<table>
<thead>
<tr>
<th>Title</th>
<th>Example of zero viscosity limit for two dimensional nonstationary Navier-Stokes flows with boundary(Mathematical Analysis of Phenomena in Fluid and Plasma Dynamics)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Author(s)</td>
<td>MATSUI, Shin'ya</td>
</tr>
<tr>
<td>Citation</td>
<td>数理解析研究所講究録 745: 102-109</td>
</tr>
<tr>
<td>Issue Date</td>
<td>1991-02</td>
</tr>
<tr>
<td>URL</td>
<td><a href="http://hdl.handle.net/2433/102194">http://hdl.handle.net/2433/102194</a></td>
</tr>
<tr>
<td>Type</td>
<td>Departmental Bulletin Paper</td>
</tr>
<tr>
<td>Textversion</td>
<td>publisher</td>
</tr>
<tr>
<td>Institution</td>
<td>Kyoto University</td>
</tr>
</tbody>
</table>
Example of zero viscosity limit for two dimensional nonstationary Navier–Stokes flows with boundary

北海道情報大学 松井伸也 (Shin'ya MATSUI)

1. INTRODUCTION

Our purpose in this report is to give an example for a flow $u^\nu$ of the nonstationary, incompressible Navier–Stokes equations in $\Omega$ which is convergent to an Euler flow $\overline{u}$ as the viscosity $\nu$ tends to zero, where $\Omega$ is a bounded domain with smooth boundary.

Let $(u^\nu(t), p^\nu(t))$ be the unique global classical solution of the Navier–Stokes equations for the viscosity $\nu$ (we omit a variable $x \in \Omega$ for simplicity):

\[ u^\nu_t - \nu \Delta u^\nu + (u^\nu, \nabla)u^\nu + \nabla p^\nu = f^\nu, \]

(NS)

\[ \text{div } u^\nu = 0 \quad \text{in } \Omega \times (0,T), \]

\[ u^\nu|_{\partial \Omega} = 0, \; u^\nu|_{t=0} = u^\nu_0, \]

where $f^\nu(t)$ and $u^\nu_0$ are outer forces and initial data which satisfy the compatibility conditions $u^\nu_0|_{\partial \Omega} = 0$ and $\text{div } u^\nu_0 = 0$ (for existence, see [3]). If $\nu \to 0$ in (NS) formally, we have the Euler equations which has the unique global classical solution $(\overline{u}(t), \overline{p}(t))$ (see, [1]):

\[ \overline{u}_t + (\overline{u}, \nabla)\overline{u} + \nabla \overline{p} = \overline{f}, \]

(EE)

\[ \text{div } \overline{u} = 0 \quad \text{in } \Omega \times (0,T), \]

\[ \overline{u} \cdot n|_{\partial \Omega} = 0, \; \overline{u}|_{t=0} = \overline{u}_0, \]

where $\overline{f}(t), \overline{u}_0$ and $n$ are outer forces, initial data and the unit outer normal to $\partial \Omega$ respectively with $\overline{u}_0$ satisfying the compatibility conditions $\overline{u}_0 \cdot n|_{\partial \Omega} = 0$ and $\text{div } \overline{u}_0 = 0$.

To prove the convergent of our flow in the example we need

**Theorem 1.** Assume

(1) \[ u^\nu_0 \to \overline{u}_0 \quad \text{as } \nu \to 0 \text{ in } L^2(\Omega), \]

(2) \[ f^\nu \to \overline{f} \quad \text{as } \nu \to 0 \text{ in } L^1(0,T; L^2(\Omega)). \]
Then the following three conditions are equivalent for \( t \in [0, T] \).

(a) \[ \| u^\nu(t) - \overline{u}(t) \|_{L^2(\Omega)} \to 0 \text{ as } \nu \to 0 \text{ uniformly (pointwisely) in } t, \]

(b) \[ \lim_{\nu \to 0} \nu \int_0^t \int_{\partial \Omega} \overline{u}(\tau) \cdot n \times \text{rot} u^\nu(\tau) \, dSd\tau = 0 \text{ uniformly (pointwisely) in } t, \]

(c) \[ \lim_{\nu \to 0} \nu \int_0^t \int_{\partial \Omega} \overline{u}(\tau) \cdot n \times \text{rot} u^\nu(\tau) \, dSd\tau = 0 \text{ uniformly (pointwisely) in } t, \]

where \( dS \) denotes surface area of \( \partial \Omega \), \( \text{rot} u = \partial u_2/\partial x_1 - \partial u_1/\partial x_2 \) for vector fields \( u(x) = (u_1(x), u_2(x)) \) in \( x = (x_1, x_2) \) and \( a \times b = (a_2b, -a_1b) \) for a vector \( a = (a_1, a_2) \) and a scholar \( b \).

Remark. (1) Shirota also obtained Theorem 1 in somewhat different statements independently of ours, which is not published.

(2) Kato\[2\] obtained other equivalent conditions to (a) for the flows in a bounded domain of \( \mathbb{R}^n \). One of them is

\[ \nu \int_0^T \| \text{grad} u^\nu(\tau) \|_{L^2(\Gamma_{c\nu})}^2 \, d\tau \to 0 \text{ as } \nu \to 0, \]

where \( \Gamma_{c\nu} \) is the boundary strip of width \( c\nu \) with \( c > 0 \) fixed.

2. Example

In this section \( \Omega \) is the open unit disk \( \{ z = (z_1, z_2) \in \mathbb{R}^2; |z| = (z_1^2 + z_2^2)^{1/2} < 1 \} \). For simplicity we denote \( r = |z| \) and \( f'(\cos \theta, \sin \theta) = z/|z| \), where \( f'(\cdot, \cdot) \) is a transported vector of \( (\cdot, \cdot) \). We note that the unit outer normal to \( \partial \Omega \) is \( z/|z| \). Furthermore we assume \( f^\nu = \overline{f} = 0 \).

We employ the stationary solution \( \overline{u} \), defined by a rotating eddy, to the Euler equations (see, [4]):

(2.1) \[ \overline{u}(x) = \overline{u_0}(x) = \left( \begin{array}{c} -\sin \theta \\ \cos \theta \end{array} \right) \frac{1}{r} \int_0^r \rho \overline{\omega_0}(\rho) \, d\rho. \]
For any function $\overline{\omega}_0 \in C([0,1])$ we have

\begin{align*}
(2.2a) & \quad \text{div } \overline{u} = 0 \quad \text{in } \overline{\Omega}, \\
(2.2b) & \quad \overline{u} \cdot n = 0 \quad \text{on } \partial \Omega, \\
(2.2c) & \quad \text{rot } \overline{u} = \overline{\omega}_0 \quad \text{in } \overline{\Omega}, \\
(2.2d) & \quad (\overline{u}, \nabla)\overline{u} = -\left(\frac{\cos \theta}{\sin \theta}\right) \frac{\overline{\varphi}^2}{r^3} = \nabla \overline{F} \quad \text{in } \overline{\Omega},
\end{align*}

where $\overline{\varphi}(r) = \int_0^r \rho \overline{\omega}_0(\rho) \, d\rho$ and $\overline{F}(r) = -\int_0^r \overline{\varphi}^2(s)/s^3 \, ds$ which is well defined in $[0,1]$, since

\begin{equation}
(2.3) \quad |\overline{\varphi}(s)|^2 \leq \int_0^s \rho^2 \, d\rho \cdot \int_0^s \overline{\varphi}^2(\rho) \, d\rho \leq \frac{1}{3} s^3 ||\overline{\omega}||^2_{L^2(0,1)}.
\end{equation}

Thus, $(\overline{u}, \overline{p})$ is the solution of (EE) for $\overline{f} = 0$, if $\overline{u}$ is in (2.1) and $\nabla \overline{p} = -\nabla \overline{F}$.

We construct a non-stationary solution of (NS) in the form:

\begin{equation}
(2.4) \quad u^\nu(x,t) = \left(-\sin \theta \cos \theta \right) \frac{1}{\tau} \int_0^\tau \rho \omega^\nu(\rho, t) \, d\rho,
\end{equation}

where $\omega^\nu(r,t)$ is unknown. We note that $u^\nu(x,t)$ in (2.4) satisfies the same identities as (2.2a)–(2.2d).

To construct $u^\nu(t)$, we reduce (NS) to an equation of

\begin{equation}
(2.5) \quad \varphi^\nu(r,t) = \int_0^\tau \rho \omega^\nu(\rho, t) \, d\rho
\end{equation}

instead of $\omega^\nu = \text{rot } u^\nu$. By (2.2d) we have

\begin{align*}
\varphi^\nu_t - \nu \Delta \varphi^\nu + (u^\nu, \nabla)u^\nu + \nabla p^\nu \\
&= \left(-\sin \theta \cos \theta \right) \frac{1}{\tau} (\varphi^\nu_t - \nu \varphi^\nu + \frac{\nu}{\tau} \varphi^\nu) + \nabla(F^\nu + p^\nu) = 0,
\end{align*}

where $F^\nu = -\int_0^\tau (\varphi^\nu)^2(s,t)/s^3 \, ds$ which is well defined because of (2.3), if $\omega^\nu \in L^\infty(0,T; L^2(\Omega))$. Since a vector field $t(-\sin \theta, \cos \theta) \Phi(r)$ is solenoidal for a radially symmetric function $\Phi(r)$, that is,

\begin{align*}
\text{div} \left(\frac{-\sin \theta}{\cos \theta} \Phi(r)\right) &= 0 \quad \text{in } \Omega \quad \text{and} \quad \left(\frac{-\sin \theta}{\cos \theta} \Phi(r)\right) \cdot n = 0 \quad \text{on } \partial \Omega,
\end{align*}
then the equation of $\varphi^\nu$ is

$$
\varphi^\nu_t - \nu \varphi^\nu_r + \frac{\nu}{r} \varphi^\nu_r = 0 \quad \text{for } (r,t) \in Q_T \equiv (0,1) \times (0,T),
$$

(E)

$$
\varphi^\nu_r|_{r=0} = 0, \quad \varphi^\nu|_{r=1} = 0 \quad \text{for } t \in (0,T),
$$

$$
\varphi^\nu|_{t=0} = \varphi^\nu_0 \equiv \int_0^r \rho \omega^\nu_0(\rho) \, d\rho \quad \text{for } r \in (0,1),
$$

here $\omega^\nu_0 = \text{rot } u^\nu_0$ is given data, $T$ is any but fixed positive number and a subscript of $\varphi^\nu$ denotes partial differential with respect to its variable.

Thus let $\varphi^\nu(t)$ be a solution of (E). Then $(u^\nu, p^\nu)$ is a solution of (NS) for $f^\nu = 0$ and $u^\nu_0 = t(-\sin \theta, \cos \theta)\varphi^\nu_0/r$, if $u^\nu(t)$ is defined by (2.4) and $p^\nu(t)$ is a solution of

$$
\Delta p^\nu = -\Delta F^\nu \quad \text{in } \Omega,
$$

$$
\nabla p^\nu \cdot n = -\nabla F^\nu \cdot n \quad \text{on } \partial \Omega.
$$

For existence of a solution to (E) we have

**Theorem 2 (Existence of the Flow).** Assume

$$
\omega^\nu_0(r) = \frac{1}{r} \partial_r (\varphi^\nu_0)(r), \quad \text{that is, } \varphi^\nu_0(r) = \int_0^r \rho \omega^\nu_0(\rho) \, d\rho
$$

for $\varphi^\nu_0 \in C^{2+\alpha}([0,1])$ with $\varphi^\nu_0(0) = \partial_r (\varphi^\nu_0)(0) = 0$ and $0 < \alpha < 1$. Then there exists an unique solution $\varphi^\nu \in C^{2,1}(Q)$ of (E), which satisfies

$$
\varphi^\nu(0,t) = 0 \quad \text{for } 0 \leq t < \infty,
$$

$$
|\varphi^\nu(r,t)| \leq \frac{\sqrt{3}}{3} \|\omega^\nu_0\|_{L^2(0,1)} \quad \text{in } Q,
$$

where $Q = \{(r,t); 0 \leq r \leq 1, \ 0 \leq t < \infty \text{ and } (r,t) \neq (1,0)\}$ and $\partial_r$ denotes the differential operator $\frac{d}{dr}$.

Here $C^{2,1}(Q)$ (resp. $C^{2+\alpha}([0,1])$) is the Banach space whose elements have second derivatives in $r$ and first derivatives in $t$ (resp. second derivatives in $r$ ). Furthermore second derivatives of the elements in $C^{2+\alpha}([0,1])$ are Hölder continuous with exponent $\alpha$ in $r \in [0,1]$. 

4
Remark. In Theorem 2 we don’t require the compatibility condition \( \varphi_{0}^\nu(1) = 0 \). Thus for the existence of a solution \( u^\nu(t) \) in (2.4) to (NS) we don’t need to assume \( u_{0}^\nu|_{\partial \Omega} = 0 \).

Hence our solution \( u^\nu(t) \) has the initial layer.

Finally our example is

\textsc{Theorem 3 (Convergence of the Flow).} Assume the same in Theorem 2 and \( \overline{w}_{0} \in C([0,1]) \) in (2.1). We put \( u_{0}^\nu = \hat{t}(-\sin \theta, \cos \theta)\varphi_{0}^\nu \) and let \( \overline{w} \) and \( u^\nu(t) \) be in (2.1) and (2.4) respectively. Finally we assume that \( u_{0}^\nu \rightarrow \overline{w}_{0} \) in \( L^{2}(\Omega) \) as \( \nu \rightarrow 0 \) and \( ||\omega_{0}^\nu||_{L^{2}(0,1)} \leq C \) independent of the viscosity \( \nu \). Then we obtain for any but fixed \( T > 0 \)

\[ ||u^\nu(t) - \overline{w}||_{L^{2}(\Omega)} \rightarrow 0 \quad \text{as} \quad \nu \rightarrow 0 \text{ uniformly in} \quad t \in [0,T]. \]

Remark. (1) Since we don’t require \( \varphi_{0}^\nu(1) = 0 \), we can take \( \overline{w}_{0} \) as the initial data of (NS).

(2) If we assume that the compatibility condition \( \varphi_{0}^\nu(1) = 0 \) in Theorem 2, then by arguments likely to the below we can obtain

\[ u^\nu \rightarrow \overline{w} \text{ in} \quad C(K) \quad \text{as} \quad \nu \rightarrow 0 \]

for any compact subset \( K \subset \overline{Q} \), even if \( |\omega_{0}^\nu(r)| \leq \nu^{-\varepsilon} \) for \( 1 - \nu^{2\varepsilon} \leq r \leq 1 \) and \( \varepsilon < 1 \) fixed. The proof is omitted in this report.

The remaining part in this section is to prove Theorem 3.

We denote by \( \psi(r,t) \), the solution of

\[ \begin{aligned}
\psi_{t} - \psi_{rr} + \frac{1}{r} \psi_r &= 0 \quad \text{for} \quad (r,t) \in Q_T, \\
\psi_r|_{r=0} = 0, \quad \psi|_{r=1} = 0 \quad \text{for} \quad t \in (0,T), \\
\psi|_{t=0} &= \varphi_{0}^\nu \equiv \int_{0}^{1} \rho \omega_{0}^\nu(\rho) \, d\rho \quad \text{for} \quad r \in (0,1).
\end{aligned} \]

Then the uniqueness of the solution to (E) implies

\textsc{Lemma 1.} Let \( \varphi^\nu(t) \) be the solution of (E) in Theorem 2. Then we obtain

\[ \varphi^\nu(r,t) = \psi(r,\nu t) \quad \text{in} \quad Q \]
for a fixed $\nu$.

The following lemma plays the essential role in the proof of Theorem 3.

**Lemma 2.** Let $\varphi^\nu(t)$ be the solution in Theorem 2. Then

$$\left| \int_0^t \varphi^\nu(1,\tau) \, d\tau \right| \leq C(\|\omega^\nu_0\|_{L^2(0,1)} + 1) \exp C(\|\omega^\nu_0\|_{L^2(0,1)} T + 1)$$

for any $t \in [0, T]$, where $C$ denotes several different positive constants independently of $\nu$ and $T$ here and after.

**Proof.** Let $\psi(t)$ be in Lemma 1 and $\nu$ be fixed. In (E') we replace $t$ by $\nu t$. Then it follows that

$$\psi_{rr}(r, \nu t) - \frac{1}{r} \psi_r(r, \nu t) - \psi_t(r, \nu t) = 0 \quad \text{in } Q.$$ 

To integrate this equation in $t$ on $(\epsilon, t)$ for any but fixed $\epsilon > 0$, then $f(r, t) = \int_\epsilon^t \psi(r, \nu \tau) \, d\tau$ satisfies

$$f_{rr} - \frac{1}{r} f_r - f_t = a_s \quad \text{in } Q_T^\epsilon = (0,1) \times (\epsilon, T),$$

$$f_r|_{r=0} = 0, f|_{r=1} = 0 \quad \text{for } t \in (\epsilon, T),$$

$$f|_{t=\epsilon} = 0 \quad \text{for } r \in (0,1),$$

where $a_s(r) = \psi(r, \nu \epsilon)$.

For $\chi \in C^\infty(\mathbb{R})$ which satisfies $0 \leq \chi(r) \leq 1$, $\chi = 1$ in $[2/3, \infty)$ and $\chi = 0$ in $(-\infty, 1/3]$, we put $z(t) = \chi^2 \exp f(t)$ and $Pz = z_{rr} - z_t$. Then we have

$$Pz = (\chi^2)^{''} e^t + 4\chi\chi' f_r e^t + \chi^2 f_r^{''} e^t + \chi^3 f_{rr} e^t - \chi^2 f_t e^t$$

$$= e^t \left\{ \chi^2 (f_{rr} - f_t) + (\chi^2)^{''} + 4\chi\chi' f_r + \chi^3 f_r^{''} \right\}$$

$$= e^t \left\{ \frac{\chi^2}{r} f_r + \chi^2 a_s + (\chi^2)^{''} + 4\chi\chi' f_r + \chi^3 f_r^{''} \right\}.$$ 

Since absolute values of $\chi^2/r$, $\chi'$ and $(\chi^2)^{''}$ are estimated by $C$ for $r \in [0, 1]$, we obtain

$$Pz \geq Ce^t \left\{ -\frac{1}{\mu} - \mu \chi^2 f_r^2 - \chi |a_s| - 1 - \frac{1}{\mu} - \mu \chi^2 f_r^2 + \chi^2 f_r^{''2} \right\}$$

6
for any $\mu > 0$.

Using the estimate $|\psi(r, \nu t)| \leq (1/\sqrt{3})||\omega_0||_{L^2(0,1)}$ for any $(r, t) \in Q$ in Theorem 2 and taking $\mu = 1/2$, then

$$Pz \geq -C(||\omega_0^\nu||_{L^2(0,1)} + 1) \exp(C||\omega_0^\nu||_{L^2(0,1)}T)$$

$$\equiv -M_1 e^{M_2}.$$ 

Putting $y = z + 2M_1 \exp(M_2 + r)$, then $Py > 0$ holds. Hence the maximum principle implies $y(t)$ does not take its maximum in $Q_T \equiv [0,1] \times [\epsilon, T]$ at $(r, t) \in (0,1) \times (\epsilon, T]$. On the other hand, at parabolic boundary of $Q_T$, $y(t)$ holds as follows:

$$y|_{r=0} = z|_{r=0} + 2M_1 e^{M_2} = 2M_1 e^{M_2},$$

$$y|_{t=\epsilon} = z|_{t=\epsilon} + 2M_1 e^{M_2+\epsilon} = 2M_1 e^{M_2+\epsilon},$$

$$y|_{t=1} = z|_{t=1} + 2M_1 e^{M_2+1} = 2M_1 e^{M_2+1},$$

Hence at the all points $(1, t)$ with $\epsilon \leq t \leq T$, $y(r, t)$ attains its maximum in $Q_T^\epsilon$. Then we conclude that

$$\frac{\partial y}{\partial r}|_{r=1} = \int_\epsilon^t \psi_\nu(r, \nu \tau) d\tau + 2M_1 e^{M_2+1} \geq 0.$$ 

Putting $\epsilon \rightarrow 0$, then

$$\int_0^t \psi_\nu(r, \nu \tau) d\tau \geq -2M_1 e^{M_2+1}.$$ 

The estimate from above of $f(r, t)$ with $\epsilon = 0$ can be established in a similar way by making the substitution $\hat{z} = \chi^2 \exp(-f)$ and considering $\hat{y} = \hat{z} - 2M_1 \exp(M_2 + \nu)$. Hence by the identity in Lemma 1 the proof of our estimate is completed.

Finally we note that in this proof we use the method of the proof to Lemma 3 of Section 3 in [6].

Now we show Theorem 3. Since

$$\text{rot } u^\nu|_{\partial \Omega} = \omega^\nu(r, t)|_{r=1} = \frac{\partial}{\partial r} \int_0^r \rho \omega^\nu(\rho, t) \, d\rho|_{r=1}$$

$$= \varphi^\nu(1, t),$$
we have
\[ \overline{u} \cdot n \times \text{rot} u(t)|_{\partial \Omega} = -\varphi^\nu(1, t) \int_0^1 \rho \overline{\omega}_0(\rho) \, d\rho. \]

Thus we obtain an identity
\[ \nu \int_0^t \int_{\partial \Omega} \overline{u} \cdot n \times \text{rot} u(\tau) \, dS \, d\tau = -2\pi \nu \int_0^1 \rho \overline{\omega}_0(\rho) \, d\rho \cdot \int_0^t \varphi^\nu(1, \tau) \, d\tau. \]

Hence it is easy to show that (c) in Theorem 1 holds because of Lemma 2. This proves Theorem 3 by Theorem 1.

For the proofs of Theorem 1 and Theorem 2, see [5].

References


