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EXISTENCE OF STEADY INCOMPRESSIBLE FLOWS PAST AN OBSTACLE

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Introduction.

The stationary exterior Navier-Stokes problem consists in the study of a flow past a compact region $\mathcal{B}$, when all data are supposed to be independent of time. If we attach at a point C interior to T a reference system $\mathcal{R}\equiv\{C,x_1,x_2,x_3\}$ having the x_1-axis parallel to the velocity v_∞ of C, the steady exterior Navier-Stokes problem is formulated as follows:

Boundary value problem:

$$\begin{aligned} &\Delta v - \nabla p = R\, v.\nabla v - f, \\ &\nabla . v = 0, \qquad \text{in } \Omega\equiv\mathbb{R}^3\setminus\mathcal{B}; \\ &v(x) = v_*(x) \quad x\in\partial\Omega; \end{aligned} \tag{1}$$

Data at infinity:

$$v(x)\to v_\infty \quad \text{as } |x|\to\infty, \tag{2}$$

where R is a Reynolds number, f the external force and v_*, v_∞ the velocity at the boundary and at infinity, respectively. Generally, the usually adopted requirement on v_* is the condition of vanishing at the boundary, when $\mathcal{B}$ is a rigid body fixed in $\mathcal{R}$. However, other conditions can be easily figured out such as: i) $\mathcal{B}$ rigidly rotates around C; ii) $\mathcal{B}$ is fixed, but there is a device, composed by sinks and sources, by which fluid is removed from or added to the boundary $\partial\Omega$ at a prescribed rate v_* . In this latter case the boundary will be no more impermeable, i.e., $v_*.n \neq 0$, where n is the outer normal to $\partial\Omega$.

The objective of this paper is to investigate when the above boundary conditions are compatible with the circumstance in which the full momentum contributed into the liquid by the boundary of the flow is zero, *momentumless flows*, for $v_\infty=0$. The momentumless flow condition is analytically formulated as a restriction upon the drag $\mathcal{F}=\mathcal{F}(v,p)$ to be zero, i.e.,

$$\mathcal{F}(v,p) \equiv \int_{\partial\Omega} \{-T(v,p)\cdot n+R\ v_* v_*\cdot n\} d\Sigma = 0$$

where T is the stress tensor with components

$$T_{ij}(v,p) = -p\delta_{ij} + D_{ij}(v) ,$$

and D_{ij} is the deformation rate tensor. It appears evident from the definition of the drag that the boundary data (2) *do not control the drag*, hence, the condition $\mathcal{F}(v,p)=0$ should be derived appropriately. In other words, a first question to be set could be the following one: "Are compatibility conditions in the exterior problem needed? If yes, what kind of conditions are they?". It has long been a general belief that, in order to obtain momentumless flows, even in linear theories, some consistency condition upon v_* should be required, cf., e.g.,Finn (1965), Avudanayagam et al. (1986), Pukhnachov (1989). However, the first complete, satisfactory answer to the problem in the linear Stokes approximation has been provided only recently by Galdi & Simader (1990), where it is rigorously proved, among other things, that *momentumless flows are possible if and only if a consistency condition is satisfied by the data*. Thus a characterization of such flows is explicitly given in the linear case.

In the present paper we furnish an answer to the momentumless problem for the *full nonlinear* Navier-Stokes equations, when $v_\infty=0$. For the sake of simplicity, we shall confine ourselves to the case of zero external forces and to $\Omega\subset\mathbb{R}^3$, since the non-homogeneous n dimensional case does not present conceptual difficulties even though the relative changes have to be clarified. The case $v_\infty\neq 0$ will be anlyzed in a forthcoming work.

The paper is organized as follows: in section 1, we recall the results of Galdi & Simader (1990) on the existence of momentumless flows for the linear Stokes problem; next, in section 2, under the assumption of small Reynolds number, we prove that flows past a fixed obstacle suffering zero drag are possible *if and only if* the data at the boundary satisfy the *same* compatibility condition derived in Galdi & Simader (1990) for the Stokes problem, *cf.* equation (5) below. For instance, when $\mathcal{B}$ is a ball, the condition becomes

$$\int_{\partial\Omega} v_* d\sigma = 0.$$

1 - The linear Stokes problem.

In the present section we give some preparatory results concerning existence of solutions to the Stokes problem, in the Lebesgue space L^q, with zero external force. This is a particular case of the general theory developed by Galdi & Simader (1990). As is well known, the Stokes problem is described by

Boundary value problem:

$$\begin{aligned} &\Delta v - \nabla p = 0, \\ &\nabla . v = 0, \\ &v(x) = v_*(x) \quad x\in\partial\Omega; \end{aligned} \tag{3}$$

Data at infinity:

$$v(x)\to 0 \quad \text{as } |x|\to\infty. \tag{4}$$

In the sequel, it is assumed Ω of class $C^{2+\delta}$, $\delta>0$.

Let us denote by $H_0^{1,q}(\Omega)$ the completion of $C_0^\infty(\Omega)$ in the norm $|w|_{1,q} \equiv (\int_\Omega |\nabla w|^q)^{1/q}$, by $H^{-1,q}(\Omega)$ its dual space, with norm $|w|_{-1,q}$. The corresponding vector spaces will be denoted with the same symbol. By $D_0^{1,q}(\Omega)$ we indicate the subspace of $H_0^{1,q}(\Omega)$ of solenoidal vector fields.

The first step concerns the study of the boundary value problem (3). Denoting by $\mathcal{S}_q$ the linear subspace of $H_0^{1,q}(\Omega)xL^q(\Omega)$ constituted by solenoidal velocity fields v ($\in H_0^{1,q}(\Omega)$) and corresponding pressure p ($\in L^q(\Omega)$) solving the homogeneous system (3), it can be proved that $\mathcal{S}_q=\{0\}$ if $q<3$, while dim $\mathcal{S}_q=3$ if $q\geq 3$, cf. Galdi & Simader (1990)). Moreover, we set

$$\mathbb{S}_q=\{D_0^{1,q}(\Omega)xL^q(\Omega)\}\setminus\mathcal{S}_q$$

and, for v_* in the trace space $W^{1-(1/q),q}(\partial\Omega)$ (cf., e.g., Adams (1974)) we put

$$|v_*|_{1-1/q,q,\partial\Omega} \equiv b_q$$

where $|.|_{1-(1/q),q,\partial\Omega}$ is the norm of $W^{1-(1/q),q}(\partial\Omega)$. It holds:

(a) If $q>3/2$, there exists one and only one solution $(v, p)\in\mathbb{S}_q$ to (3). This solution verifies

$$\|v\|^{\sim}_{1,q} + \|p\|^{\sim}_{q} \leq cb_q ,$$

where the left hand side denotes the norm of (v,p) in the quotient space $\mathbb{S}_q$.

(b) If $q\leq 3/2$, the problem has a solution *if and only if*

$$\int_{\partial\Omega} v_*\cdot T(h,\pi)\cdot n d\Sigma=0 \qquad \text{for all } (h,\pi)\in\mathcal{S}_q \tag{5}$$

In such a circumstance the following estimate holds

$$|v|_{1,q} + |p|_q \leq cb_q .$$

As a corollary to the case (b), we have $v\in L^p$, $p=3q/(3-q)\leq 3$, *if and only if* the compatibility condition (5) is satisfied.

Let us, now, recall some asymptotic representation formula for the solution to the boundary value problem (3). To this end, let us denote by $U\equiv\{U_{i,j}\}$, $q\equiv\{q_i\}$ the fundamental solution of the Stokes system $(3)_{1,2}$, i.e.,

$$\begin{aligned} U_{i,j}(x) &= c_1[\delta_{i,j}|x|^{-1}+\frac{x_i x_j}{|x|^3}] \\ q_i(x) &= c_2\frac{x_i}{|x|^3} . \end{aligned} \tag{6}$$

with $c_i = c_i(n)$. Thus for all v, p $\in C^\infty(\Omega)$ solutions to $(3)_{1,2}$ corresponding to $f \in C_0^\infty(\Omega)$ with $v \in H_0^{1,q}(\Omega)$, $1<q<3$, the following asymptotic representation formula holds, see Chang & Finn (1961) and Galdi & Simader (1990).

$$\begin{aligned} v(x) &= U(x).\tau + \sigma(x) \\ p(x) &= q(x).\tau + \eta(x), \end{aligned} \qquad \text{as } |x| \to \infty \tag{7}$$

where $\tau = \int_{\partial\Omega} T(v,p).n d\Sigma$, and the derivatives of order $m \geq 0$ of $\sigma(x)$, $\eta(x)$ are infinitesimal of order $|x|^{-2-m}$ and $|x|^{-3-m}$, respectively. From the representation (7) we recognize at once that solutions with $v \in L^q(\Omega)$, $q \leq 3$, can exist if and only if the drag τ is zero. Thus, the linearized Stokes problem provides an explicit asymptotic behavior and momentumless flows are thus characterized in the Lebesgue space L^q, $q \leq 3$.

In the next section, we shall characterize the class of momentumless flows for the full non-linear system. To reach this goal, we shall need to recall some properties of the Green tensor. As is well known, this tensor is defined by the formulae

$$\begin{aligned} G_{i,j}(x,y) &\equiv U_{i,j}(x,y) + g_{i,j}(x,y) \\ \gamma_i(x,y) &\equiv q_i(x,y) + g_i(x,y) \end{aligned} \tag{8}$$

where $U_{i,j}, q_i$ is the singular solution (6), and $g_{i,j}$, g_i is the regular solution to

$$\Delta_y g_{i,j}(x,y) + \nabla_y g_j(x,y) = 0$$

$$\nabla_y . g_{i,j}(x,y) = 0, \qquad x,y \in \Omega$$

$$g_{i,j}(x,y)|_{y\in\partial\Omega} = U_{i,j}(x,y)|_{y\in\partial\Omega}, \quad \lim_{|y|\to\infty} g_{i,j}(x,y) = 0, \quad \forall x \in \Omega.$$

The Green's tensor exists, cf., e.g., Finn (1965), and verifies the Stokes problem with $v_* = 0$. Moreover, we have, see, e.g., Babenko (1972)

$$|D_k^\alpha G_{ij}(x,y)| \leq M/|x-y|^{1+\alpha}, \quad |\gamma_i| \leq M/|x-y|^2, \tag{9}$$

for all $x,y \in \Omega$, $i,j,k=1,2,3$ and $\alpha=0,1$.

n.2 The non-linear Navier-Stokes problem.

Concerning the non-linear Navier-Stokes problem (1), (2), it is well known that a smooth solution always exists, for any large data; furthermore, it possesses a finite Dirichlet integral and tends to zero at infinity, cf. Leray (1933), Finn (1959). We call these solutions D-*solutions*. This remarkable result could not have been predicted from any known experimental observation (bifurcation of the flow), nor it is in any way obvious from the mathematical structure of the equations. However, since the speed at which $v-v_\infty$ tends to zero at infinity is in general not given, it cannot be proved that the structure of the flow at infinity fits the one known for the linear problem[1].

As is well known, the following representation formula for D-solution p,v of problem (1), (2) holds

$$\begin{aligned} v(x) &= U(x).\tau + \int_\Omega v(y).\nabla v(y).U(x-y)dy + \sigma(x) \\ p(x) &= U(x).\tau + \eta(x), \end{aligned} \tag{10}$$

with τ, σ and η defined after (7). From (10) it can be proved that $v\in L^q$, $q\le 9/2$, implies $|v|=O(|x|^{-\alpha})$, $\alpha\ge 1/2$, cf. Galdi, forthcoming. Till now, the problem of the asymptotic decay of the D-solutions has been solved with the additional assumption of summability with power q greater than 9/2, (cf., e.g. Galdi, forthcoming, also for more general n-dimensional results). An alternative resolution to the problem can be given by changing the existence class and it is just the direction pursued in this work. To this end, we first prove the following key regularity lemma:

Lemma 1- *Let v, p be a smooth solution to the non-homogeneous Stokes problem*

$$\begin{aligned} \Delta v - \nabla p &= \operatorname{div} F, \\ \nabla . v &= 0, \end{aligned} \tag{11}$$

[1]To be precise, this problem arises only when $v_\infty=0$, since, if $v_\infty\neq 0$, D-solutions behave asymptotically as solutions to the linear Oseen equations, cf. Babenko (1973).

with $F\in L^3(\Omega)$. *Then, if* $v\in L^3(\Omega)$ *it follows* $\nabla v, p\in L^{3/2}(\Omega)$.

Proof- In order to prove $\nabla v, p\in L^{3/2}(\Omega)$ it is enough to show $\nabla v, p\in L^{3/2}(\Omega_R)$, $\Omega_R\equiv\{x\in\Omega: |x|>R\}$ for sufficiently large R. To this end, we apply the Helmoltz decompositon, see Solonnikov (1977), to the vector $F_{ij}e_j\in L^{3/2}(\Omega)$, where $\{e_j\}$ is a basis in $\mathfrak{R}$. It follows (in the distributional sense)

$$F = V + \nabla(u+\nabla\alpha),$$

$$\sum_{i=1}^{3} \frac{\partial}{\partial x_i} V_{ij} = 0,$$

$$\Delta u = \operatorname{div} F + \nabla\pi , \qquad (12)$$

$$\operatorname{div} u=0,$$

$$\frac{d}{dn} u = 0 \ , \text{ on } \partial\Omega,$$

and $\nabla u\in L^{3/2}(\Omega)$, $u\in L^3(\Omega)$. Subtracting $(11)_1$ from $(12)_3$ we thus get, in particular,

$$\Delta(v-u)=\nabla p_*,$$
$$\operatorname{div}(v-u) = 0, \qquad (13)$$

Observing that, by well-known results on the Stokes problem, $(v-u)\in C^\infty(\Omega)\cap L^3(\Omega)$, see *e.g.* Cattabriga (1961), we can employ the representation formula (7) which holds, in particular, for any solution to the problem $(13)_{1,2}$ which is summable in a neighbourhood of infinity:

$$(v-u)(x) = U(x).\tau + \sigma(x)$$
$$p_*(x) = U(x).\tau + \eta(x). \qquad (14)$$

for suitable vector τ. Since $\sigma\in L^3(\Omega_R)$, while $U(x)\cdot\tau\notin L^3(\Omega_R)$ for all non-zero τ, it must be $\tau=0$. As immediate consequence we deduce p_*, $\nabla(v-u)\in L^{3/2}(\Omega_R)$ which in turn imply $\nabla v, p\in L^{3/2}(\Omega_R)$. The proof of the lemma is then completed.

We are, now, in position to prove the following

Theorem 1- *Assume* R *sufficiently small. Then a solution* v, p *to the problem* (1),(2) *with* $v_\infty=0$ *such that*

$$v \in L^3(\Omega),\ p, \nabla v \in L^{3/2}(\Omega),$$

exists if the consistency condition (5) *is satisfied.*

Remark 1. The consistency condition (5) is exactly the same one required in the linear case.

Remark 2. Since the functions h, π are explicitly given, it is not difficult to check, that in the case of a flow past a sphere condition (5) reduces to

$$\int_{\partial\Omega} v_* d\Sigma = 0,$$

cf. Galdi & Simader (1990).

Proof -Let the consistency condition (5) be satisfied, we shall prove here that there exists a solution v,p to the full non-linear Navier-Stokes problem such that $v \in L^3(\Omega)$, and $p, \nabla v \in L^{3/2}(\Omega)$. To this end it is suitable to transform the starting differential problem to the following non-linear Fredholm integral system, cf. Finn (1965):

$$v(x) = v_0(x) + \int_\Omega u(y).\nabla G(x,y).u(y)dy;$$

$$p(x) = p_0(x) + \int_\Omega u(y).\nabla \gamma(x,y).u(y)dy. \tag{15}$$

Here, $v_0 \in L^3(\Omega)$, $p_0 \in L^{3/2}(\Omega)$ denotes the solution to the linear Stokes problem (3), (4), which exists because of (5). Moreover, $G(x,y)$ is the Green tensor defined in (8). We prove now that the non-linear integral Fredholm operator $I(u) \equiv v$ defined via $(15)_1$ is a bounded contraction in the space W of functions having gradients in $L^{3/2}(\Omega)$ and vanishing (in the mean) at infinity. It is known that if $w \in W$ then $w \in L^3(\Omega)$, see, e.g., Galdi, forthcoming. The first step is to prove that $I(u) \in L^3(\Omega)$. This is a consequence of the estimate (9) and of the Sobolev theorem. Specifically, from (9) we deduce that

$$\int_\Omega u(y).\nabla G(x,y).u(y)dy \le M \int_\Omega \frac{u^2}{|x-y|^2} \equiv A(u^2),$$

where $A(u^2)$ is a weakly singular integral. Therefore, the Sobolev theorem applies to show

$$u^2 \in L^{3/2}(\Omega) \Rightarrow A(u^2) \in L^p(\Omega), \qquad p = \frac{3.(3/2)}{3-(3/2)} = 3.$$

and in particular there exists a positive constant c such that

$$\|A(u^2)\|_3 \le c\|u^2\|_{3/2} = \|u\|_3^2. \tag{16}$$

Relation (16) furnishes the boundedness of the operator $I(u)$ in $L^3(\Omega)$, in fact we have

$$\|I(u)\|_3 \le \|v_0\|_3 + c\|u\|_3^2.$$

The second step is to show that $I(u) \in W$. To see this, assume u smooth. Differentiating relation $(15)_1$ we obtain that the solution of the integral equation $(15)_1$ satisfies also the differential equations

$$\Delta v = \nabla p + \mathrm{div}(u \otimes u),$$
$$\mathrm{div} v = 0,$$
$$v|_{\partial\Omega} = v_*,$$

with $u \otimes u \in L^{3/2}(\Omega)$. By known results on the Stokes system, e.g., Cattabriga (1961), v,p is smooth in Ω and since $v \in L^3(\Omega)$ we can apply Lemma 1 to deduce that $p, \nabla v \in L^{3/2}(\Omega)$. In addition, the following estimate holds, cf. Galdi & Simader (1990)

$$|v|_{1,3/2} + |p|_3 \le c(b_3 + |u|_3^2) \le c' b_3.$$

Clearly, since the smooth functions are dense in W, this latter inequality continues to hold for all $u \in W$. From this we easily obtain the validity of the inequality

$$|I(u_2 - u_1)|_{1,3/2} \le c(\|u_1\|_3 + \|u_2\|_3)\|u_2 - u_1\|_3$$

and we conclude that the operator I defines a contraction in W, provided v_* (i.e. the Reynolds number) is sufficiently small. The proof is then completed.

We now give an important regularity result for our solution which cannot be deduced from the classical results of Cattabriga (1961) nor from the reasoning of Temam (1979, p.172)[2]. Specifically, we can prove

[2]Actually, Temam's regularity proof is based on the following recurrence argument. If v is a solution (in the

Lemma 2 - *Let* v,p *be a weak solution to the nonlinear Navier-Stokes equations such that* $\nabla v,p\in L^{3/2}(\Omega)$, *then* $v,p\in C^{\infty}(C)$. *Moreover,* $\nabla v\in L^{2}(\Omega)$.

Proof - The first step is the proof that $\nabla v,p\in L^{3/2}(\Omega)$ implies $D^2v\in L^{3/2}(C)$, for any compact set C contained in Ω. To this end, we notice that, given any vector $v\in L^3(\Omega)$, for all arbitrary small positive constant ε there exists a decomposition of v into the sum of two vectors $v_1\in L^3(\Omega)$, $v_2\in L^{\infty}(\Omega)$ such that

$$\|v_1\|_3<\varepsilon, \qquad \|v_2\|_{\infty}<c, \tag{17}$$

with c positive constant, depending on v . Moreover, denoting by C' a compact set contained in Ω with $\overline{C}\subset C'$, we let $\varphi\in C^{\infty}(\Omega)$ be one in C and zero in C'. Writing p', v' in place of $p\varphi$, $v\varphi$ and multiplying by φ equations (1) one easily recognize that v',p' verify the equations

$$\begin{aligned} &\Delta v'-\nabla p' = F'+v.\nabla v', \\ &\operatorname{div} v' = g, \qquad\qquad \text{in } C' \\ &v'|_{\partial C'}=0, \end{aligned} \tag{18}$$

where

$$F'\equiv \operatorname{div}[\nabla\varphi\otimes v]+\nabla\varphi.\nabla v-p\nabla\varphi-(v.\nabla\varphi)v, \qquad g\equiv v.\nabla\varphi.$$

The summability properties of the solution v,p delivers

$$F'\in L^{3/2}(\Omega), \quad g\in W^{1,3/2}(\Omega).$$

The key tool of our proof is a Lemma of Galdi, forthcoming, where a regularity result is proved for the following linearized version of the problem (1):

$$\begin{aligned} \Delta w-\nabla\pi &= v.\nabla w + F' \\ \operatorname{div} w &= g \qquad \text{in } C' \\ w &=0 \qquad \text{at } \partial C'. \end{aligned} \tag{19}$$

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sense of distribution) to $\Delta v=\nabla p+v\cdot\nabla v$, starting with $v\cdot\nabla v\in L^{q}_{loc}(\Omega)$, $q>1$, by Cattabriga's results on the Stokes problem, one deduces $v\in W^{2,q}_{loc}(\Omega)$. This implies that $v\cdot\nabla v\in L^{r}_{loc}(\Omega)$ for some $r>q$. Iterating this procedure as many times as we please, we finally obtain $v\in C^{\infty}(\Omega)$. In our case, the above argument fails since $q=1$ and the Cattabriga's results do not hold.

In particular, for three-dimensional flows the Lemma asserts that if $F'\in L^q(C')$, $g\in W^{1,q}(C')$, $6/5\le q<3$, and the solenoidal vector v can be decomposed into the sum of two vectors $v_1\in L^3(C')$, $v_2\in L^\infty(C')$ satisfying (17), there exists one solution w, π to system (19) with $w\in W^{2,q}(\Omega)$, $\pi\in W^{1,q}(\Omega)$ enjoying the estimate

$$\|w\|_{2,q}+\|\pi\|_{1,q}$$
$$\le c[\|F'\|_q+\|g\|_{1,q}+\|w_*\|_{2-1/q,q,\partial C}](1+\|v_2\|_\infty). \tag{20}$$

Furthermore, this solutions is unique in the class of generalized solutions to (19) having $\nabla w\in L^q(\Omega)$. Taking v as the solution of equations (1) constructed in Theorem 1, it follows $\nabla v\in L^{3/2}(\Omega)$, and so it is easy to recover that the hypotheses of the Lemma are satisfied with $q=3/2$. Therefore, we have $\nabla v'\in L^2(C)$ and, by classical results, we conclude $v',p'\in C^\infty(C)$.

We next prove $\nabla v\in L^2(\Omega)$. Multiplying $(1)_1$ by v and integrating over Ω delivers

$$\int_{\partial\Omega} v_*[-v_*.nv_*+\frac{d}{dn}v+pn]d\Sigma+\int_{\Omega_R}\nabla v:\nabla v dx+\int_{\Sigma_R} v.[\frac{d}{dn}v+pn+v.nv]d\Sigma = 0$$

where Ω_R is the intersection of Ω with a sphere of radius R. From the properties $v\in L^3(\Omega)$, $p,\nabla v\in L^{3/2}(\Omega)$ and by the Hölder inequality we have

$$\int_{\Sigma_R} v.[T.n+v.nv]d\Sigma\le(\int_{\Sigma_R} v^3)^{1/3}(\int_{\Sigma_R}[T.n+v.nv]^{3/2}d\Sigma)^{2/3},$$

and we conclude that the integral over Σ_R tends to zero as R^{-1}, when $R\to\infty$ (at least along a sequence), proving the finiteness of $\|\nabla v\|_2$. The proof of the Lemma is thus completed.

A further regularity property regards the asymptotic behavior of the solutions. Let us prove, in fact, that any solution $v\in L^3(\Omega)$ is such that

$$|v|=O(|x|^{-1}). \tag{21}$$

This will be achieved by using the representation formula (8) and a method used in elasticity for studying the Saint-Venant problem. in unbounded domains, see , *e.g.*,

Galdi, Knops & Rionero (1985). Specifically, from the representation (8) we have only to increase appropriately the non-linear term:

$$\int_\Omega v.\nabla v.U dx \le \int_{\Omega_1} \frac{|v.\nabla v|}{|x-y|} dy + \int_{\Omega_2} \frac{|v.\nabla v|}{|x-y|} dy,$$

where Ω_1 is the intersection of Ω with a sphere of fixed radius R, sufficiently large to include the boundary $\partial\Omega$, while $\Omega_2 \equiv \mathbb{R}^3/\Omega_1$ and contains the point x for $|x|\to\infty$. We use different inequalities for the two integrals. Precisely, we have

$$\begin{aligned} &\int_{\Omega_1} \frac{|v.\nabla v|}{|x-y|} dy \le \frac{1}{|x-y_*|} \|v\|_3 \|\nabla v\|_{3/2}, \qquad y_* \in \partial\Omega_1; \\ &\int_{\Omega_2} \frac{|v.\nabla v|}{|x-y|} dy \le \left(\int_{\Omega_2} \frac{v^2}{|x-y|^2} dy \int_{\Omega_2} |\nabla v|^2 dy \right)^{1/2}. \end{aligned} \tag{22}$$

The first integral behaves as $|x|^{-1}$ at infinity, because y_* varies in a bounded set. In order to prove the rate of decay for the integral over Ω_2, we recall the following inequality

$$\int_{\Omega_2} \frac{v^2}{|x-y|^2} dy \le C \int_{\Omega_2} |\nabla v|^2 dy,$$

cf. *e.g.* Finn (1965), so that $(22)_2$ yields

$$\int_{\Omega_2} \frac{|v.\nabla v|}{|x-y|} dy \le C \int_{\Omega_2} |\nabla v|^2 dy. \tag{23}$$

Set

$$G(R) \equiv \int_R^\infty \int_{\Sigma_R} |\nabla v|^2 d\Sigma dr$$

and observe that

$$G'(R) = -\int_{\Sigma_R} |\nabla v|^2 < 0. \tag{24}$$

Multiplying $(1)_1$ by v and integrating over Σ_r for $r\in(R,\infty)$ and then integrating over R for $R\in(0,\infty)$ furnishes

$$\int_0^\infty \int_R^\infty \int_{\Sigma_r} |\nabla v|^2 d\Sigma dr dR + \int_0^\infty \int_{\Sigma_R} v.T(v,p).n d\Sigma dR = 0. \qquad (25)$$

From identity (25) we infer, at once, that

$$\int_0^\infty G(R)dR < \infty \qquad (26)$$

because $v \in L^3$ and $T \in L^{3/2}$. In view of (24), (26) it follows $G(R) \leq C/R$ and from (22), (23) we are thus allowed to conclude that also the integral over Ω_2 behaves as $|x|^{-1}$ at infinity. The proof is therefore completed.

The following concluding result gives necessary conditions for the existence of solutions determined in Theorem 1. In particular, it shows that *the momentumless condition* $\mathcal{F}(v,p)=0$ *must be satisfied.*

Theorem 2 - *Assume that there exists a solution* v,p *with* $\nabla v \in L^3$ *of the nonlinear Navier-Stokes problem* (1), (2), *then necessarily:*

(i) v,p *verify the momentumless condition* $\mathcal{F}(v,p)=0$;

(ii) v_* *satisfies the consistency condition* (5).

Proof - To show (i) we observe that from Lemma 2 we have $\nabla v \in L^2(\Omega)$ and so v obeys the asymptotic representation (10). Integrating by parts in this relation the nonlinear term and taking into account the definition of the drag $\mathcal{F}$, with the aid of (21) we deduce

$$v(x) = U(x).\mathcal{F} - \int_\Omega v(y).\nabla U(x-y).v(y)dy + \sigma(x).$$

Since v, σ and the nonlinear term belong to $L^3(\Omega)$, it follows $U(x).\mathcal{F} \in L^3(\Omega)$ which is possible only if $\mathcal{F}=0$.The proof of (ii) is a consequence of the integral representation formula for v

$$v(x) = v_0(x) + \int_\Omega v(y).\nabla G(x,y).v(y)dy$$

which now tells us that v_0 is in L^3, and of a statement of Galdi & Simader (1990), recalled as (b) in section 1 of this paper, which ensures that, under such a circumstance,

necessarily v_* verifies (5).

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