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Introduction to Mathematical Theory of Brownian Motion

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§1. Prologue.

The molecular phenomenon which is now named after *Robert Brown* has been known for many years. The meaning of his work done in 1828 is that the phenomenon concerns not the biological problem but physical one. The motion of pollen obeys the randomness as time goes on.

Let $(\Omega, \mathcal{F}, P)$ be a probability space, and

$$X \ : \ \Omega \ni \omega \mapsto X(\omega) \in \mathbf{R}^1$$

be a random variable (namely a measurable function) on $(\Omega, \mathcal{F}, P)$. This means that

$$\forall a \in \mathbf{R} \ ; \ \{\omega \in \Omega \ : \ X(\omega) < a\} \in \mathcal{F}.$$

A (stochastic) process is a family of random variables parametrized by nonnegative real numbers: $(X_t)_{t \geq 0}$. In 1923, N. Wiener gave the mathematical foundation to the Brownian motion in terms of probability theory.

DEFINITION 1. A *Brownian motion* is a process $(B_t)_{t \geq 0}$ satisfying the following three conditions:

(i) $\forall \omega \in \Omega \ ; \ t \mapsto B_t(\omega)$ is continuous and $B_0(\omega) = 0$,

(ii) $0 < \forall t_1 < \forall t_2 < \cdots < \forall t_n \ ; \ \{B_{t_1}, B_{t_2} - B_{t_1}, \ldots, B_{t_n} - B_{t_{n-1}}\}$ is independent,

and

(iii) $0 \leq s < t \Rightarrow B_t - B_s \in N(0, t-s)$, namely

$$P(B_t - B_s \in \Lambda) = \int_\Lambda \frac{1}{\sqrt{2\pi(t-s)}} e^{-\frac{x^2}{(t-s)}} dx,$$

for any Borel set Λ.

REMARK. The first condition is a mathematical representation of the continuous path with no derivatives at any point, which was pointed out by J. Perrin in his famous book "Les Atomes". The second one, the distribution at present time is completely independent from the amount of change appears in a paper of P. Langevin. The third one comes from the central limit theorem.

The spatial symmetry yields that $m = 0$, while $\sigma^2 = t - s$ was proved by A. Einstein.

The theory has an application to signal processing. The Brownian motion represents the noise in information theory. For the filtering theory, cf. H. Kunita [4].

Not only the engineers but also mathematicians are interested in this theory, e.g., the works of H. Kaneko, A. Sakai, T. Kôno and the others.

§2. Three constructions of the Brownian Motion.

Let

$$\begin{aligned} &\Omega := C([0,\infty) \to \mathbf{R}^1), \\ &\omega \in \Omega, \\ &B_t(\omega) := \omega(t) \qquad (t \in [0,\infty)). \end{aligned}$$

Then the family $(B_t)_{0 \le t \le \infty}$ is a stochastic process. In this case, the path $t \mapsto B_t(\omega)$ is nothing but ω itself.

For each

$$\begin{aligned} &0 < t_1 < t_2 < \cdots < t_n, \\ &\Lambda_1, \Lambda_2, \ldots, \Lambda_n \ : \ \text{Borel}, \end{aligned}$$

define a cylinder set $A \subset \Omega$ by

$$A := \left\{ \omega \in \Omega \ : \ 1 \le \forall i \le n \ ; \ B_{t_i} \in \Lambda_i \right\},$$

and the σ-field $\mathcal{F}$ generated by those cylinder sets. Obviously $(\Omega, \mathcal{F})$ is a measurable space. The Gaussian distribution

$$g(t,x,y) := \frac{1}{\sqrt{2\pi t}} e^{-\frac{(y-x)^2}{2t}} \quad (t > 0, x, y \in \mathbf{R})$$

on the cylinder set defined by

$$P(A) := \int \cdots \int \int_{\Lambda_1 \times \Lambda_2 \times \cdots \times \Lambda_n} g(t_1, 0, x_1) g(t_2 - t_1, x_1, x_2) \cdots g(t_n - t_{n-1}, x_{n-1}, x_n) dx_1 dx_2 \cdots dx_n$$

can be extended to a probability measure on the σ-field $\mathcal{F}$ by Kolmogorov's extension theorem.

Recall that

$$\int_{\mathbf{R}} g(t,0,x)dx = 1$$

yields that $P(\Omega) = 1$.

The process *is* a Brownian motion in the preceding sense. In fact, the first condition is shown as follows:

(i)

$$\begin{aligned}
&P(B_0 = 0) = \lim_{n\to\infty} P\left(|B_0| < \frac{1}{n}\right)\\
&= \lim_{n\to\infty}\lim_{t\to 0} P\left(|B_t| < \frac{1}{n}\right)\\
&= \lim_{n\to\infty}\lim_{t\to 0}\int_{-1/n}^{1/n} \frac{1}{\sqrt{2\pi t}} e^{-\frac{x^2}{2t}} dx\\
&= \lim_{n\to\infty}\lim_{t\to 0}\int_{-1/n\sqrt{t}}^{1/n\sqrt{t}} \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}} dy\\
&=1.
\end{aligned}$$

The third condition comes as follows: Let

$$\phi_n \ : \ \mathbf{R}^n \ni (x_1, x_2, \ldots, x_n) \mapsto (y_1, y_2, \ldots, y_n) \in \mathbf{R}^n$$

be defined by

$$\begin{aligned}
y_1 &= x_1\\
y_j &= \sum_{i=1}^{j} x_i \qquad (j \geq 2).
\end{aligned}$$

This is a homeomorphism and hence maps the Borel sets to the Borel ones. Let $0 \leq s < t$. Then one has

$$\begin{aligned}
&P(B_s \in \Lambda_1, B_t - B_s \in \Lambda_2)\\
=&P((B_s, B_t) \in \phi_2(\Lambda_1 \times \Lambda_2))\\
=&\int\int_{\phi_2(\Lambda_1\times\Lambda_2)} g(s,0,y_1)g(t-s,y_1,y_2)dy_1dy_2
\end{aligned}$$

by change of variables $x_1 = y_1, x_2 = y_2 - x_2$,

$$\begin{aligned}
&= \int\int_{\Lambda_1\times\Lambda_2} g(s,0,x_1)g(t-s,0,x_2)dx_1dx_2\\
&= \int_{\Lambda_1} g(s,0,x_1)dx_1 \int_{\Lambda_2} g(t-s,0,x_2)dx_2
\end{aligned}$$

and in particular, ($\Lambda_1 = \mathbf{R}^1$), one has

$$\begin{aligned} & P(B_t - B_s \in \Lambda_2) \\ &= \int_{\Lambda_2} g(t-s, 0, x_2)dx_2 \\ &= \int_{\Lambda_2} \frac{1}{\sqrt{2\pi(t-s)}} e^{-\frac{x^2}{(t-s)}} dx. \end{aligned}$$

Hence $B_t - B_s \in N(0, t-s)$.

The second condition is obtained by calculating the following simultaneous distribution:

$$(B_{t_1}, B_{t_2}, \ldots, B_{t_n}) = \phi_n(B_{t_1}, B_{t_2} - B_{t_1}, \ldots, B_{t_n} - B_{t_{n-1}}).$$

$$\begin{aligned} & P(B_{t_1} \in \Lambda_1, B_{t_2} - B_{t_1} \in \Lambda_2, \ldots, B_{t_n} - B_{t_{n-1}} \in \Lambda_n) \\ &= \int \cdots \int_{\phi_n(\Lambda_1 \times \cdots \Lambda_n)} \\ & \qquad g(t_1, 0, y_1) g(t_2 - t_1, y_1, y_2) \cdots g(t_n - t_{n-1}, y_{n-1}, y_n) dy_1 \cdots dy_n \\ & \qquad \text{by change of variables } x_1 = y_1, x_j = y_j - x_j - 1 \quad (j \geq 2), \\ &= \int_{\Lambda_1} \cdots \int_{\Lambda_n} g(t_1, 0, x_1) g(t_2 - t_1, 0, x_2) \cdots g(t_n - t_{n-1}, 0, x_n) dx_1 \cdots dx_n \\ &= \int_{\Lambda_1} g(t_1, 0, x_1)dx_1 \int_{\Lambda_2} g(t_2 - t_1, 0, x_2)dx_2 \cdots \int_{\Lambda_n} g(t_n - t_{n-1}, 0, x_n)dx_n \\ &= P(B_{t_1} \in \Lambda_1) P(B_{t_2} - B_{t_1} \in \Lambda_2) \cdots P(B_{t_n} - B_{t_{n-1}} \in \Lambda_n). \end{aligned}$$

Thus (B_t) is a Brownian motion. The next step is to change the starting point of the motion:

$$P_a(B_t \in \Lambda) = P(B_t + a \in \Lambda).$$

In fact,

$$P_a(B_0 = a) = P(B_0 + a = a) = P(B_0 = 0) = 1,$$

and so on. In general, one may consider for any initial distribution μ,

$$P_\mu(A) := \int_A P_x(A)\mu(dx).$$

This is the construction of the Brownian motion given by N. Wiener himself. There are other constructions. The first one is due to R.E.A.C. Paley and N. Wiener [16].

Consider a sequence X_n of independent random variables subject to

$$N(0,1).$$

Then the process B_t defined by

$$B_t(\omega) := \frac{t}{\sqrt{\pi}} X_0(\omega) + \sum_{n=1}^{\infty} X_n(\omega) \frac{\sin \pi t}{n},$$

gives a Brownian motion on finite time interval $[0,\pi]$. Since X_n's form an ONS in $L^2(\Omega)$, the convergence is obvious. The convergence is uniform on $[0,\pi]$ for $\forall \omega \in \Omega$. The construction of those independent sequences are realized by

$$X = \sqrt{-2\log U_1}\cos(2\pi U_2),$$

where U_i is a pair of random variables with uniform distribution on $(0,1)$.

The third construction. Consider a sequence of ± 1 valued random variables w_n with

$$P(w_j = 1) = \frac{1}{2} = P(w_j = -1).$$

This is obtained as a sequence of coin tossing. The first amount of the coin tossing

$$W_n = \sum_{j=1}^{n} w_j$$

is called the random walk. Let (B_t) be a Brownian motion. Set

$$\begin{aligned} &\tau_0^j = 0, \\ &\tau_n^j(\omega) := \min\{t > \tau_{n-1} \ : \ |B_t - B_{\tau_{n-1}}| = \frac{1}{\sqrt{j}}\}. \end{aligned}$$

Then one has

$$P(\forall t \ ; \ |B_t| \leq G) \leq P(|B_t| \leq G) \leq \frac{2G}{\sqrt{2\pi t}} \to 0.$$

And hence

$$W_n^j := \sqrt{j} B_{\tau_n^j}$$

is a random walk. In fact, F. Knight [12] proved that

$$P\left(\lim_{j\to\infty}\max_{0\le t\le 1}|B_t^j - B_t| = 0\right) = 1.$$

The converse is the construction of a Brownian motion.

§3. Sample Properties of the Brownian Motion.

THEOREM 1.

$$P\left(\left\{\omega\in\Omega\ :\ t\mapsto B_t(\omega)\text{ differentiable at sometime }t\right\}\right) = 0.$$

A beautiful proof of A. Dovoretsky, P. Erdös, and S. Kakutani [5] was told. The details are omitted here.

THEOREM 2 (N. Wiener).

$$\overline{\forall}\omega\in\Omega\ ;\ \{t\mapsto B_t(\omega)\}\notin BV[0,1].$$

Using the elementary inequality

$$e^{-x}\le 1-x+\frac{x^2}{2}\quad (x\ge 0),$$

one has

$$\mathbf{E}\left[\exp\left(\sum_{j=1}^{2^n}|B_{j/2^n}-B_{(j-1)/2^n}|\right)\right]\to 0$$

as $n\to\infty$ and the result follows.

THEOREM 3.

$$\overline{\forall}\omega\in\Omega\ ;\ \sum_{j=1}^{2^n}\left(B_{jt/2^n}-B_{(j-1)t/2^n}\right)^2\longrightarrow t\qquad ((\text{as } n\to\infty)).$$

For $h > 0$, let

$$C(h) := \sup_{|s-t|\leq h} |B_s - B_t|.$$

THEOREM 4 (P. Lévy, 1937).

$$P\left(\overline{\lim}_{h\to\infty} \frac{C(h)}{\sqrt{2h\log\frac{1}{h}}} = 1\right) = 1.$$

§4. **Transition Operators.**

Let $C_\infty(\mathbf{R}^1 \to \mathbf{R}^1)$ be the totality of bounded continuous functions on the real line. For $f \in C_\infty(\mathbf{R}^1 \to \mathbf{R}^1)$, and a time parameter $t \geq 0$, let

$$(T_t f)(x) = \mathbf{E}_x[f(B_t)] = \int_{-\infty}^{\infty} f(y)P_x(B_t \in dy) = \int_{-\infty}^{\infty} f(y)g(t,x,y)dy.$$

It is obvious that $(T_0 f)(x) = f(x)$.

THEOREM 5.

(i) $T_t \ : \ C_\infty \ni f \mapsto T_t f \in C_\infty$ *is a linear operator, and*

$$\|T_t\| = 1,$$

(ii) $T_{t+s} = T_s T_t = T_t T_s$.

The semigroup property ii) comes from the well known Chapman - Kolmogorov's identity:

$$\int_{-\infty}^{\infty} g(s,x,y)g(t,y,z)dy = g(s+t,x,z).$$

And the semigroup property applied to an indicator function $f = I_\Lambda$ yields also that

$$P_x\left(B_{s+t} \in \Lambda\right) = \int_{-\infty}^{\infty} P_y\left(B_t \in \Lambda\right) P_x\left(B_s \in dy\right).$$

A physical interpretation of this identity is that the entrance probability into Λ at time $s+t$ of a particle starting from x is identical to the total sum of the entrance probability into Λ at time $s+t$ starting from x and passing through dy at time s.

§5. Kakutani's Theorem.

Let $B = (B_t)$ be a two dimensional Brownian motion, namely an ordered pair of mutually independent Brownian motions, say, $B = (B_t) = (B'_t, B''_t)$. Suppose further that $z \in \mathbf{R}^2$ and $\mathbf{R}^2 \supset \Lambda$ is a Borel set. Since

$$P_z(B_t \in \Lambda) = \int_\Lambda \frac{1}{2\pi t} e^{-\frac{|x-z|^2}{2t}} dx,$$

one has that

$$0 \leq P_z(\forall t \; ; \; B_t \in \Lambda) \leq P_z(B_t \in \Lambda) \leq \frac{|\Lambda|}{2\pi t} \to 0 \qquad (\text{as } t \to \infty).$$

For the upper half plain

$$\mathbf{R}^2_+ := \{z = (x, y) \in \mathbf{R}^2 \; : \; y > 0\},$$

the first exit time will be denoted by τ :

$$\tau(\omega) := \inf\{t \; : \; B_t(\omega) \in \partial \mathbf{R}^2_+\}.$$

S. Kakutani [11] proved the following theorem which gives the probability of the event that a particle starting from z_0 and going out the half plain from left to right.

THEOREM 6 (S. Kakutani, 1944).

$$\forall z_0 = (x_0, y_0) \in \mathbf{R}^2_+ \; ; \; P_{z_0}(B_\tau \leq \lambda) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{y_0}{(x_0 - t)^2 + y_0^2} dt.$$

There is a close relationship between this theorem and the Dirichlet problem. If $f \in C(\partial \mathbf{R}^2_+)$, then

$$f(x, y) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{y}{(x-t)^2 + y^2} dt = \mathbf{E}_z[f(B_\tau)]$$

gives a solution to the Dirichlet problem.

LEMMA. *For any angular domain $D := \{0 < \alpha < 2\pi\}$, let σ be the first exit time from D. Then*

$$p(z) := P_z(B_\sigma \in \overrightarrow{OB}) = \frac{\theta}{\alpha},$$

where $\theta = \arg z, \alpha = \arg B$.

The conformal image of this theorem goes as follows: Let $D := \{|z| < 1\}$ and $\zeta := \inf\{t \ : \ B_t \notin D\}$ be the first exit time from D.

THEOREM 7. *If Γ be a Borel set on the one dimensional torus, then*

$$P_z(B_\zeta \in \Gamma) = \frac{1}{2\pi} \int_\Gamma \frac{1 - r^2}{1 - 2r\cos(\theta - t) + r^2} dt.$$

The Cayley transform is available for the proof.

Let $m(dt)$ be a measure on ∂D.

COROLLARY 1.

$$P_0(B_\zeta \in \Gamma) = \frac{m(\Gamma)}{2\pi}.$$

COROLLARY 2. *If h is continuous on $\overline{D}$ and harmonic in D, then*

$$h(z) = \mathbf{E}_z[h(B_\zeta)].$$

This is a representation of the solution of Dirichlet problem using the Brownian motion.

§6. Applications.

EXAMPLE 1. Let $0 < r < R$ and $D_{r,R}$ be the corresponding annulus. Then the function h defined by

$$h(z) = \begin{cases} \frac{\log R - \log|z|}{\log R - \log r} & \text{if } z \in D_{r,R}, \\ 0 & \text{if } |z| = R, \\ 1 & \text{if } |z| = r, \end{cases}$$

is continuous on $\overline{D_{r,R}}$ and harmonic in $D_{r,R}$. If η is the first exit time from $D_{r,R}$, then

$$h(z) = \mathbf{E}_z[h(B_\eta)] = 1 \cdot P_z(|B_\eta| = r) + 0 \cdot P_z(|B_\eta| = R) = P_z(|B_\eta| = r).$$

Letting $R \to \infty$, one has $h(z) \to 1$, and hence the particle falls down into the smaller ball as time goes on.

EXAMPLE 2 (T. Kôno, 1984) The theory of Brownian motion is available to prove the fundamental theorem of algebra. Let $f(z) \in \mathbf{C}[z]$. Since the function $f(z)$ on the complex plain is non-constant and entire, P. Lévy's theorem is applicable to show that $(f(B_t))$ is again a Brownian motion with a time change. Since

$$S(\epsilon) := \{z \in \mathbf{C} \ : \ |f(z)| < \epsilon\}$$

is a compact set, one has from the preceding example that

$$P(\exists t > 0 \ ; \ f(B_t) \in S(\epsilon)) = 1.$$

Hence $S(\epsilon) \neq \emptyset$, and

$$\bigcap_{\epsilon > 0} S(\epsilon) \neq \emptyset,$$

which completes the proof.

Q.E.D.

The last example which is a converse of the mean value theorem, is obtained recently by H. Kaneko and A. Sakai.

THEOREM 8 (H. Kaneko and A. Sakai, 1989). *Let a domain D has a C^1-smooth boundary curve ∂D, and $z_0 \in D$. If*

$$h(z_0) = \frac{1}{m(\partial D)} \int_{\partial D} f(\zeta) m(d\zeta)$$

for any function h that is continuous on the closure of D and harmonic in D, then D is a ball around z_0.

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