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# Operator Theoretical Approach to Selberg Inequality

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## §0. Introduction.

In any first course of functional analysis, every student will learn the Schwarz' inequality. The inequality is so fundamental that there are several variants and generalizations. Among them, Bessel's generalization of the Schwarz' inequality is essential in studying the geometry of Hilbert spaces. In fact, Bessel's inequality concerns the dimensions of Hilbert spaces, or the cardinality of orthonormal systems.

It is often pointed out in the course of functional analysis that the orthogonality is convenient in comparison with other spaces like Banach spaces. And the orthogonality is essential in studying harmonic analysis, which concerns the decomposition of any oscillation into harmonic ones $\exp in\theta$. These harmonic ones are mutually orthogonal, while non harmonic ones $\exp i\lambda_n\theta$ not.

On the other hand, the analytic number theorists have been a user of the geometry of Hilbert space and they developed the theory of *almost or-*

*thonormal* systems to study sieve method. R.P. Boas [2] and R. Bellman [1] are the pioneers of those studies. Their inequalities include Bessel's one. The most useful of these is the inequality Selberg found:

THEOREM S (cf. [3]). *If $x_1, x_2, \ldots, x_n$, and $x$ are non zero vectors in an inner product space, then*

$$\sum_{i=1}^{n} \frac{|<x \mid x_i>|^2}{\sum_{j=1}^{n} |<x_i \mid x_j>|} \leq \|x\|^2.$$

Recall that Bessel's inequality is a special case of Selberg's inequality. When $x_1, x_2, \ldots, x_n$ form an orthonormal system, Selberg's inequality is reduced to

$$\sum_{i=1}^{n} |<x \mid x_i>|^2.$$

In this talk at 1988-RIMS meeting on operator theory, another proof of this inequality was given and several generalizations are discussed. So in the present talk, two more generalizations will be announced. In §2, those inequalities are applied to Wirtinger type inequality. In the last section, a continuous analogy of Bombieri's estimate will be discussed. The details are published later in other place.

### §1. Generalizations.

Since Selberg's inequality was developed as an inequality for almost orthonormal system with Bessel's one in mind, it is natural to consider the inequality for infinite almost orthonormal systems. At first, one obtains the countably infinite case as the limit of the monotone increasing sequence

$$\sum_{i=1}^{n} \frac{|<x \mid x_i>|^2}{\sum_{j=1}^{\infty} |<x_i \mid x_j>|} \leq \|x\|^2.$$

THEOREM 1. *If $x_1, x_2, \ldots, x_n, \ldots,$ and $x$ are non zero vectors in an inner product space, then*

$$\sum_{i=1}^{\infty} \frac{|<x \mid x_i>|^2}{\sum_{j=1}^{\infty} |<x_i \mid x_j>|} \leq \|x\|^2.$$

The convergence of vector sequence in an inner product space can generalized to the concept of summable system of vectors. As Halmos [4] pointed out, the concept of the summabilty is more of a notational convenience than a great generalization of the more elementary concept of infinite series. His point-out still alive for Selberg type inequality for non orthonormal uncountable systems. But in any way, the finite case can be generalized to uncountable case in the sense of summability:

THEOREM 2. *If $\{x_\iota\}_{\iota \in J}$ , and $x$ are non zero vectors in an inner product space, then*

$$\sum_{\iota \in J} \frac{|<x \mid x_\iota>|^2}{\sum_{\kappa \in J} |<x_\iota \mid x_\kappa>|} \leq \|x\|^2.$$

It is an elementary exercise that if $x, y$ form an orthonormal pair, then their distance is exactly $\sqrt{2}$. Thus there are no continuous functions whose values are orthonormal vectors in $\mathcal{H}$ on a topological space, say $[0,1]$. However, Selberg's inequality needs no orthonormality. Thus one may generalize the inequality for continuous functions on some topological space.

THEOREM 3. *Let $\mu$ be a regular Borel measure on a compact Hausdorff space $X$ and $\mathcal{H}$ be an inner product space. If a never vanishing function*

$$x(\cdot) : X \longrightarrow \mathcal{H}$$

*is continuous, then*

$$\int_X \frac{|<x \mid x(t)>|^2}{\int_X |<x(t) \mid x(s)>| d\mu(s)} d\mu(t) \leq \|x\|^2.$$

REMARK. A function

$$x(\cdot) : X \longrightarrow \mathcal{H}$$

is said to be *never vanishing* iff

$$\forall t \in X \ ; \ x(t) \neq 0.$$

The conditions of this theorem are direct analogies of THEOREM S.

## §2. Application 1.

There are some applications of THEOREM 3. At first, an inequality of Wirtinger's type will be discussed.

Let $X = [0, 2\pi]$ with the usual topology and the normalized Lebesgue measure $dt/2\pi$. A $\ell^2$-valued never vanishing function is defined by

$$x(t) = \left(\frac{\exp(int)}{n}\right) \qquad (t \in [0, 2\pi])$$

because

$$\sum_{n=1}^{\infty} \left|\frac{\exp(int)}{n}\right|^2 = \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6} < +\infty.$$

Suppose that $(\xi_n)_{n=1}^{\infty} \in \ell^2$ satisfies further that

$$(n\xi_n)_{n=1}^{\infty} \in \ell^2.$$

The $\mathrm{H}^2(\mathbf{T})$-function with Fourier coefficient $\xi_n$ will be denoted by $f$:

$$f(t) = \sum_{n=1}^{\infty} \xi_n \exp(int) \qquad (\text{in } \mathrm{L}^2 - \text{sense}).$$

Then by applying THEOREM 3 to $x = (n\xi_n)_{n=1}^{\infty} \in \ell^2$, one has

$$\begin{aligned}
&\int_0^{2\pi} |< x(t) \mid x(s) > | ds/2\pi \\
&= \int_0^{2\pi} \left|\sum_{n=1}^{\infty} \frac{1}{n^2} \exp(int)\overline{\exp(ins)}\right| ds/2\pi \\
&= \int_0^{2\pi} \left|\sum_{n=1}^{\infty} \frac{1}{n^2} \exp\left(in(t-s)\right)\right| ds/2\pi,
\end{aligned}$$

and

$$| < x \mid x(t) > |^2 = \left|\sum_{n=1}^{\infty} n\xi_n \cdot \frac{\exp(int)}{n}\right|^2 = \left|\sum_{n=1}^{\infty} \xi_n \exp(int)\right|^2 = |f(t)|^2.$$

Thus

$$\begin{aligned}
\int_0^{2\pi} \frac{|f(t)|^2}{\int_0^{2\pi} \left|\sum_{n=1}^{\infty} \frac{1}{n^2} \exp\left(in(t-s)\right)\right| ds/2\pi} dt/2\pi &\le \sum_{n=1}^{\infty} n^2 |\xi_n|^2 \\
&= \int_0^{2\pi} |f'(t)|^2 dt/2\pi.
\end{aligned}$$

Since the normalized measure $ds/2\pi$ is translation invariant, one has the following inequality.

THEOREM 4. *If $f, f' \in L^2$, then*

$$\int_0^{2\pi} |f(t)|^2 dt/2\pi \leq \left(\int_0^{2\pi} \left|\sum_{n=1}^{\infty} \frac{1}{n^2} e^{int}\right| dt/2\pi\right) \left(\int_0^{2\pi} |f'(t)|^2 dt/2\pi\right).$$

§3. **Application 2.**

In this section, a continuous analogy of Bombieri's inequality will be given.

In Bombieri's text [3], the following estimate of truncated $\ell^2$-norm of truncated Fourier series is given:

THEOREM B. *Let $x_1, x_2, \ldots, x_R$ is a sequence of real numbers of modulus 1, that satisfies*

$$\min\{\mathrm{dist}(x_i - x_j, \mathbf{Z}) \; : \; 1 \leq i \neq j \leq R\} \geq \delta > 0,$$

*and $N \in \mathbf{Z}$, with $N \geq 0$. Then for any $a_{-N}, a_{-N+1}, \ldots, a_{N-1}, a_N \in \mathbf{C}$,*

$$\sum_{j=1}^{R} \left|\sum_{n=-N}^{N} a_n \exp(2\pi i n x_j)\right|^2 \leq \left(2N + \frac{2}{\delta}\right) \sum_{n=-N}^{N} |a_n|^2.$$

The core of his proof is Selberg's inequality in an inner product space. Thus the truncated $L^2$-norm of truncated Fourier transform will be estimated in terms of $L^2$-norm of the original function.

**Theorem 5.** *Let $T, L, R \in \mathbf{R}$ be positive real numbers satisfying $T + L \leq R$. If $f$ is a square integrable function on the real line $\mathbf{R}$ with support* $\mathrm{supp}(f) \subset [-T, T]$, *then*

$$\int_{-R}^{R} \left| \int_{-T}^{T} f(s)e^{ist}ds \right|^2 dt \leq 8\sqrt{\frac{2T+L}{L}} \int_{-T}^{T} |f(t)|^2 dt.$$

**Proof.** Let

$$X = [-R, R], \ d\mu \ : \text{Lebesgue measure on } [-R, R], \ \mathcal{H} = L^2(X),$$

in **Theorem** 3. And the constant vector $x$ in sc Theorem 3 is the restriction of $f$ onto $X = [-R, R]$. And the never vanishing function $x(\cdot)$ in **Theorem** 3 will be chosen in the following manner:

$$\phi_t(s) := \begin{cases} \exp(-its) & \text{if } |s| \leq T, \\ \left(\frac{1}{L}(T + L - |s|)\right)^{1/2} \exp(-its) & \text{if } T < |s| \leq T + L, \\ 0 & \text{if } T + L < |s|. \end{cases}$$

And

$$\forall t \in X \ ; \ x(t) := \phi_t \in L^2(X) =: \mathcal{H}.$$

The numerator of the inequality in **Theorem** 3 is given by

$$| < x \mid x(t) > |^2 = \left| \int_{-R}^{R} f(s)\overline{\phi_t(s)}ds \right|^2$$

$$= \left| \int_{-T}^{T} f(s)\overline{\phi_t(s)}ds \right|^2 = \left| \int_{-T}^{T} f(s)\exp(its)ds \right|^2.$$

The right hand side of the inequality in **Theorem** 3 is given by

$$||x||^2 = ||f||^2 = \int_{-T}^{T} |f(t)|^2 dt.$$

The denominator of the inequality is left. For each $t, u \in X$, one has

$$
\begin{aligned}
&< x(t) \mid x(u) > \\
&= < \phi_t \mid \phi_u > \\
&= \int_{-T-L}^{T+L} \phi_t(s)\overline{\phi_u(s)}ds \\
&=\frac{1}{L}\int_{-T-L}^{-T}(T+L-|s|)e^{i(u-t)s}ds \\
&+\int_{-T}^{T} e^{i(u-t)s}ds \\
&+\frac{1}{L}\int_{T}^{T+L}(T+L-|s|)e^{i(u-t)s}ds \\
&=\frac{1}{L}\left\{\int_{-(T+L)}^{T+L}(T+L-|s|)e^{i(u-t)s}ds-\int_{-T}^{T}(T-|s|)e^{i(u-t)s}ds\right\}.
\end{aligned}
$$

To avoid a redundancy, one has only to calculate the following:

$$
\begin{aligned}
&\int_{-M}^{M}(M-|s|)e^{ias}ds \\
&=\int_{-M}^{0}(M-|s|)e^{ias}ds+\int_{0}^{M}(M-|s|)e^{ias}ds \\
&=\int_{M}^{0}(M-t)e^{-iat}(-1)dt+\int_{0}^{M}(M-s)e^{ias}ds \\
&=\int_{0}^{M}(M-t)e^{-iat}dt+\int_{0}^{M}(M-t)e^{iat}dt \\
&=2\int_{0}^{M}(M-t)\cos at dt \\
&=2\left\{\left[\frac{M}{a}\sin at\right]_0^M-\left[\frac{t\sin at}{a}+\frac{\cos at}{a^2}\right]_0^M\right\} \\
&=2\left\{\frac{1}{a^2}-\frac{\cos aM}{a^2}\right\} \\
&=\left(\frac{\sin\frac{a}{2}M}{\left(\frac{a}{2}\right)}\right)^2.
\end{aligned}
$$

Thus one has further that

$$
\begin{aligned}
&< x(t) \mid x(u) > \\
&= < \phi_t \mid \phi_u > \\
&= \frac{1}{L}\left\{\left(\frac{\sin\frac{u-t}{2}(T+L)}{(\frac{u-t}{2})}\right)^2 - \left(\frac{\sin\frac{u-t}{2}T}{(\frac{u-t}{2})}\right)^2\right\}.
\end{aligned}
$$

Recalling the graph of the function

$$\frac{\sin t}{t},$$

the integrand of the denominator has a bound even in the neighborhood of zero:

$$
\begin{aligned}
&| < \phi_t \mid \phi_u > | = \frac{1}{L}\left|\left(\frac{\sin\frac{u-t}{2}(T+L)}{(\frac{u-t}{2})}\right)^2 - \left(\frac{\sin\frac{u-t}{2}T}{(\frac{u-t}{2})}\right)^2\right| \\
&\leq \frac{(T+L)^2 - T^2}{L} = \frac{2TL + L^2}{L} =: K.
\end{aligned}
$$

It has another bound if $u \neq t$ :

$$
\begin{aligned}
&| < \phi_t \mid \phi_u > | \\
&= \frac{1}{L}\left|\left(\frac{\sin\frac{u-t}{2}(T+L)}{(\frac{u-t}{2})}\right)^2 - \left(\frac{\sin\frac{u-t}{2}T}{(\frac{u-t}{2})}\right)^2\right| \\
&= \frac{4}{L(u-t)^2}\left|\left(\sin\frac{u-t}{2}(T+L)\right)^2 - \left(\sin\frac{u-t}{2}T\right)^2\right| \\
&\leq \frac{4}{L(u-t)^2}.
\end{aligned}
$$

The next step is to obtain the integral of $| < x(t) \mid x(u) > |$ over

$u \in [-R, R]$. By a translation $r := u - t$, the integral is given by

$$
\begin{aligned}
&\int_{-R}^{R} | < \phi_t \mid \phi_u > | du \\
=&\frac{1}{L}\int_{-R}^{R}\left|\left(\frac{\sin\frac{u-t}{2}(T+L)}{(\frac{u-t}{2})}\right)^2-\left(\frac{\sin\frac{u-t}{2}T}{(\frac{u-t}{2})}\right)^2\right|du \\
=&\frac{1}{L}\int_{-R-t}^{R-t}\left|\left(\frac{\sin\frac{r}{2}(T+L)}{(\frac{r}{2})}\right)^2-\left(\frac{\sin\frac{r}{2}T}{(\frac{r}{2})}\right)^2\right|dr \\
=&\frac{1}{L}\left(\int_{-R-t}^{-\epsilon}+\int_{-\epsilon}^{\epsilon}+\int_{\epsilon}^{R-t}\right)\left|\left(\frac{\sin\frac{r}{2}(T+L)}{(\frac{r}{2})}\right)^2-\left(\frac{\sin\frac{r}{2}T}{(\frac{r}{2})}\right)^2\right|dr \\
\leq&\left(\int_{-R-t}^{-\epsilon}+\int_{\epsilon}^{R-t}\right)\frac{4}{Lr^2}dr+\int_{-\epsilon}^{\epsilon}\frac{2TL+L^2}{L}dr \\
=&\frac{4}{L}\left[\frac{-1}{r}\right]_{-R-t}^{-\epsilon}+\frac{4}{L}\left[\frac{-1}{r}\right]_{\epsilon}^{R-t}+2\epsilon\frac{2TL+L^2}{L} \\
=&\frac{4}{L}\left(\frac{2}{\epsilon}-\left(\frac{1}{R-t}+\frac{1}{R+t}\right)\right)+2\epsilon\frac{2TL+L^2}{L} \\
\leq&8\sqrt{\frac{2T+L}{L}},
\end{aligned}
$$

by choosing

$$\epsilon = \frac{2}{\sqrt{2TL+L^2}}.$$

From **Theorem** 3, one obtains that

$$\int_{-R}^{R}\frac{\left|\int_{-T}^{T} f(s)e^{ist}ds\right|^2}{8\sqrt{\frac{2T+L}{L}}} \leq \|f\|^2 = \int_{-T}^{T}|f(t)|^2dt,$$

which is the desired result.

Q.E.D.

## References

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[4] P.R. Halmos, *Introduction to Hilbert Space and the Theory of Spectral Multiplicity*, Chelsea, New York, 1951.