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Periodic solutions of Boussinesq equations

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Let \( \Omega \) be a bounded domain in \( \mathbb{R}^2 \) with the boundary \( \partial \Omega \) such that
\[
\partial \Omega = \Gamma_1 \cup \Gamma_2, \quad \Gamma_1 \cap \Gamma_2 = \phi.
\]
We consider the following initial boundary value problem:

\[
\begin{aligned}
\frac{\partial u}{\partial t} + (u \cdot \nabla) u &= -\frac{1}{\rho} \nabla p + \nu \Delta u + \beta g \theta \\
\text{div} u &= 0, \\
\frac{\partial \theta}{\partial t} + (u \cdot \nabla) \theta &= \chi \Delta \theta,
\end{aligned}
\]
\( x \in \Omega, t > 0, \) \hspace{1cm} (1)

\[
\begin{aligned}
u(x,t) &= 0, \theta(x,t) = \xi(x,t), \\
u(x,t) &= 0, \frac{\partial}{\partial n} \theta(x,t) = \eta(x,t), \\
u(x,0) &= a_0(x), \\
\theta(x,0) &= \tau_0(x),
\end{aligned}
\]
\( x \in \Gamma_1, t > 0, \) \hspace{1cm} (2)

where \( u = (u_1, u_2) \) is the fluid velocity, \( p \) is the pressure, \( \theta \) is the temperature, \( u \cdot \nabla = \sum_{j=1}^{2} u_j \frac{\partial}{\partial x_j} \) denotes the outer normal derivative of \( \theta \) at \( x \) to \( \partial \Omega \), \( g(x,t) \) is the gravitational vector function, and \( \rho \) (density), \( \nu \) (kinematic viscosity), \( \beta \) (coefficient of volume expansion), \( \chi \) (thermal diffusivity) are positive constants. \( \xi(x,t) \) (resp. \( \eta(x,t) \)) is a function defined on \( \Gamma_1 \times (0,T) \) (resp. \( \Gamma_2 \times (0,T) \)) and \( a_0(x) \) (resp. \( \tau_0(x) \)) is a vector (resp. scalar) function defined on \( \Omega \).

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In order to state our results, we introduce some **Function spaces**([1],[2],[3]).

$L^p(\Omega)$ and the Sobolev space $W^l_p(\Omega)$ are defined as usual. We also denote $L^p(\Omega) = L^p(\Omega) \times L^p(\Omega)$, $H^l(\Omega) = W^l_p(\Omega)$. Whether the elements of the space are scalar or vector functions is understood from the contexts unless stated explicitly.

\[ \begin{align*}
D_\sigma &= \{ \text{vector function } \varphi \in C^\infty(\Omega) \mid \text{supp}\varphi \subset \Omega, \text{div}\varphi = 0 \text{ in } \Omega\}, \\
H &= \text{completion of } D_\sigma \text{ under the } L^2(\Omega)-\text{norm}, \\
V &= \text{completion of } D_\sigma \text{ under the } H^1(\Omega)-\text{norm}, \\
D_0 &= \{ \text{scalar function } \varphi \in C^\infty(\Omega) \mid \varphi \equiv 0 \text{ in a neighborhood of } \Gamma_1\}, \\
W &= \text{completion of } D_0 \text{ under the } H^1(\Omega)-\text{norm}, \\
V', W' \text{ are dual space of } V, W.
\end{align*} \]

**Definition 1**

\{u, \theta\} is called a weak solution of evolutational problem (1),(2) if, for some function $\theta_0$ such that 

\[ \theta_0 \in L^2(0, T : H^1(\Omega)), \quad \theta_0 = \xi \text{ on } \Gamma_1, \]

\{u, \theta\} satisfies following conditions:

\[ \begin{align*}
\frac{d}{dt} (u,v) + \nu(\nabla u, \nabla v) + ((u \cdot \nabla)u, v) - (\beta g \theta, v) &= 0, \quad \forall v \in V, \\
\frac{d}{dt} (\theta, \tau) + \chi(\nabla \theta, \nabla \tau) + ((u \cdot \nabla)\theta, \tau) - \chi(\eta, \tau)_{\Gamma_2} &= 0, \quad \forall \tau \in W,
\end{align*} \]

where 

\[ (\eta, \tau)_{\Gamma_2} = \int_{\Gamma_2} \eta(x')\tau(x')d\sigma. \]
As for the smoothness of $\partial \Omega$, we suppose

**Condition (H)**

$\partial \Omega$ is of class $C^1$ and divided as follows:

$$\partial \Omega = \Gamma_1 \cup \Gamma_2, \quad \Gamma_1 \cap \Gamma_2 = \phi, \quad \text{measure of } \Gamma_1 \neq 0,$$

and the intersection $\overline{\Gamma}_1 \cap \overline{\Gamma}_2$ consists of finite points.

In [3], we showed the existence and the uniqueness of weak solution of evolutional problem for $2 \leq n \leq 4$. For $n = 2$, we have the following result:

**Theorem A**

Let $\Omega$ be a bounded domain in $\mathbb{R}^2$ with $C^1$ boundary satisfying Condition (H). If the function $g$ is in $L^\infty(\Omega \times (0,T))$, $\xi \in C^1(\overline{\Gamma}_1 \times [0,T])$, $\eta \in L^2(\Gamma_2 \times (0,T))$, $a_0 \in H$, $\tau_0 \in L^2(\Omega)$, then there exists one and only one weak solution $\{u, \theta\}$ of (1), (2) satisfying the initial condition (3). Furthermore

$$u \in C([0,T] : H), \quad \theta \in C([0,T] : L^2(\Omega)).$$

**Definition 2**

$\{u, \theta\}$ is called a periodic weak solution of (1), (2) with period $T_0$, if $\{u, \theta\}$ is a weak solution of (1), (2) for $T = T_0$ satisfying

$$u(x, T_0) = u(x, 0), \quad \theta(x, T_0) = \theta(x, 0). \quad (5)$$

We also obtained the existence of periodic weak solutions([3]).

**Theorem B**

Let $\Omega$ be a bounded domain in $\mathbb{R}^2$ with $C^1$ boundary satisfying Condition (H). Let $g(x, t), \xi(x, t), \eta(x, t)$ be periodic with respect to $t$ with period $T_0$, satisfying $g \in L^\infty(\Omega \times (0,T_0))$, $\xi \in C^1(\overline{\Gamma}_1 \times [0,T_0])$ and $\eta \in L^2(\Gamma_2 \times (0,T_0))$.

Set $g_\infty = ||g||_{L^\infty(\Omega \times (0,T_0))}$. If $\frac{\beta g_\infty}{\sqrt{\nu \chi}}$ is sufficiently small, then there exists a periodic weak solution of (1), (2) with period $T_0$. Furthermore

$$u \in C([0,\infty) : H), \quad \theta \in C([0,\infty) : L^2(\Omega)).$$

Now we can state our results. As for the uniqueness of periodic weak solutions, we obtained:
Theorem 1
Let \( \{u_{\pi}, \theta_{\pi}\} \) be a weak periodic solution of (1), (2) with period \( T_{0} \) such that for some \( p > 2 \),
\[
\operatorname{ess.\sup}_{t}\{c\|u_{\pi}(t)\|_{p} + \frac{1}{4\chi}(c\|\theta_{\pi}(t)\|_{p} + c'\beta g_{\infty})^{2}\} < \nu,
\]
where \( c \) and \( c' \) are constants depending on \( \Omega \). If \( \{u_{\pi} + u, \theta_{\pi} + \theta\} \) is a weak periodic solution of (1), (2) with period \( T_{0} \), then \( u = 0, \theta = 0 \).

Let \( g \in L^{\infty}(\Omega \times (0, \infty)) \), \( \xi \in C^{1}(\overline{\Gamma}_{1} \times [0, \infty)) \), \( \eta \in L^{2}(\Gamma_{2} \times (0, \infty)) \), \( a_{0} \in H, \tau_{0} \in L^{2}(\Omega) \). Let \( T \) be any positive number. Then there exists one and only one weak solution \( \{u_{T}, \theta_{T}\} \) of (1), (2) satisfying (3). Therefore, for \( T < T' \),
\[
u_{T}(t) = u_{T'}(t), \quad \theta_{T}(t) = \theta_{T'}(t) \quad \text{for} \quad \forall t \in (0, T)
\]
hold, and we can omit \( T \). This solution is called a global weak solution. We obtained the asymptotic property of solutions of Boussinesq equations as follows.

Theorem 2
Let \( g, \xi, \eta \) satisfy the condition of Theorem B, \( a_{0} \in H, \tau_{0} \in L^{2}(\Omega) \). Let \( \{u, \theta\} \) be a global weak solution of (1), (2) satisfying (3), \( \{u_{\pi}, \theta_{\pi}\} \) a periodic weak solution satisfying (6). Then
\[
\lim_{t \to \infty}\{\|u(t) - u_{\pi}(t)\|^{2} + \|\theta(t) - \theta_{\pi}(t)\|^{2}\} = 0.
\]

Remark
(i) Since \( u_{\pi} \in L^{2}(0, T : V) \cap C([0, T] : H) \), \( u_{\pi} \) belongs to the space \( L^{2p/(p-2)}(0, T : L^{p}(\Omega)) \) for \( \forall p > 2 \). Similarly \( \theta_{\pi} \) is in \( L^{2p/(p-2)}(0, T : L^{p}(\Omega)) \). The condition (6) is stronger than this one.
(ii) When (6) holds, such periodic solution is unique (Theorem 1).

References
