Periodic solutions of Boussinesq equations

MORIMOTO, Hiroko (森本浩子) *

Let $\Omega$ be a bounded domain in $\mathbb{R}^2$ with the boundary $\partial \Omega$ such that
$$\partial \Omega = \Gamma_1 \cup \Gamma_2, \quad \Gamma_1 \cap \Gamma_2 = \phi.$$ We consider the following initial boundary value problem:

\begin{align*}
\frac{\partial u}{\partial t} + (u \cdot \nabla)u &= -\frac{1}{\rho} \nabla p + \nu \Delta u + \beta g \theta \\
\text{div} u &= 0, \quad x \in \Omega, \; t > 0, \quad (1) \\
\frac{\partial \theta}{\partial t} + (u \cdot \nabla)\theta &= \chi \Delta \theta, \\
\left\{ \begin{array}{l}
u(x,t) = 0, \quad \theta(x,t) = \xi(x,t), \\
u(x,t) = 0, \quad \frac{\partial \theta(x,t)}{\partial n} = \eta(x,t),
\end{array} \right. \quad x \in \Gamma_1, \; t > 0, \quad (2) \\
\left\{ \begin{array}{l}
u(x,0) = a_0(x), \\
\theta(x,0) = \tau_0(x),
\end{array} \right. \quad x \in \Omega, \quad (3)
\end{align*}

where $u = (u_1, u_2)$ is the fluid velocity, $p$ is the pressure, $\theta$ is the temperature, $u \cdot \nabla = \sum_{j=1}^{2} u_j \frac{\partial}{\partial x_j}$ denotes the outer normal derivative of $\theta$ at $x$ to $\partial \Omega$, $g(x,t)$ is the gravitational vector function, and $\rho$ (density), $\nu$ (kinematic viscosity), $\beta$ (coefficient of volume expansion), $\chi$ (thermal diffusivity) are positive constants. $\xi(x,t)$ (resp. $\eta(x,t)$) is a function defined on $\Gamma_1 \times (0,T)$ (resp. $\Gamma_2 \times (0,T)$) and $a_0(x)$ (resp. $\tau_0(x)$) is a vector (resp. scalar) function defined on $\Omega$.

*School of Science and Technology, Meiji University
In order to state our results, we introduce some function spaces ([1],[2],[3]).

$L^p(\Omega)$ and the Sobolev space $W^s_p(\Omega)$ are defined as usual. We also denote $L^p(\Omega) = L^p(\Omega) \times L^p(\Omega)$, $H^s(\Omega) = W^s_p(\Omega)$. Whether the elements of the space are scalar or vector functions is understood from the contexts unless stated explicitly.

$$D_\sigma = \{\text{vector function } \varphi \in C^\infty(\Omega) \mid \text{supp}\varphi \subset \Omega, \text{div}\varphi = 0 \text{ in } \Omega\},$$

$H = \text{completion of } D_\sigma \text{ under the } L^2(\Omega)\text{-norm},$

$V = \text{completion of } D_\sigma \text{ under the } H^1(\Omega)\text{-norm},$

$D_0 = \{\text{scalar function } \varphi \in C^\infty(\overline{\Omega}) \mid \varphi \equiv 0 \text{ in a neighborhood of } \Gamma_1\},$

$W = \text{completion of } D_0 \text{ under the } H^1(\Omega)\text{-norm},$

$V', W'$ are dual space of $V, W$.

**Definition 1**

$\{u, \theta\}$ is called a weak solution of evolitional problem (1),(2) if, for some function $\theta_0$ such that

$$\theta_0 \in L^2(0, T : H^1(\Omega)), \quad \theta_0 = \xi \text{ on } \Gamma_1,$$

$\{u, \theta\}$ satisfies following conditions:

$$ u \in L^2(0, T : V), \quad \theta - \theta_0 \in L^2(0, T : W),$$

$$ \begin{cases} 
  \frac{d}{dt}(u, v) + \nu(\nabla u, \nabla v) + ((u \cdot \nabla)u, v) - (\beta g \theta, v) = 0, \quad \forall v \in V, \\
  \frac{d}{dt}(\theta, \tau) + \chi(\nabla \theta, \nabla \tau) + ((u \cdot \nabla)\theta, \tau) - \chi(\eta, \tau)_{\Gamma_2} = 0, \quad \forall \tau \in W,
\end{cases}$$

where

$$ (\eta, \tau)_{\Gamma_2} = \int_{\Gamma_2} \eta(x')\tau(x')d\sigma. $$
As for the smoothness of $\partial \Omega$, we suppose

**Condition (H)**

$\partial \Omega$ is of class $C^1$ and devided as follows:

$$\partial \Omega = \Gamma_1 \cup \Gamma_2, \quad \Gamma_1 \cap \Gamma_2 = \emptyset,$$

and the intersection $\overline{\Gamma}_1 \cap \overline{\Gamma}_2$ consists of finite points.

In [3], we showed the existence and the uniqueness of weak solution of evoluntional problem for $2 \leq n \leq 4$. For $n = 2$, we have the following result:

**Theorem A**

Let $\Omega$ be a bounded domain in $R^2$ with $C^1$ boundary satisfying Condition (H). If the function $g$ is in $L^\infty(\Omega \times (0, T))$, $\xi \in C^1(\overline{\Gamma}_1 \times [0, T])$, $\eta \in L^2(\Gamma_2 \times (0, T))$, $a_0 \in H$, $\tau_0 \in L^2(\Omega)$, then there exists one and only one weak solution $\{u, \theta\}$ of (1), (2) satisfying the initial condition (3). Furthermore

$$u \in C([0, T] : H), \quad \theta \in C([0, T] : L^2(\Omega)).$$

**Definition 2**

$\{u, \theta\}$ is called a periodic weak solution of (1), (2) with period $T_0$, if $\{u, \theta\}$ is a weak solution of (1), (2) for $T = T_0$ satisfying

$$u(x, T_0) = u(x, 0), \quad \theta(x, T_0) = \theta(x, 0). \tag{5}$$

We also obtained the existence of periodic weak solutions([3]).

**Theorem B**

Let $\Omega$ be a bounded domain in $R^2$ with $C^1$ boundary satisfying Condition (H). Let $g(x, t)$, $\xi(x, t)$, $\eta(x, t)$ be periodic with respect to $t$ with period $T_0$, satisfying $g \in L^\infty(\Omega \times (0, T_0))$, $\xi \in C^1(\overline{\Gamma}_1 \times [0, T_0])$ and $\eta \in L^2(\Gamma_2 \times (0, T_0))$. Set $g_\infty = \|g\|_{L^\infty(\Omega \times (0, T_0))}$. If $\frac{\beta g_\infty}{\sqrt{\nu \chi}}$ is sufficiently small, then there exists a periodic weak solution of (1), (2) with period $T_0$. Furthermore

$$u \in C([0, \infty) : H), \quad \theta \in C([0, \infty) : L^2(\Omega)).$$

Now we can state our results. As for the uniqueness of periodic weak solutions, we obtained:
Theorem 1
Let \(\{u_{\pi}, \theta_{\pi}\}\) be a weak periodic solution of (1), (2) with period \(T_{0}\) such that for some \(p > 2\),
\[
\text{ess.sup}_{t}\{c\|u_{\pi}(t)\|_{p} + \frac{1}{4\chi}(c\|\theta_{\pi}(t)\|_{p} + c'\beta g_{\infty})^{2}\} < \nu,
\]
where \(c\) and \(c'\) are constants depending on \(\Omega\). If \(\{u_{\pi} + u, \theta_{\pi} + \theta\}\) is a weak periodic solution of (1), (2) with period \(T_{0}\), then \(u = 0, \theta = 0\).

Let \(g \in L^\infty(\Omega \times (0, \infty))\), \(\xi \in C^{1}(\Gamma_{1} \times [0, \infty))\) \(\eta \in L^{2}(\Gamma_{2} \times (0, \infty))\), \(a_{0} \in H, \tau_{0} \in L^{2}(\Omega)\). Let \(T\) be any positive number. Then there exists one and only one weak solution \(\{u_{T}, \theta_{T}\}\) of (1), (2) satisfying (3). Therefore, for \(T < T'\),
\[
u_{T}(t) = u_{T'}(t), \quad \theta_{T}(t) = \theta_{T'}(t) \quad \text{for } \forall t \in (0, T)
\]
hold, and we can omit \(T\). This solution is called a global weak solution. We obtained the asymptotic property of solutions of Boussinesq equations as follows.

Theorem 2
Let \(g, \xi, \eta\) satisfy the condition of Theorem B, \(a_{0} \in H, \tau_{0} \in L^{2}(\Omega)\). Let \(\{u, \theta\}\) be a global weak solution of (1), (2) satisfying (3), \(\{u_{\pi}, \theta_{\pi}\}\) a periodic weak solution satisfying (6). Then
\[
\lim_{t \rightarrow \infty}\{\|u(t) - u_{\pi}(t)\|^{2} + \|\theta(t) - \theta_{\pi}(t)\|^{2}\} = 0.
\]

Remark
(i) Since \(u_{\pi} \in L^{2}(0, T : V) \cap C([0, T] : H)\), \(u_{\pi}\) belongs to the space \(L^{2p/(p-2)}(0, T : L^{p}(\Omega))\) for \(\forall p > 2\). Similarly \(\theta_{\pi}\) is in \(L^{2p/(p-2)}(0, T : L^{p}(\Omega))\). The condition (6) is stronger than this one.
(ii) When (6) holds, such periodic solution is unique (Theorem 1).

References
