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Kyoto University
Example of zero viscosity limit for two dimensional nonstationary Navier–Stokes flows with boundary

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1. INTRODUCTION

Our purpose in this report is to give an example for a flow $u^\nu$ of the nonstationary, incompressible Navier–Stokes equations in $\Omega$ which is convergent to an Euler flow $\overline{u}$ as the viscosity $\nu$ tends to zero, where $\Omega$ is a bounded domain with smooth boundary.

Let $(u^\nu(t), p^\nu(t))$ be the unique global classical solution of the Navier–Stokes equations for the viscosity $\nu$ (we omit a variable $x \in \Omega$ for simplicity):

\begin{align}
&\frac{\partial u^\nu}{\partial t} - \nu \Delta u^\nu + (u^\nu, \nabla) u^\nu + \nabla p^\nu = f^\nu, \\
&(\text{NS}) \quad \text{div } u^\nu = 0 \quad \text{in } \Omega \times (0,T), \\
&u^\nu|_{\partial \Omega} = 0, \quad u^\nu|_{t=0} = u_0^\nu,
\end{align}

where $f^\nu(t)$ and $u_0^\nu$ are outer forces and initial data which satisfy the compatibility conditions $u_0^\nu|_{\partial \Omega} = 0$ and $\text{div } u_0^\nu = 0$ (for existence, see [3]). If $\nu \to 0$ in (NS) formally, we have the Euler equations which has the unique global classical solution $(\overline{u}(t), \overline{p}(t))$ (see, [1]):

\begin{align}
&\overline{u}_t + (\overline{u}, \nabla) \overline{u} + \nabla \overline{p} = \overline{f}, \\
&(\text{EE}) \quad \text{div } \overline{u} = 0 \quad \text{in } \Omega \times (0,T), \\
&\overline{u} \cdot n|_{\partial \Omega} = 0, \quad \overline{u}|_{t=0} = \overline{u}_0,
\end{align}

where $\overline{f}(t), \overline{u}_0$ and $n$ are outer forces, initial data and the unit outer normal to $\partial \Omega$ respectively with $\overline{u}_0$ satisfying the compatibility conditions $\overline{u}_0 \cdot n|_{\partial \Omega} = 0$ and $\text{div } \overline{u}_0 = 0$.

To prove the convergent of our flow in the example we need

**Theorem 1.** Assume

\begin{align}
(1) \quad &u_0^\nu \to \overline{u}_0 \quad \text{as } \nu \to 0 \text{ in } L^2(\Omega), \\
(2) \quad &f^\nu \to \overline{f} \quad \text{as } \nu \to 0 \text{ in } L^1(0,T; L^2(\Omega)).
\end{align}
Then the following three conditions are equivalent for $t \in [0,T]$.

(a) \[ \|u^\nu(t) - \bar{u}(t)\|_{L^2(\Omega)} \to 0 \text{ as } \nu \to 0 \text{ uniformly (pointwisely) in } t, \]

(b) \[ \lim_{\nu \to 0} \nu \int_0^t \int_{\partial \Omega} \bar{u}(\tau) \cdot n \times \text{rot } u^\nu(\tau) \, dSd\tau = 0 \text{ uniformly (pointwisely) in } t, \]

(c) \[ \lim_{\nu \to 0} \nu \int_0^t \int_{\partial \Omega} \bar{u}(\tau) \cdot n \times \text{rot } u^\nu(\tau) \, dSd\tau = 0 \text{ uniformly (pointwisely) in } t, \]

where $dS$ denotes surface area of $\partial \Omega$, $\text{rot } u = \partial u_2/\partial x_1 - \partial u_1/\partial x_2$ for vector fields $u(x) = (u_1(x), u_2(x))$ in $x = (x_1, x_2)$ and $a \times b = (a_2 b, -a_1 b)$ for a vector $a = (a_1, a_2)$ and a scalar $b$.

Remark. (1) Shirota also obtained Theorem 1 in somewhat different statements independently of ours, which is not published.

(2) Kato[2] obtained other equivalent conditions to (a) for the flows in a bounded domain of $\mathbb{R}^n$. One of them is

\[ \nu \int_0^T \|\text{grad } u^\nu(\tau)\|_{L^2(\Gamma_{c\nu})}^2 \, d\tau \to 0 \text{ as } \nu \to 0, \]

where $\Gamma_{c\nu}$ is the boundary strip of width $c\nu$ with $c > 0$ fixed.

2. Example

In this section $\Omega$ is the open unit disk \( \{z = (x_1, x_2) \in \mathbb{R}^2; |z| = (x_1^2 + x_2^2)^{1/2} < 1\} \).

For simplicity we denote $r = |z|$ and \( t(\cos \theta, \sin \theta) = z/|z|, \) where \( t(\cdot, \cdot) \) is a transported vector of \( (\cdot, \cdot) \). We note that the unit outer normal to $\partial \Omega$ is $z/|z|$. Furthermore we assume $f^\nu = \bar{f} = 0$.

We employ the stationary solution $\bar{u}$, defined by a rotating eddy, to the Euler equations (see, [4]):

\[ \bar{u}(x) (= \bar{u}_0(x)) = \left( \begin{array}{c} -\sin \theta \\ \cos \theta \end{array} \right) \frac{1}{r} \int_0^r r \bar{\omega}_0 (\rho) \, d\rho. \]
For any function $\overline{\omega}_{0} \in C([0,1])$ we have

\begin{align}
(2.2a) & \quad \text{div } \overline{u} = 0 \quad \text{in } \overline{\Omega}, \\
(2.2b) & \quad \overline{u} \cdot n = 0 \quad \text{on } \partial \Omega, \\
(2.2c) & \quad \text{rot } \overline{u} = \overline{\omega}_{0} \quad \text{in } \overline{\Omega}, \\
(2.2d) & \quad (\overline{u}, \nabla)\overline{u} = -\left(\frac{\cos \theta}{\sin \theta} \frac{\overline{\varphi}^{2}}{r^{3}}\right) = \nabla \overline{F} \quad \text{in } \overline{\Omega},
\end{align}

where $\overline{\varphi}(r) = \int_{0}^{r} \overline{\mu}_{0}(\rho) d\rho$ and $\overline{F}(r) = -\int_{0}^{r} \overline{\varphi}^{2}(s)/s^{3} ds$ which is well defined in $[0,1]$, since

\begin{equation}
|\overline{\varphi}(s)|^{2} \leq \int_{0}^{s} \rho^{2} d\rho \cdot \int_{0}^{r} \overline{\omega}^{2}(\rho) d\rho \leq \frac{1}{3}s^{2}\|\overline{\omega}\|_{L^{2}(0,1)}^{2}.
\end{equation}

Thus, $(\overline{u},\overline{p})$ is the solution of (EE) for $\overline{f} = 0$, if $\overline{u}$ is in (2.1) and $\nabla \overline{p} = -\nabla \overline{F}$.

We construct a non–stationary solution of (NS) in the form:

\begin{equation}
\overline{u}^{\nu}(x,t) = \left(\frac{-\sin \theta}{\cos \theta}\right) \frac{1}{r} \int_{0}^{r} \rho \omega^{\nu}(\rho,t) d\rho,
\end{equation}

where $\omega^{\nu}(r,t)$ is unknown. We note that $\overline{u}^{\nu}(x,t)$ in (2.4) satisfies the same identities as (2.2a)–(2.2d).

To construct $u^{\nu}(t)$, we reduce (NS) to an equation of

\begin{equation}
\overline{\varphi}^{\nu}(r,t) = \int_{0}^{r} \rho \omega^{\nu}(\rho,t) d\rho
\end{equation}

instead of $\omega^{\nu} = \text{rot } u^{\nu}$. By (2.2d) we have

\begin{align}
\overline{u}^{\nu}_{r} - \nu \Delta u^{\nu} + (u^{\nu}, \nabla)u^{\nu} + \nabla p^{\nu} \\
= \left(\frac{-\sin \theta}{\cos \theta}\right) \frac{1}{r} \left(\phi_{r}^{\nu} - \nu \phi_{r}^{\nu} + \frac{\nu}{r} \phi_{r}^{\nu}\right) + \nabla (F^{\nu} + p^{\nu}) = 0,
\end{align}

where $F^{\nu} = -\int_{0}^{r} (\phi^{\nu})^{2}(s,t)/s^{3} ds$ which is well defined because of (2.3), if $\omega^{\nu} \in L^{\infty}(0,T;L^{2}(\Omega))$. Since a vector field $\overline{t}(-\sin \theta, \cos \theta) \Phi(r)$ is solenoidal for a radially symmetric function $\Phi(r)$, that is,

\text{div}\left\{\left(\frac{-\sin \theta}{\cos \theta}\right) \Phi(r)\right\} = 0 \text{ in } \Omega \text{ and } \left(\frac{-\sin \theta}{\cos \theta}\right) \Phi(r) \cdot n = 0 \text{ on } \partial \Omega,$
then the equation of $\varphi^\nu$ is

$$
\varphi^\nu_t - \nu \varphi^\nu_{rr} + \frac{\nu}{r} \varphi^\nu_r = 0 \quad \text{for } (r, t) \in Q_T = (0, 1) \times (0, T),
$$

(E)

$$
\varphi^\nu_r |_{r=0} = 0, \quad \varphi^\nu_t |_{r=1} = 0 \quad \text{for } t \in (0, T),
$$

$$
\varphi^\nu |_{t=0} = \varphi_0^\nu = \int_0^1 \rho \omega_0^\nu(\rho) \, d\rho \quad \text{for } r \in (0, 1),
$$

here $\omega_0^\nu = \text{rot} u_0^\nu$ is given data, $T$ is any but fixed positive number and a subscript of $\varphi^\nu$ denotes partial differential with respect to its variable.

Thus let $\varphi^\nu(t)$ be a solution of (E). Then $(u^\nu, p^\nu)$ is a solution of (NS) for $f^\nu = 0$ and

$$
u^\nu = t(-\sin \theta, \cos \theta) \varphi_0^\nu/r,$

if $u^\nu(t)$ is defined by (2.4) and $p^\nu(t)$ is a solution of

$$
\Delta p^\nu = -\Delta F^\nu \quad \text{in } \Omega,
$$

$$
\nabla p^\nu \cdot n = -\nabla F^\nu \cdot n \quad \text{on } \partial \Omega.
$$

For existence of a solution to (E) we have

**THEOREM 2 (EXISTENCE OF THE FLOW).** Assume

$$
\omega_0^\nu(r) = \frac{1}{r} \delta_r(\varphi_0^\nu)(r), \quad \text{that is, } \varphi_0^\nu(r) = \int_0^r \rho \omega_0^\nu(\rho) \, d\rho
$$

for $\varphi_0^\nu \in C^{2+\alpha}([0, 1])$ with $\varphi_0^\nu(0) = \delta_r(\varphi_0^\nu)(0) = 0$ and $0 < \alpha < 1$. Then there exists an unique solution $\varphi^\nu \in C^{2,1}(Q)$ of (E), which satisfies

$$
\varphi^\nu(0, t) = 0 \quad \text{for } 0 \leq t < \infty,
$$

$$
|\varphi^\nu(r, t)| \leq \frac{\sqrt{3}}{3} ||\omega_0^\nu||_{L^2(0, 1)} \quad \text{in } Q,
$$

where $Q = \{(r, t); 0 \leq r \leq 1, \ 0 \leq t < \infty \text{ and } (r, t) \neq (1, 0)\}$ and $\delta_r$ denotes the differential operator $\frac{d}{dr}$.

Here $C^{2,1}(Q)$ (resp. $C^{2+\alpha}([0, 1])$) is the Banach space whose elements have second derivatives in $r$ and first derivatives in $t$ (resp. second derivatives in $r$). Furthermore second derivatives of the elements in $C^{2+\alpha}([0, 1])$ are Hölder continuous with exponent $\alpha$ in $r \in [0, 1]$.
Remark. In Theorem 2 we don’t require the compatibility condition $\varphi^\nu_0(1) = 0$. Thus for the existence of a solution $u^\nu(t)$ in (2.4) to (NS) we don’t need to assume $u^\nu_0|_{\partial \Omega} = 0$. Hence our solution $u^\nu(t)$ has the initial layer.

Finally our example is

THEOREM 3 (CONVERGENCE OF THE FLOW). Assume the same in Theorem 2 and $\overline{\omega}_0 \in C([0,1])$ in (2.1). We put $u^\nu_0 = \left( -\sin \theta, \cos \theta \right) \varphi^\nu_0/r$ and let $\overline{u}$ and $u^\nu(t)$ be in (2.1) and (2.4) respectively. Finally we assume that $u^\nu_0 \rightarrow \overline{u}_0$ in $L^2(\Omega)$ as $\nu \rightarrow 0$ and $||\omega^\nu_0||_{L^2(0,1)} \leq C$ independent of the viscosity $\nu$. Then we obtain for any but fixed $T > 0$

$$||u^\nu(t) - \overline{u}||_{L^2(\Omega)} \rightarrow 0 \quad \text{as} \ \nu \rightarrow 0 \ \text{uniformly in} \ t \in [0,T].$$

Remark. (1) Since we don’t require $\varphi^\nu_0(1) = 0$, we can take $\overline{u}_0$ as the initial data of (NS).

(2) If we assume that the compatibility condition $\varphi^\nu_0(1) = 0$ in Theorem 2, then by arguments likely to the below we can obtain

$$u^\nu \rightarrow \overline{u} \ \text{in} \ C(K) \ \text{as} \ \nu \rightarrow 0$$

for any compact subset $K \subset \overline{Q}$, even if $|\omega^\nu_0(r)| \leq \nu^{-\varepsilon}$ for $1 - \nu^{2\varepsilon} \leq r \leq 1$ and $\varepsilon < 1$ fixed.

The proof is omitted in this report.

The remaining part in this section is to prove Theorem 3.

We denote by $\psi(r,t)$, the solution of

$$\psi_t - \psi_{rr} + \frac{1}{r} \psi_r = 0 \quad \text{for} \ (r,t) \in Q_T,$$

\[ \left( E' \right)\]

$$\psi|_{r=0} = 0, \ \psi|_{r=1} = 0 \quad \text{for} \ t \in (0,T),$$

$$\psi|_{t=0} = \varphi^\nu_0 \equiv \int_0^r \rho \omega^\nu_0(\rho) \ d\rho \quad \text{for} \ r \in (0,1).$$

Then the uniqueness of the solution to $(E)$ implies

**LEMMA 1.** Let $\varphi^\nu(t)$ be the solution of $(E)$ in Theorem 2. Then we obtain

$$\varphi^\nu(r,t) = \psi(r,\nu t) \ \text{in} \ Q$$
The following lemma plays the essential role in the proof of Theorem 3.

**Lemma 2.** Let $\varphi^\nu(t)$ be the solution in Theorem 2. Then

$$|\int_0^t \varphi^\nu(1, \tau) \, d\tau| \leq C(||\omega^\nu_0||_{L^2(0,1)} + 1) \exp C(||\omega^\nu_0||_{L^2(0,1)}T + 1)$$

for any $t \in [0, T]$, where $C$ denotes several different positive constants independently of $\nu$ and $T$ here and after.

**Proof.** Let $\psi(t)$ be in Lemma 1 and $\nu$ be fixed. In (E') we replace $t$ by $\nu t$. Then it follows that

$$\psi_{rr}(r, \nu t) - \frac{1}{r} \psi_r(r, \nu t) - \psi_t(r, \nu t) = 0 \quad \text{in } Q.$$

To integrate this equation in $t$ on $(\epsilon, t)$ for any but fixed $\epsilon > 0$, then $f(r, t) = \int_\epsilon^t \psi(r, \nu \tau) \, d\tau$ satisfies

$$f_{rr} - \frac{1}{r} f_r - f_t = a_\nu \quad \text{in } Q^\epsilon_T = (0, 1) \times (\epsilon, T),$$

$$f_r|_{r=0} = 0, f|_{r=1} = 0 \quad \text{for } t \in (\epsilon, T),$$

$$f|_{t=\epsilon} = 0 \quad \text{for } r \in (0, 1),$$

where $a_\nu(r) = \psi(r, \nu \epsilon)$.

For $\chi \in C^\infty(\mathbb{R})$ which satisfies $0 \leq \chi(r) \leq 1$, $\chi = 1$ in $[2/3, \infty)$ and $\chi = 0$ in $(-\infty, 1/3]$, we put $z(t) = \chi^2 \exp f(t)$ and $Pz = z_{rr} - z_t$. Then we have

$$Pz = (\chi^2)' e + 4\chi' \chi f_r e + \chi^2 f_r^2 e + \chi^2 f_r e - \chi^2 f_r e$$

$$= \chi' \left\{ \chi^2(f_{rr} - f_t) + (\chi^2)'' + 4\chi' \chi f_r + \chi^2 f_r^2 \right\}$$

$$= \chi' \left\{ \frac{\chi^2}{r} f_r + \chi^2 a_\nu + (\chi^2)'' + 4\chi' \chi f_r + \chi^2 f_r^2 \right\}.$$ 

Since absolute values of $\chi^2/r$, $\chi'$ and $(\chi^2)''$ are estimated by $C$ for $r \in [0, 1]$, we obtain

$$Pz \geq Ce^t \left\{ -\frac{1}{\mu} - \mu \chi^2 f_r^2 - \chi^2 |a_\nu| - 1 - \frac{1}{\mu} - \mu \chi^2 f_r^2 + \chi^2 f_r^2 \right\}$$
for any $\mu > 0$.

Using the estimate $|\psi(r, \nu t)| \leq (1/\sqrt{3}) ||\omega_0||_{L^2(0,1)}$ for any $(r, t) \in Q$ in Theorem 2 and taking $\mu = 1/2$, then

$$Pz \geq -C(||\omega_0^\nu||_{L^2(0,1)} + 1) \exp(C||\omega_0^\nu||_{L^2(0,1)}T)$$

$$\equiv -M_1e^{M_2}.$$

Putting $y = z + 2M_1 \exp(M_2 + r)$, then $Py > 0$ holds. Hence the maximum principle implies $y(t)$ does not take its maximum in $Q_T^\epsilon \equiv [0, 1] \times [\epsilon, T]$ at $(r, t) \in (0, 1) \times (\epsilon, T]$. On the other hand, at parabolic boundary of $Q_T^\epsilon$, $y(t)$ holds as follows:

$$y|_{r=0} = z|_{r=0} + 2M_1 e^{M_2} = 2M_1 e^{M_2},$$

$$y|_{t=\epsilon} = z|_{t=\epsilon} + 2M_1 e^{M_2 + r} = 2M_1 e^{M_2 + r},$$

$$y|_{r=1} = z|_{r=1} + 2M_1 e^{M_2 + 1} = 2M_1 e^{M_2 + 1},$$

Hence at all points $(1, t)$ with $\epsilon \leq t \leq T$, $y(r, t)$ attains its maximum in $Q_T^\epsilon$. Then we conclude that

$$\frac{\partial y}{\partial r}|_{r=1} = \int_\epsilon^t \psi_r(r, \nu \tau) d\tau + 2M_1 e^{M_2 + 1} \geq 0.$$

Putting $\epsilon \rightarrow 0$, then

$$\int_0^t \psi_r(r, \nu \tau) d\tau \geq -2M_1 e^{M_2 + 1}.$$

The estimate from above of $f(r, t)$ with $\epsilon = 0$ can be established in a similar way by making the substitution $\hat{\tau} = \chi^2 \exp(-f)$ and considering $\hat{y} = \hat{\tau} - 2M_1 \exp(M_2 + r)$.

Hence by the identity in Lemma 1 the proof of our estimate is completed.

Finally we note that in this proof we use the method of the proof to Lemma 3 of Section 3 in [6]. □

Now we show Theorem 3. Since

$$\text{rot } u^\nu|_{\partial \Omega} = \omega^\nu(r, t)|_{r=1} = \frac{\partial}{\partial r} \int_0^r \rho \omega^\nu(\rho, t) d\rho|_{r=1}$$

$$= \varphi^\nu_r(1, t),$$
we have
\[ \overline{u} \cdot n \times \text{rot} \overline{u}'(t)|_{\partial \Omega} = -\varphi' \nu(1,t) \int_0^1 \rho \overline{\omega}_0(\rho) \, d\rho. \]

Thus we obtain an identity
\[ \nu \int_0^t \int_{\partial \Omega} \overline{u} \cdot n \times \text{rot} \overline{u}'(\tau) \, dSd\tau = -2\pi \nu \int_0^1 \rho \overline{\omega}_0(\rho) \, d\rho \cdot \int_0^t \varphi'(1,\tau) \, d\tau. \]

Hence it is easy to show that (c) in Theorem 1 holds because of Lemma 2. This proves Theorem 3 by Theorem 1.

For the proofs of Theorem 1 and Theorem 2, see [5].

References


