The singularity on interpolation by rational spline functions

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Let $P_n$ be the collection of all polynomials of degree $n$, and $R_{r}^{l}$ the collection of all rational functions with the form $p(x)/q(x)$, where $p \in P_r$, and $q \in P_l$.

Denote by $T$ the following partition of the interval $[a, b]$:

$$T : \quad a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b.$$ 

If a real function $R(x)$ defined on $[a, b]$ satisfies

1°  $R(x) \in R_{r}^{l}$,  in each interval $[x_j, x_{j+1}]$;

2°  $R(x) \in C^{s}[a, b]$,

then $R(x)$ is said to be a rational spline of type $-(r, l)^s$ with respect to the partition $T$.

In this paper, we shall discuss the rational splines of type $(2, 1)^1$ and type $(2, 1)^2$ with the forms

$$R(x) = p_{1j}(x) + \frac{(x - x_j)(x - x_{j+1})}{q_{1j}(x)}, \quad x_j \leq x \leq x_{j+1}, \quad j = 0, \cdots, n - 1,$$  

(1)

where $p_{1j}(x)$ and $q_{1j}(x) \in P_1$.

Suppose that the interpolation conditions are

$$\left\{ \begin{array}{l}
R(x_j) = y_j, \\
R'(x_j) = y'_j,
\end{array} \right. \quad \left\{ \begin{array}{l}
R(x_{j+1}) = y_{j+1}, \\
R'(x_{j+1}) = y'_{j+1};
\end{array} \right. \quad (2)$$

$$\left\{ \begin{array}{l}
R(x_j) = y_j, \\
R(x_{j+1}) = y_{j+1},
\end{array} \right. \quad \left\{ \begin{array}{l}
R'(x_j) = [f(x_{j-1}, x_j) + f(x_j, x_{j+1})]/2, \\
R'(x_{j+1}) = [f(x_j, x_{j+1}) + f(x_{j+1}, x_{j+2})]/2;
\end{array} \right. \quad (3)$$

$$\left\{ \begin{array}{l}
R(x_j) = y_j, \\
R(x_{j+1}) = y_{j+1},
\end{array} \right. \quad \left\{ \begin{array}{l}
R'(x_j) = y'_j, \\
R''(x_j) = y''_j;
\end{array} \right. \quad (4)$$
respectively, where \( f(x_j, x_{j+1}) \) denotes the divided difference of the first degree, etc.

Denote by \( R_{1j}(x) \), \( R_{2j}(x) \), and \( R_{3j}(x) \) the rational spline functions of satisfying the interpolation conditions (1)–(2), (1)–(3), and (1)–(4) respectively.

R.H. Wang and S.T. Wu ([1],[2]) have obtained the following rational piecewise functions which are satisfying the interpolation conditions (1)–(2), (1)–(3), and (1)–(4) respectively,

\[
R_{1j}(x) = y_j + f(x_j, x_{j+1})(x - x_j) \frac{(x - x_j)(x - x_{j+1})[y_j' - f(x_j, x_{j+1})][y_{j+1}' - f(x_j, x_{j+1})]}{(x - x_j)[y_j' - f(x_j, x_{j+1})] + (x - x_{j+1})[y_{j+1}' - f(x_j, x_{j+1})]},
\]

\[
R_{2j}(x) = y_j + f(x_j, x_{j+1})(x - x_j) + \frac{\{(x - x_j)(x - x_{j+1})[f(x_{j-1}, x_j) - f(x_j, x_{j+1})][f(x_{j+1}, x_{j+2}) - f(x_j, x_{j+1})]\}}{2[f(x_{j-1}, x_j) + f(x_{j+1}, x_{j+2}) - 2f(x_j, x_{j+1})]x}.
\]

\[
R_{3j}(x) = y_j + f(x_j, x_{j+1})(x - x_j) + \frac{2[y_j' - f(x_j, x_{j+1})]^2(x - x_j)(x_{j+1} - x)}{2[y_j' - f(x_j, x_{j+1})](x_{j+1} - x) + y_j'(x_j - x_{j+1})(x - x_j)}.
\]

Denote by \( R_i(x) \) \( (i = 1, 2, 3) \) the rational spline functions:

\[
R_i(x) = \{R(x) \in (2, 1)^1 | R(x)|_{[x_j, x_{j+1}]} = R_{ij}(x), \ j = 0, \ldots, n - 1\}, \ i = 1, 2;
\]

\[
R_3(x) = \{R(x) \in (2, 1)^2 | R(x)|_{[x_j, x_{j+1}]} = R_{3j}(x), \ j = 0, \ldots, n - 1\}.
\]

It is not hard to prove the following lemmas:

[Lemma 1] \( R_1(x) \) has the singular point in the interval \([x_j, x_{j+1}]\), if and only if

\[
\text{sign}\{[y_j' - f(x_j, x_{j+1})] \cdot [y_{j+1}' - f(x_j, x_{j+1})]\} > 0.
\]

[Lemma 2] \( R_2(x) \) has the singular point in the interval \([x_j, x_{j+1}]\), if and only if

\[
\text{sign}\{[f(x_{j-1}, x_j) - f(x_j, x_{j+1})] \cdot [f(x_{j+1}, x_{j+2}) - f(x_j, x_{j+1})]\} > 0.
\]
In fact, by the formula of $R_1(x)$ on $[x_j, x_{j+1}]$ given in (5), the singularity of $R_1(x)$ on $[x_j, x_{j+1}]$ can appear only at point

$$x^* = \frac{[y_j' - f(x_j, x_{j+1})]x_j + [y_{j+1}' - f(x_j, x_{j+1})]x_{j+1}}{[y_j' - f(x_j, x_{j+1})] + [y_{j+1}' - f(x_j, x_{j+1})]} := \lambda x_j + (1 - \lambda)x_{j+1},$$

where $\lambda = \frac{[y_j' - f(x_j, x_{j+1})]}{[y_j' - f(x_j, x_{j+1})] + [y_{j+1}' - f(x_j, x_{j+1})]}$. So, it is easy to see that $x^* \in (x_j, x_{j+1})$ if and only if

$$\text{sign}[(y_j' - f(x_j, x_{j+1})) \cdot (y_{j+1}' - f(x_j, x_{j+1}))] > 0.$$ 

By the similar argument, we can prove Lemma 2.

It notes that if $y_j' - f(x_j, x_{j+1})$ or $y_{j+1}' - f(x_j, x_{j+1}) = 0$, then $R_1(x)$ will be a linear function in the interval $[x_j, x_{j+1}]$, so it should has no singular point.

By using the above Lemmas, we have

[Theorem 1] Let the interpolation function $y = f(x) \in C^3[a, b]$. If $R_i(x)$ $(i = 1, 2)$ exists the singular point in the interval $[x_j, x_{j+1}]$, then $y = f(x)$ has the inflection point in the open interval $(x_j, x_{j+1})$.

Proof Let $R_1(x)$ exist the singular point in $[x_j, x_{j+1}]$. By Lemma 1, without the loss of generality, suppose that the following inequalities hold

$$y_j' - f(x_j, x_{j+1}) > 0, \quad y_{j+1}' - f(x_j, x_{j+1}) > 0. \quad (10)$$

It follows Lagrange's mean value theorem, that there exists $\xi \in (x_j, x_{j+1})$, such that

$$f'(\xi) = f(x_j, x_{j+1}). \quad (11)$$

By (10) and (11), there exist $\eta$ and $\zeta$ of satisfying

$$f'(x_j) - f(x_j, x_{j+1}) = f''(\eta)(x_j - \xi), \quad f'(x_{j+1}) - f(x_j, x_{j+1}) = f''(\zeta)(x_{j+1} - \xi)$$

respectively, where $x_j < \eta < \xi < \zeta < x_{j+1}$. Hence

$$f''(\eta) \cdot f''(\zeta) < 0,$$

and there exists at least one inflection point of $f(x)$ in $(\eta, \zeta)$. This completes the proof of this theorem for $R_1(x)$.

Similarly we can prove the theorem for the case of $R_2(x)$.
[Theorem 2] Let the interpolation function $y = f(x) \in C^2[a, b]$. If $R_3(x)$ exists the singular point in the interval $[x_j, x_{j+1}]$, then the original interpolation function $y = f(x)$ has the inflection point in the open interval $(x_j, x_{j+1})$.

In fact, because of the singular point of $R_3(x)$ may be only appearing at

$$\bar{x} = \frac{y''(x_{j+1} - x_j)x_j + 2(y'_j - f(x_j, x_{j+1}))(x_{j+1})}{y''(x_j - x_{j+1}) + 2(y'_j - f(x_j, x_{j+1}))}.$$ 

By the same argument shown in the proof of Lemma 1, we have

$$\text{sign}\{[y''(x_{j+1} - x_j)] \cdot [y'_j - f(x_j, x_{j+1})]\} > 0,$$

provided that $R_3(x)$ exists the singular point in $[x_j, x_{j+1}]$.

It follows Lagrange's mean value theorem, that there exists $\xi \in (x_j, x_{j+1})$, such that

$$f'(\xi) = f(x_j, x_{j+1}).$$

By Lagrange's mean value theorem once again, there is a point $\eta \in (x_j, \xi)$, such that

$$f'(x_j) - f'(\xi) = f''(\eta)(x_j - \xi).$$

So

$$\text{sign}\{f''(x_j) \cdot f''(\eta)\} < 0.$$ 

Hence, there exists at least one inflection point of $f(x)$ in $(x_j, \eta)$.

This complete the proof of this theorem.

In addition, we may prove that although the interpolation function $f(x)$ has the inflection points in $[a, b]$, however, provided that we are taking all inflection points as the knots of the rational spline function, then the singularity can be avoided to appear. For example, let $x_j$ be an inflection point of $f(x)$, $x_{j+1}$ be not, and there is no another inflection point between $x_j$ and $x_{j+1}$. Then the first derivative of $f(x)$ will be monotone in the interval $[x_j, x_{j+1}]$. So the inequality (8) will be not satisfied. By Lemma 1, hence, there is no singular point in $(x_j, x_{j+1})$.

Therefore, we have
Theorem 3] For any given interpolation function $f(x) \in C^2[a, b]$, we can construct a partition $T$ of the interval $[a, b]$, such that the rational spline function $R_1(x)$, $R_2(x)$, and $R_3(x)$ based on the partition $T$ have no singularity.

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References