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On a constrained noncooperative n-person game

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§1. Introduction

In this paper, a noncooperative equilibrium point in a constrained n-person game is investigated. we show that there exits an equilibrium point in the n-person game if and only if some set valued mapping has a fixed point. Then, using the proposition which is derived from Ekeland's theorem, we shall discuss the conditions under which there exists a fixed point of the mapping.

§2. Formulation of a noncooperative n-person game

We define a noncooperative n-person game by the following strategic form

$$(N, X, F), \tag{2.1}$$

where

(i) $N = \{ 1, 2, \cdots, n \}$ is the set of n players.

(ii) $X = \prod_{i=1}^{n} X^i \subset U = \prod_{i=1}^{n} U^i$, for each $i \in N$, X^i is the subset of a complete metric space U^i and is called the strategy set of each player i.

(iii) $F = (f^1, f^2, \cdots, f^n) : X \to R^n$, is a multiloss operator and, for each $i \in N$, $f^i: X \to R$, denotes a loss function for player i.

In this paper, denoting by $\hat{i} = N - i$ the coalition adverse to each player i, the multistrategy set, $X = \prod_{i=1}^{n} X^i$ is split as follows

$$X = X^i \times X^{\hat{i}} \quad \text{and} \quad X^{\hat{i}} = \prod_{j \neq i} X^j.$$

If π^i and $\pi^{\hat{i}}$ denote the projection from X into X^i and $X^{\hat{i}}$, we set $x^i = \pi^i x$ and $x^{\hat{i}} = \pi^{\hat{i}} x$ for a multistrategy $x = (x^i, x^{\hat{i}}) \in X$.

Now, we define, for each $i \in N$,

$$\alpha^i = \inf_{x \in X} f^i(x)$$

and, throughout this paper, we assume that $\alpha^i > -\infty$ for all $i \in N$. In this case, the game is bounded below and $\alpha = (\alpha^1, \alpha^2, \cdots , \alpha^n)$ is called *shadow minimum* of the game. Then, we have

$$F(X) \subset \alpha + R^n_+ ,$$

where

$$F(X) = \{ F(x) \in R^n ; \text{ for all } x = (x^1, x^2, \cdots , x^n) \in X \}$$

and

$$R^n_+ = \{ x = (x^1, x^2, \cdots, x^n) \in R^n ; x^i \geq 0 \text{ for all } i \in N \}.$$

If $\alpha = F(\bar{x})$ belongs to F(X), the multistrategy $\bar{x} \in X$ attains to the minimum of the loss function f^i for each player i. In this case, $\bar{x}$ is the best solution for each player. But, this situation is seldom the case and we have to investigate other solution concepts. So, we consider especially noncooperative equilibrium point.

Definition 2.1 A multistrategy $x = (x^1, x^2, \cdots , x^n)$

$\in X$ is said to be a noncooperative equilibrium point (Nash equilibrium point) if, for all $i \in N$,

$$f^i(x) = \inf_{\substack{y \in X \\ \pi^{\hat{i}} y = x^{\hat{i}}}} f^i(y). \tag{2.2}$$

Such a noncooperative equilibrium point shows that given the complementary coalition's choice $x^{\hat{i}}$, the player i responds by playing a strategy $x^i \in X^i$ which minimizes $f^i(\cdot, x^{\hat{i}})$ on X^i.

Remark 2.1 If $N = \{ 1,2 \}$ and $f^1(x) + f^2(x) = 0$ for all multistrategies $x \in X = X^1 \times X^2$, an equilibrium point is called saddle point.

We define the correspondences U^i mapping $X^{\hat{i}}$ into X^i which assign to the choice $x^{\hat{i}}$ of strategies of players $j \neq i$ the subset of feasible strategies for the i player:

$$U_i(x^{\hat{i}}) = \{ y^i \in X^{\hat{i}} \ ; \ (y^i, x^{\hat{i}}) \in X \}. \tag{2.3}$$

So, we can rewrite the definition of a noncooperative equilibrium point in the following way:

$$\text{(i)} \quad \bar{x}^i \in U_i(x^{\hat{i}}) \qquad \text{for all } i \in N$$

$$\text{(ii)} \quad f_i(x^i, x^{\hat{i}}) = \min_{y^i \in U_i(x^{\hat{i}})} f_i(y^i, x^{\hat{i}}) \qquad \text{for all } i \in N. \tag{2.4}$$

Under such the situation, we introduce the following notations: for all $i \in N$,

$$S_i(x^{\hat{i}}) = \{ x^i \in U_i(x^{\hat{i}}) \ ; \ f_i(x^i, x^{\hat{i}}) = \min_{y^i \in U_i(x^{\hat{i}})} f_i(y^i, x^{\hat{i}}) \}$$

and

$$S(x) = \prod_{i=1}^{n} S_i(x^{\hat{i}}) : X \to 2^X, \tag{2.5}$$

where 2^X denotes the set of all subsets in X. Then we can show the relationships between a fixed point of $S : X \to 2^X$ and a noncooperative equilibrium point in the game.

Proposition 2.1 A multistrategy $\bar{x} \in X$ is a nonncooperative equilibrium point if and only if $\bar{x}$ is a fixed point of S, that is,

$$\bar{x} \in S(\bar{x}).$$

Proof. Using (i) and (ii) in (2.4) and the definition of the set valued mapping S, we can easily prove the proposition.

Thus, it is very important to show that there exists a fixed point of the mapping S in X. In next section, we shall study the existence of a fixed point of the mapping S.

§3. A fixed point of the mapping S from U into 2^U

In this section, we assume that the strategy set of each player i is U^i, that is, $X^i = U^i$ for all $i \in N$. In order to show that there exists a noncooperative equilibrium point in the n-person game (2.1), we shall discuss a fixed point of mapping S. Let P(U) be the family of nonempty subset of the complete metric space (U,d), $2^U - \{\phi\}$. For a point $x \in U$ and a member $A \in P(U)$, we define the distance d from x to A as follows:

$$d(x,A) = \inf_{a \in A} d(x,a).$$

Further, given two members $A,B \in P(U)$, we introduce the following notation:

$$\delta(A,B) = \sup_{a \in A} d(a,B). \qquad (3.1)$$

The function δ may be infinite valued on $P(U)\times P(U)$. So, throughout the section, we restrict δ to $P'(U) = \{ A \in P(U);$ A is close and bounded in U $\}$ and, then we can prove the following proposition.

Proposition 3.1 For any A, B, and C in P'(U),

(1) If $\delta(A,B) = 0$, $A \subset B$

(2) $\delta(A,B) \leq \delta(A,C) + \delta(C,B)$.

Proof. The proof of (1) is obvious from the definition of δ. In order to prove (2), for any $a \in A$, $b \in B$, and $c \in C$, we have

$$d(a,b) \leq d(a,c) + d(c,b). \tag{3.2}$$

Taking infimum of both sides of (3.2) on B, we obtain

$$d(a,B) \leq d(a,c) + d(c,B),$$

that is, for any $c \in C$,

$$d(a,B) - d(c,B) \leq d(a,c). \tag{3.3}$$

So, from (3.3), for any $a \in A$,

$$d(a,B) \leq d(a,C) + \delta(C,B). \tag{3.4}$$

(3.4) shows that (2) holds. Thus, the proof is completed.

Hence, given two members A, B in P'(U), we define

$$H(A,B) = \max\,[\delta(A,B), \delta(B,A)].$$

Using Proposition 3.1, we show that the function $H: P'(U) \times P'(U) \to [0,\infty)$ satisfies all properties of a metric on P'(U), that is, (P'(U),H) is a metric space and H is well known as the Hausdorff metric.

Proposition 3.2 For any $x \in U$ and any B, C in P'(U),

(1) $d(x,B) - d(x,C) \leq \delta(C,B)$

(2) $|\, d(x,B) - d(x,C) \,| \leq H(B,C)$.

Proof. For any $\varepsilon > 0$, there exists $c \in C$ such that

$$d(x,c) \leq d(x,C) + \varepsilon. \tag{3.5}$$

From (3.5), it follows that, for all $b \in B$,

$$d(x,B) - d(x,C) < d(x,b) - d(x,c) + \varepsilon$$
$$\le d(b,c) + \varepsilon. \qquad (3.6)$$

Taking infimum of left side in (3.6) on B and, then, supremum of left side in (3.6) on C, we obtain

$$d(x,B) - d(x,C) \le \delta(C,B) + \varepsilon.$$

Since ε is arbitrary, we get

$$d(x,B) - d(x,C) \le \delta(C,B). \qquad (3.7)$$

Further, using the definition of H, it follows from (3.7) that

$$d(x,B) - d(x,C) \le H(C,B). \qquad (3.8)$$

Interchanging B with C in (3.8), we have

$$d(x,C) - d(x,B) \le H(B,C) \ (= H(C,B)). \qquad (3.9)$$

Thus, (3.8) and (3.9) complete the proof of the proposition.

Then, we give the concepts of H-continuity (H-c.), H-upper semicontinuity (H-u.s.c.), and H-lower semicontinuity (H-l.s.c.) of the set valued mapping $S: U \to P'(U)$.

Definition 3.1 The mapping S is called H-c. at $x \in U$ if for any each sequence $\{x^k\}, k=1,2,\cdots$, in U converging to x, it follows that $\{H(S(x^k),S(x))\}, k=1,2,\cdots$, converges to 0. Further, when S is H-c. at every point in U, S is called H-c. in U.

Definition 3.2 The mapping S is called H-u.s.c. at $x \in U$ if for each sequence $\{x^k\}, k=1,2,\cdots$, in U converging to x, it follows that $\{\delta(S(x^k),S(x))\}, k=1,2,\cdots$, converges to 0. The mapping S is called H-l.s.c. at $x \in U$ if for each sequence $\{x^k\}, k=1,2,\cdots$, in U converging to x, it follows that $\{\delta(S(x),S(x^k))\}, k=1,2,\cdots$, converges to 0. The mapping S is called H-u.s.c.(resp.H-l.s.c.) in U if it is H-u.s.c. (resp. H-l.s.c.) at every point in U.

Remark 3.1 If the mapping S is H-c., it is obvious that S is H-u.s.c. and H-l.s.c..

The mapping S is called a contraction one if there exists a real numbers $r \in [0,1)$ such that for all $x, y \in U$,

$$H(S(x),S(y)) \le rd(x,y). \tag{3.10}$$

Then, introducing the function $G(x) = d(x,S(x))$: $U \to \mathbb{R}$, which plaies an important role in the section, we can prove H-c. of G.

Proposition 3.3 Suppose S is a contraction mapping. Then, the function G is H-c. in U.

Proof. From the propositions 3.1 and 3.2, it follows that

$$\begin{aligned} |G(x) - G(y)| &\le |d(x,S(x)) - d(y,S(y))| \\ &\le |d(x,S(x)) - d(y,S(x))| + |d(y,S(x)) - d(y,S(y))| \\ &\le d(x,y) + H(S(x),S(y)). \end{aligned} \tag{3.11}$$

Since H is contraction, we obtain

$$|G(x) - G(y)| \le (1+r)d(x,y),$$

which completes the proof.

Further, in order to show that there exists a fixed point of the mapping S, we shall need the following proposition which is derived from Ekeland's theorem (see [4] and [5]).

Proposition 3.4 Let (U,d) be a complete metric space, and $G: U \to R\cup\{+\infty\}$, a l.s.c. function, $\not\equiv +\infty$, bounded from below. For any $\varepsilon > 0$, there is some point $\bar{x} \in U$ with:

$$G(\bar{x}) \le \inf_{x\in U} G(x) + \varepsilon$$

$$G(x) \ge G(\bar{x}) - \varepsilon d(\bar{x},x).$$

Theorem 3.1 Suppose that $S: U \to P'(U)$, is a contraction mapping, that is, S satisfies (3.10). Then, S has a fixed point

$\bar{x} \in U$ satisfying $\bar{x} \in S(\bar{x})$.

Proof. Since $G(x) = d(x,S(x))$ is H-c. in U by Proposition 3.3 and $G(x) \geq 0$, using Proposition 3.4 for any $\varepsilon \in (0,1-r)$, there exists a point $\bar{x}$ such that for all $x \in U$,

$$G(x) + \varepsilon d(\bar{x},x) \geq G(\bar{x}). \tag{3.12}$$

From (3.10), it follows that for all $x \in S(\bar{x})$,

$$G(\bar{x}) \leq (r+\varepsilon)d(\bar{x},x),$$

that is,

$$G(\bar{x}) \leq (r+\varepsilon)G(\bar{x}).$$

Thus, $G(\bar{x}) = 0$ because $r+\varepsilon < 1$. This completes the proof.

Proposition 3.5 If S is H-u.s.c. in U, $G(x) = d(x,S(x))$ is l.s.c..

Proof. For any sequence $\{x^k\}, k=1,2,\cdots$, in U converging to x, we make use of the following notations:

$$F_1(x^k) = G(x^k) - d(x,x^k)$$

$$F_2(x^k) = d(x,S(x^k)) - G(x).$$

Then, we have

$$\begin{aligned} G(x^k) - G(x) &= d(x^k,S(x^k)) - d(x,S(x)) \\ &= G(x^k) - d(x,S(x^k)) + d(x,S(x^k)) - G(x) \\ &= F_1(x^k) + F_2(x^k). \end{aligned} \tag{3.13}$$

Using (3.4) in the proof of Proposition 3.1 for $F_1(x^k)$ and $F_2(x^k)$, we get

$$F_1(x^k) \geq -d(x,x^k), \qquad F_2(x^k) \geq -\delta(S(x^k),S(x)). \tag{3.14}$$

From (3.13) and (3.14), it follows that

$$G(x^k) - G(x) \geq -d(x,x^k) -\delta(s(x^k),S(x)). \tag{3.15}$$

Since S is H-u.s.c., (3.15) shows that

$$\liminf_{k \to \infty} G(x^k) \ge G(x).$$

Whence, the proof is complete.

Theorem 3.2 Suppose that S is H-u.s.c. and there exist two positive real numbers α_1, α_2, $\alpha_1+\alpha_2< 1$, such that for all x, y $\in$ U,

$$H(S(x),S(y)) \le \alpha_1 G(x) + \alpha_2 G(y). \qquad (3.16)$$

Then, there exixts a fixed point $\bar{x} \in U$ of S satisfying

$$\bar{x} \in S(\bar{x}).$$

Proof. Since S is H-u.s.c., $G(x) = d(x,S(x))$ is l.s.c. from Proposition 3.5. Using Proposition 3.4 for $\varepsilon \in (0, 1-\alpha_1/(1-\alpha_2))$, we have a point $\bar{x} \in U$ such that for all $x \in U$,

$$G(x) \ge G(\bar{x}) - \varepsilon d(\bar{x},x). \qquad (3.17)$$

From (3.17), it follows that for all $x \in S(\bar{x})$,

$$\begin{aligned} G(\bar{x}) &\le G(x) + \varepsilon d(\bar{x},x) \\ &\le H(S(\bar{x}),S(x)) + \varepsilon d(\bar{x},x) \\ &\le \alpha_1 G(\bar{x}) + \alpha_2 G(x) + \varepsilon d(\bar{x},x) \\ &\le \alpha_2 H(S(\bar{x}),S(x)) + \alpha_1 G(\bar{x}) + \varepsilon d(\bar{x},x) \\ &\le \alpha_2[\alpha_1 G(\bar{x}) + \alpha_2 G(x)] + \alpha_1 G(\bar{x}) + \varepsilon d(\bar{x},x) \\ &\le \alpha_2^2 G(x) + \alpha_1(\alpha_2+1)G(\bar{x}) + \varepsilon d(\bar{x},x) \\ &\quad \cdot \\ &\quad \cdot \\ &\le \alpha_2^{n+1}G(x) + \alpha_1(\alpha_2^n + \alpha_2^{n-1} + \cdots + \alpha_2 + 1)G(\bar{x}) + \varepsilon d(\bar{x},x). \end{aligned}$$

Since $\alpha_2^{n+1} \to 0$ as $n \to \infty$, we arrive at

$$G(\bar{x}) \le \alpha_1/(1-\alpha_2)G(\bar{x}) + \varepsilon d(\bar{x},x) \quad \text{for all } x \in S(\bar{x}). \qquad (3.18)$$

From (3.18), we get

$$G(\bar{x}) \le \alpha_1/(1-\alpha_2)G(\bar{x}) + \varepsilon G(\bar{x}),$$

that is,

$$[\ 1 - \alpha_1/(1-\alpha_2) - \varepsilon\ ]G(\bar{x}) \le 0.$$

This shows that $G(\bar{x}) = 0$ because $[\ 1 - \alpha_1/(1-\alpha_2) - \varepsilon\ ] > 0$. Thus, the proof is completed.

Theorem 3.3 Suppose that $F: U \to R$, a l.s.c. function, bounded from below and there exists a positive real number r such that

$$F(x) - F(y) \ge rd(x,y) \quad \text{for all } y \in S(x) \text{ and all } x \in U. \quad (3.19)$$

Then, S has a fixed point $\bar{x} \in U$ satisfying

$$S(\bar{x}) = \{\bar{x}\}.$$

Proof. Using Proposition 3.4 for $\varepsilon \in (0,r)$, there exists some point $\bar{x} \in U$ such that

$$F(x) \ge F(\bar{x}) - \varepsilon d(\bar{x},x) \quad \text{for all } x \in U.$$

Thus, we get

$$F(y) + \varepsilon d(\bar{x},y) \ge F(\bar{x}) \quad \text{for all } y \in S(\bar{x}). \quad (3.20)$$

From (3.19) and (3.20), it follows that

$$(r-\varepsilon)d(\bar{x},y) \le 0 \quad \text{for all } y \in S(\bar{x}).$$

This shows that $d(\bar{x},y) = 0$ for all $y \in S(\bar{x})$, because $r-\varepsilon > 0$. Whence the proof is completed.

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