$$2H(\sigma) \leq \log[M : N].$$

Fuethermore, if the inclusion $N \subset M$ has hinite depth, then

$$H(M \mid N) = 2H_M(\sigma) = 2H(\sigma) = \log[M : N],$$

where $H_M(\sigma)$ is the entropy of σ as a *-endomorphism of M.

As an example of a basic *-endomorphism, we have the *-endomorphism σ in Example 2.

We shall show that another good example of a basic *-endomorphisms is the canonical shift on the tower of relarive commutants in §7.

Proposition 25. Let $M \supset N$ be type II_1-factors with the finite index and finite depth. Then the canonical shift Γ for the inclusion $M \supset N$ is a basic *-endomorphism of $A = (\bigcup_j (M' \cap M_j)_j)''$.

In [2], we proved that some kinds of *-endomorphisms are extended to ergodic *-automorphisms of big algebras with same values as entropies. Here we shall show it also holds for the canonical shifts.

Let R be the von Neumann algebra which is generated by the standard tower obtained from $A \supset \Gamma(A)$. Since Γ is a basic *-endomorphism of A, there exists a *-automorphism of R which is an extension of Γ. We denote it by Θ.

Theorem 26. Let $N \subset M$ be type II_1-factor with finite index. Then the automorphism Θ induced by the canonical shift Γ for the inclusion $N \subset M$ is ergodic. If $N \subset M$ has finite depth

$$H(M \mid N) = H(\Theta) = H(\Gamma) = \log[M : N]^{-1}.$$

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