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The Nonlinear Schrödinger Limit and the Initial Layer of the Zakharov Equations

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A simple system of equations describing the propagation of Langmuir turbulance in an unmagnetized, completely ionized hydrogen plasma was first obtained by Zakharov [12] by means of a two-fluid description of the plasma. The system consisting of the ion sound equation and the electric field propagation equation with nonlinear coupling terms is derived from Maxwell's equations and linearlized hydrodynamical equations. Another derivation from a Lagrangian formalism is given by Gibbons, Thornhill, Wardrop & ter Haar [5].

In a suitably scaled coordinates, the Zakharov equations are the following:

$$i\partial_t E + \Delta E = nE, \tag{1.1}$$

$$\lambda^{-2}\partial_t^2 n - \Delta n = \Delta|E|^2, \tag{1.2}$$

where E and n are functions on the time-space $\mathbb{R}\times\mathbb{R}^N$ with values in $\mathbb{C}^N$ and $\mathbb{R}$, respectively, and $\lambda > 0$ is a parameter. In these equations E is the slowly varying complex amplitude of the electric field $\mathcal{E}$ of the Langmuir waves with plasma frequency ω_p:

$$\mathcal{E}(t,x) = \mathrm{Re}(E(t,x)\exp(-it\omega_p)),$$

n is the deviation of the ion density from its equilibrium, λ is the ion sound speed, the RHS of (1.1) describes the shift of plasmon frequency caused by the slow density variation n, and the RHS of (1.2) describes the driving force caused by the pressure of plasmon gas.

In the limit $\lambda \longrightarrow \infty$ in (1.2) we formally get the equation $\Delta(n + |E|^2) = 0$ so that $n = -|E|^2$ if $n + |E|^2$ vanishes at infinity. Therefore in the limit the Zakharov equations (1.1)-(1.2) reduce the nonlinear Schrödinger equation

$$i\partial_t E + \Delta E = -|E|^2 E.$$

Thus the Zakharov equations can be regarded as a natural extension of the nonlinear Schrödinger equation in order to take a finite response time of the nonlinear medium of the ion part of the plasma into account and the limit $\lambda \longrightarrow \infty$ turns out to be related to an instant response of the medium.

We consider the Cauchy problem for the Zakharov equations and the nonlinear Schrödinger equation and examine the rate of convergence of the solutions as $\lambda \longrightarrow \infty$. The Cauchy problem for the Zakharov equations are written with the subscript λ by

$$\left.\begin{aligned} & i\partial_t E_\lambda + \Delta E_\lambda = n_\lambda E_\lambda, \qquad t > 0,\ x \in \mathbb{R}^N, \\ & \lambda^{-2}\partial_t^2 n_\lambda - \Delta n_\lambda = \Delta|E_\lambda|^2, \quad t > 0,\ x \in \mathbb{R}^N, \\ & E_\lambda(0,x) = E_0(x),\ n_\lambda(0,x) = n_0(0,x),\ \partial_t n_\lambda(0,x) = n_1(x), \end{aligned}\right\} \qquad (Z_\lambda)$$

where $\lambda > 1$ and (E_0, n_0, n_1) are given initial data. We consider the Cauchy problem for the nonlinear Schrödinger equation with the same initial condition E_0 as in (Z_λ):

$$\left.\begin{aligned} & i\partial_t E + \Delta E = -|E|^2 E,\ t > 0,\ x \in \mathbb{R}^N, \\ & E(0,x) = E_0(x) \end{aligned}\right\} \qquad (NLS)$$

From now on we assume that the initial data (E_0, n_0, n_1) are in the Schwartz space $\mathcal{S}$ for simplicity.

It was shown in H. Added & S. Added [1][2], Ozawa & Tsutsumi [7], Schochet & Weinstein [9], and C. Sulem & P. L. Sulem [10] that if $n_1 \in \dot{H}^{-1}$ and $1 \le N \le 3$, then (Z_λ) and (NLS) have unique solutions n_λ, E_λ, $E \in C^\infty([0,T_{max}); H^\infty)$ with the maximal existence time T_{max}, such that for any T with $0 < T < T_{max}$ and any $m \in \mathbb{N}$ there exist two positive constants C_0 and λ_0 satisfying

$$\sup_{\lambda \ge \lambda_0} \sup_{0 \le t \le T} (\|E_\lambda(t)\|_{H^{m+1}} + \|n_\lambda(t)\|_{H^m}) \le C_0, \qquad (1.3)$$

where $\dot{H}^{-1} = \{\psi \in \mathcal{S}'; (-\Delta)^{-1/2}\psi \in L^2\}$, $H^\infty = \bigcap_{k\geq 0} H^k$, T_{max} is independent of λ, and in particular, $T_{max} = \infty$ if $N = 1$. Moreover, it is shown in [2] that if $n_1 = \nabla\cdot\phi$ with $\phi \in \mathcal{S}$, then for any T with $0 < T < T_{max}$ and any $m \in \mathbb{N}$ there exist two positive constants C_1 and λ_1 such that for any $\lambda \geq \lambda_1$

$$\sup_{0\leq t\leq T} (\|E_\lambda(t)-E(t)\|_{H^m} + \|n_\lambda(t)+|E_\lambda(t)|^2-(\cos\lambda t(-\Delta)^{1/2})(n_0+|E_0|^2)\|_{H^{m-1}})$$

$$\leq \begin{cases} C_1\lambda^{-1/2} & \text{if } n_0+|E_0|^2 \neq 0 \text{ and } N \leq 2, & (1.4) \\ C_1\lambda^{-1}\log\lambda & \text{if } n_0+|E_0|^2 \neq 0 \text{ and } N = 3, & (1.5) \\ C_1\lambda^{-1} & \text{if } n_0+|E_0|^2 = 0. & (1.6) \end{cases}$$

There is a discrepancy between the non-compatible case $n_0+|E_0|^2 \neq 0$ and the compatible case $n_0+|E_0|^2 = 0$ concerning the rate of convergence as $\lambda \longrightarrow \infty$. This corresponds to an initial layer phenomenon and the term $Q^{(1)}(\lambda t) = (\cos \lambda t(-\Delta)^{1/2})(n_0+|E_0|^2)$ represents the first initial layer for (Z_λ). We note that $Q^{(1)}(t)$ solves the wave equation

$$\left.\begin{aligned} &\partial_t^2 Q^{(1)} - \Delta Q^{(1)} = 0, \ t > 0, \ x \in \mathbb{R}^N, \\ &Q^{(1)}(0,x) = n_0(x) + |E_0(x)|^2, \ \partial_t Q^{(1)}(0,x) = 0. \end{aligned}\right\}$$

On the other hand, if we perform a formal perturbation method under the assumption

$$\left.\begin{aligned} E_\lambda &= E^{(0)} + \lambda^{-1}E^{(1)} + \lambda^{-2}E^{(2)} + O(\lambda^{-3}), \\ n_\lambda &= n^{(0)} + \lambda^{-1}n^{(1)} + \lambda^{-2}n^{(2)} + O(\lambda^{-3}), \end{aligned}\right\} \quad (1.7)$$

as $\lambda \longrightarrow \infty$ in a suitable sense, with smooth functions $E^{(j)}$, $n^{(j)}$ on $\mathbb{R}\times\mathbb{R}^N$ independent of λ, we get

<u>zeroth order equations</u>

$$n^{(0)} = - |E^{(0)}|^2, \ i\partial_t E^{(0)} + \Delta E^{(0)} = - |E^{(0)}|^2 E^{(0)};$$

first order equations

$$n^{(1)} = -2\mathrm{Re}(\overline{E^{(0)}}\cdot E^{(1)}),$$

$$i\partial_t E^{(1)} + \Delta E^{(1)} = -|E^{(0)}|^2 E^{(1)} - 2(\mathrm{Re}(\overline{E^{(0)}}\cdot E^{(1)}))E^{(0)};$$

second order equations

$$\partial_t^2 n^{(0)} - \Delta n^{(2)} = \Delta(|E^{(1)}|^2 + 2\mathrm{Re}(\overline{E^{(0)}}\cdot E^{(2)})),$$

$$i\partial_t E^{(2)} + \Delta E^{(2)} = -|E^{(0)}|^2 E^{(2)} - 2(\mathrm{Re}(\overline{E^{(0)}}\cdot E^{(1)}))E^{(1)} + n^{(2)}E^{(0)};$$

with the initial conditions $n^{(0)}(0) = n_0$, $\partial_t n^{(0)}(0) = n_1$, $E^{(0)}(0) = E_0$, and $n^{(j)}(0) = 0$, $E^{(j)}(0) = 0$ for $j = 1, 2$. From the equation for $E^{(0)}$ we see that $E^{(0)}$ is equal to the solution E to (NLS). For the equation of $E^{(1)}$, we compute the time derivative of $\|E^{(1)}(t)\|_2^2$ and use Gronwall's inequality to get $E^{(1)} = 0$, where $\|\cdot\|_2$ denotes the L^2- norm. This leads to

$$\left.\begin{aligned} E_\lambda &= E + \lambda^{-2}E^{(2)} + O(\lambda^{-3}), \\ n_\lambda + |E_\lambda|^2 &= \lambda^{-2}(n^{(2)} + 2\mathrm{Re}(\bar{E}\cdot E^{(2)})) + O(\lambda^{-3}), \end{aligned}\right\} \quad (1.8)$$

which is exactly the same condition as the one used in Gibbons [4] as the first step of his formal derivation of (NLS) from (Z_λ).

Under the assumption (1.7) we have for any $k \geq 0$

$$\partial_t^k(n^{(0)} + |E^{(0)}|^2)(0) = 0,$$

which implies

$$n_0 + |E_0|^2 = 0,$$

$$n_1 + (\partial_t|E|^2)(0) = n_1 + 2\mathrm{Im}\sum_{j=1}^{N} \partial_j(\overline{E_0}\cdot\partial_j E_0) = 0,$$

and so on. Hence we necessarily take a strong compatible case so that this is a main drawback of the formal perturbation method, but (1.8) suggests that the RHS of (1.6) should be replaced by $C_1\lambda^{-2}$ in this strong compatible case.

Our aim is therefore on a detailed analysis of the rate of convergence of solutions to (Z_λ). We use the weighted Sobolev space $H^{m,s}$, m, s $\in \mathbb{R}$, defined by

$$H^{m,s} = \{\psi \in \mathscr{S}';\ \|\psi\|_{m,s} = \|(1+|x|^2)^{s/2}(1-\Delta)^{m/2}\psi\|_2 < \infty\}.$$

Theorem 1. Let $N \le 3$. Let E_0, $n_0 \in \mathscr{S}$ and let $n_1 \in \mathscr{S}\cap\dot{H}^{-1}$. Let (E_λ, n_λ) and E be the solutions to (Z_λ) and (NLS), respectively, with the maximal existence time T_{max}. Then:

(1) For any T with $0 < T < T_{max}$ and any $m \in \mathbb{N}$ there exist two positive constants C and λ_0 such that for any $\lambda \ge \lambda_0$

$$\sup_{0\le t\le T} \|n_\lambda(t)+|E_\lambda(t)|^2-Q^{(1)}(\lambda t)-\lambda^{-1}Q^{(2)}(\lambda t)\|_{m,0} \le C\lambda^{-1}, \tag{1.9}$$

where $Q^{(1)}(t) = (\cos t(-\Delta)^{1/2})(n_0+|E_0|^2)$ and $Q^{(2)}(t) = (-\Delta)^{-1/2}(\sin t(-\Delta)^{1/2})(n_1 + 2\mathrm{Im}\sum_{j=1}^{N}\partial_j(\overline{E_0}\cdot\partial_j E_0))$. In particular, for any $\lambda \ge \lambda_0$

$$\sup_{0\le t\le T} \|n_\lambda(t)+|E_\lambda(t)|^2-Q^{(1)}(\lambda t)\|_{m,0} \le C\lambda^{-1}. \tag{1.10}$$

(2) Assume $n_0 + |E_0|^2 \ne 0$. Then for any T with $0 < T < T_{max}$ and any $m \in \mathbb{N}$ there exist two positive constants such that for any $\lambda \ge \lambda_0$

$$\sup_{0\le t\le T} \|E_\lambda(t)-E(t)\|_{m,0} \le C\lambda^{-1}. \tag{1.11}$$

(3) Assume $n_0 + |E_0|^2 = 0$. When $N \le 2$ assume that n_1 takes the form $n_1 = \nabla\cdot\phi$ for some $\phi \in \mathscr{S}$. Then for any T with $0 < T < T_{max}$ and any $m \in \mathbb{N}$ there exist two positive constants C and λ_0 such that for any $\lambda \ge \lambda_0$

$$\sup_{0\le t\le T} \|E_\lambda(t)-E(t)\|_{m,0} \le C\lambda^{-2}. \tag{1.12}$$

Remark 1. (1) The assumption $n_1 \in \mathscr{S}\cap\dot{H}^{-1}$ is redundant when $N = 3$ since $\mathscr{S} \subset \dot{H}^{-1}$ for $N \ge 3$. This fact follows by using the Hardy inequality in the Fourier space.

(2) $Q_2(t)$ in part (1) solves the wave equation

$$\left.\begin{array}{l} \partial_t^2 Q_2 - \Delta Q_2 = 0, \ t > 0, \ x \in \mathbb{R}^N \\ Q_2(0,x) = 0, \ \partial_t Q_2(0,x) = n_1(x) + 2\mathrm{Im} \sum_{j=1}^{N} \partial_j(\overline{E_0}\cdot\partial_j E_0)(x) \end{array}\right\}$$

The term $\lambda^{-1}Q_2(\lambda t)$ in (1.9) represents the second initial layer for (Z_λ).

The rate of convergence is turned out to be better when we consider the problem locally in space or away from the initial time.

Theorem 2. Let $N \le 3$. Let E_0, $n_0 \in \mathcal{S}$ and let $n_1 \in \mathcal{S}\cap\dot{H}^{-1}$. Let (n_λ, E_λ) and E be the solutions to (Z_λ) and (E), respectively, with the maximal existence time T_{max}. Then:

(1) For any T with $0 < T < T_{max}$, any $m \in \mathbb{N}$, and any $s > 0$ there exist two positive constants C and λ_0 such that for any $\lambda \ge \lambda_0$

$$\sup_{0\le t\le T} \|n_\lambda(t)+|E_\lambda(t)|^2 - Q^{(1)}(\lambda t) - \lambda^{-1}Q^{(2)}(\lambda t)\|_{m,-\tilde{s}} \tag{1.13}$$

$$\le \begin{cases} C\lambda^{-2} & \text{if} \quad s > 1, \\ C\lambda^{-1}\log\lambda & \text{if} \quad s = 1, \\ C\lambda^{-1-s} & \text{if} \quad 0 < s < 1, \end{cases}$$

where $\tilde{s} = s$ when $N = 1$ and $\tilde{s} = s+1$ when $N \ge 2$.

(2) For any T with $0 < T < T_{max}$, any $m \in \mathbb{N}$, and any ε, $s > 0$ there exist two positive constants C and λ_0 such that for any $\lambda \ge \lambda_0$

$$\sup_{\varepsilon\le t\le T} \|n_\lambda(t)+|E_\lambda(t)|^2\|_{m,-\tilde{s}} \le C\lambda^{-s}. \tag{1.14}$$

Remark 2. In addition to the assumptions above, assume that $n_0 + |E_0|^2 = n_1 + 2\mathrm{Im} \sum_{j=1}^{N} \partial_j(\overline{E_0}\cdot\partial_j E_0) = 0$. Then part (1) implies that for any T with $0 < T < T_{max}$, any $m \in \mathbb{N}$, and any s with

$0 < s < 1$ there exist two positive constants C and λ_0 such that for any $\lambda \geq \lambda_0$

$$\sup_{0\leq t\leq T} \|n_\lambda(t)+|E_\lambda(t)|^2\|_{m,-\tilde{s}} \leq \begin{cases} C\lambda^{-2} & \text{if } s > 1, \\ C\lambda^{-1}\log\lambda & \text{if } s = 1, \\ C\lambda^{-1-s} & \text{if } 0 < s < 1. \end{cases}$$

Some of the results in Theorems 1 and 2 are optimal concerning the rate of convergence with respect to λ.

<u>Theorem 3.</u> Let $N \leq 3$. Let E_0, $n_0 \in \mathcal{S}$ and let $n_1 \in \mathcal{S}\cap\dot{H}^{-1}$. Let (E_λ, n_λ) and E be the solutions to (Z_λ) and (NLS), respectively, with the maximal existence time T_{max}.

(1) Let $\partial_t^2|E|^2(0) \neq 0$. Then for any T with $0 < T < T_{max}$, any $m \in \mathbb{N}$, and any $s \geq 0$

$$\liminf_{\lambda \to \infty} \lambda^2 \sup_{0\leq t\leq T} \|n_\lambda(t)+|E_\lambda(t)|^2-Q^{(1)}(\lambda t)-\lambda^{-1}Q^{(2)}(\lambda t)\|_{m,-s} > 0. \tag{1.15}$$

(2) Let $(n_0+|E_0|^2)E_0 \neq 0$. Then for any T with $0 < T < T_{max}$ and any $m \in \mathbb{N}$

$$\liminf_{\lambda \to \infty} \lambda \sup_{0\leq t\leq T} \|E_\lambda(t)-E(t)\|_{m,0} > 0. \tag{1.16}$$

(3) Let $n_0 + |E_0|^2 = 0$ and let $(n_1 + 2\mathrm{Im} \sum_{j=1}^{N} \partial_j(\overline{E_0}\cdot\partial_j E_0))E_0 \neq 0$. Then for any T with $0 < T < T_{max}$ and any $m \in \mathbb{N}$

$$\liminf_{\lambda \to \infty} \lambda^2 \sup_{0\leq t\leq T} \|E_\lambda(t)-E(t)\|_{m,0} > 0. \tag{1.17}$$

The theorems above give a detailed description of formation of initial layers with almost optimal rate of convergence for solutions to the Zakharov equations. For any T with $0 < T < T_{max}$, E_λ behaves like E on the interval $[0,T]$ and n_λ behaves like $-|E|^2$ on the interval $(0,T]$. This difference between the time intervals for convergence of these solutions is due to the initial layer phenomena. The formation of initial layers also reflects the rate of convergence for solutions to the Zakhrov equations.

We should note that in the theorems above we only consider the problem up to time T strictly less than the maximal time T_{max}. It cannot be expected that $T = T_{max}$ even when λ is large enough since there are discrepancies between global behaviors in time of solutions to (Z_λ) and those of (NLS). For example in one dimension (NLS) has an infinite number of conservation laws and has n-soliton solutions with arbitrary n, whereas (Z_λ) has only three conserved quantities for plasmon number, momentum, and energy, and has only one set of single soliton solutions, which are usually called Langmuir solitons (see Gibbons, Thornhill, Wardrop & ter Haar [5], Makhankov [6], Zakharov [12]). In two and three dimensions (NLS) has blow-up solutions and (Z_λ) is conjectured to have blow-up solutions. We have, however, no results on the blow-up problem for (Z_λ), especially, on the relation between the maximal existence time of solutions to (NLS) and that of (Z_λ).

Our method of the proofs of the theorems above depends essentially on the special propagation properties of acoustic waves and of nonlinear Schrödinger waves. An outline of the proofs is roughly given as follows. By setting $Q_\lambda = n_\lambda + |E_\lambda|^2$, (Z_λ) becomes

$$\left.\begin{aligned} &i\partial_t E_\lambda + \Delta E_\lambda = -|E_\lambda|^2 E_\lambda + Q_\lambda E_\lambda, \\ &\lambda^{-2} Q_\lambda - \Delta Q_\lambda = \lambda^{-2}\partial_t^2 |E_\lambda|^2, \\ &E_\lambda(0) = E_0(x), \qquad Q_\lambda(0,x) = n_0(x) + |E_0(x)|^2, \\ &\partial_t Q_\lambda(0,x) = n_1(x) + 2\mathrm{Im}\sum_{j=1}^{N} \partial_j(\overline{E_0}\cdot\partial_j E_0)(x), \end{aligned}\right\} \tag{1.18}$$

which in turn yields the system of integral equations

$$E_\lambda(t) = U(t)E_0 + i\int_0^t U(t-s)(|E_\lambda|^2 E_\lambda - Q_\lambda E_\lambda)(s)ds, \tag{1.19}$$

$$\begin{aligned} Q_\lambda(t) = {} & Q^{(1)}(\lambda t) + \lambda^{-1} Q^{(2)}(\lambda t) \\ & + \lambda^{-1}\int_0^t (-\Delta)^{-1/2}(\sin\lambda(t-s)(-\Delta)^{1/2})\partial_s^2 |E_\lambda|^2(s)ds, \end{aligned} \tag{1.20}$$

where $U(t) = e^{it\Delta}$ and $Q^{(j)}$ is as in Theorem 1. Similarly, (NLS) is rewritten as

$$E(t) = U(t)E_0 + i\int_0^t U(t-s)|E|^2 E(s)ds. \tag{1.21}$$

Then the first step is to estimate the terms in the RHS of (1.20) to conclude that $Q_\lambda(t)$ behaves like $Q^{(1)}(\lambda t)+\lambda^{-1}Q^{(2)}(\lambda t)$ as $\lambda \longrightarrow \infty$. The next step is to consider the integral equation

$$E_\lambda(t) - E(t) = i \int_0^t U(t-s)(|E_\lambda|^2 E_\lambda - |E|^2 E)(s)ds \qquad (1.22)$$
$$- i \int_0^t U(t-s)Q_\lambda(s)E_\lambda(s)ds,$$

which follows from (1.20) and (1.21). Our task is to estimate the second integral in the RHS of (1.22) since the first integral is easily estimated as $C \int_0^t \|E_\lambda(s)-E(s)\|_{m,0}\, ds$. The integrand $Q_\lambda E_\lambda$ in (1.22) corresponds the interaction between acoustic wave Q_λ and nonlinear Schrödinger wave E_λ. Q_λ propagates according to the Huygens principle. $Q_\lambda(t,x)$ is localized in a neighborhood of the sphere $|x| = \lambda t$ so that Q_λ propagates very fast as $\lambda \longrightarrow \infty$. On the other hand, E_λ propagates with group velocity independent of λ. More precisely, the map $E_0 \longmapsto E_\lambda(t)$ leaves $H^{0,m}$ invariant for any $t > 0$ and any $m \in \mathbb{N}$, and furthermore, $\|E_\lambda(t)\|_{0,m}$ is bounded by a constant depending on t, m, $\|E_0\|_{0,m}$, but not on λ. Therefore $E_\lambda(t,x)$ is localized (not in the strict sense) in a bounded domain with radius independent of λ. Hence the product $Q_\lambda E_\lambda$ should be proved to be small as $\lambda \longrightarrow \infty$ without using explicit L^∞-decay estimates for the wave equation. This is a main idea of our method, and is different from those of H. Added & S. Added [2], Asano [3], and Ukai [11], which rely heavily on explicit L^∞-estimates for solutions to the linearized problem.

For detailed arguments see Ozawa & Tsutsumi [8].

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