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<th>Game Theoretic Analysis for an Optimal Stopping Problem in Some Class of Distribution Functions</th>
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1. Introduction

Let $X_1, X_2, \ldots, X_n, \cdots$ be mutually independent and identically distributed random variables with a common cdf $F(t) = P\{X \leq t\}$ such that $E[X^+] = \int_{R_+} t dF(t) < \infty$, where $R = (-\infty, \infty), R_+ = [0, \infty)$. A positive observation cost $c(\in R_{++} = (0, \infty))$ is incurred to the observation of each $X_n, n \geq 1$. If the observation process is stopped after $X_n$ is observed, a reward $X_n - nc$ is received. The optimal stopping time $N$ is necessarily of the form; to stop at $N = \min\{n \mid X_n \in S\}$ for some stopping set $S \subset R$, and $S$ is stationary and of a control-limit-type $\{X \geq x\}$ or $\{X > x\}$ for some $x \in R$, where $x$ is called a stopping level. For this, we define that a stopping level $x$ (or $x - 0$) means a stopping set $\{X > x\}$ (or $\{X \geq x\}$) respectively.

For any stopping level $x$ and for any cdf $F$, we define an expected reward $\phi(x, F) = E[X_N - cN]$ of the stopping problem by

\begin{equation}
\phi(x, F) = x + \frac{\int_{(x, \infty)} t dF(t) - c}{\overline{F}(x)} = x + \frac{\int_{(x, \infty)} (t - x) dF(t) - c}{\overline{F}(x)} ,
\end{equation}

where $\overline{F}(x) = 1 - F(x)$. Note that $\overline{F}(x) \to 0$ and $\phi \to -\infty$ as $x \to \infty$ and that $\overline{F}(x) \to 1$ and $\phi \to \mu_F - c$ as $x \to -\infty$ where $\mu_F = E[X] = \int_R t dF(t)$.

By the assumption $E[X^+] < \infty$, define $T_F(x)$,

\begin{equation}
T_F(x) = \int_x^\infty (t - x) dF(t) = \int_x^\infty \overline{F}(t) dt .
\end{equation}

Lemma 1. $T_F(x)$ is continuous, non-negative, convex and non-increasing function of $x$. It satisfies that $T_F(x) \geq (\mu_F - x)^+$ for any $x \in R$ and that $T_F(x) \to +\infty$ as $x \to -\infty$ and $T_F(x) \to 0$ as $x \to +\infty$. $T_F$ has a derivative a.e. Moreover, if $T_F(x)$ is positive at any point $x$, it is strictly decreasing at $x$.

Now, redefining the expected reward $\phi(x, F)$ by (1.1') for any stopping level $x$ and for any cdf $F$, we will have the optimal expected reward $\phi^o(F)$ for any cdf $F$.

\begin{equation}
\phi(x, F) = x + \frac{T_F(x) - c}{F(x)} .
\end{equation}
\[
\phi^o(F) \overset{\text{def}}{=} \sup_{x \in R} \phi(x, F).
\]

(1.3)

\[
\frac{d\phi(x, F)}{dF(x)} = \frac{T_F(x) - c}{\overline{F}^2(x)}.
\]

(1.4)

The right hand side of (1.4) changes the sign from + to − at most one time as \(x\) goes from \(-\infty\) to \(+\infty\). From Lemma 1, the equation \(T_F(x) = c\) for any fixed \(c(c > 0)\) has a unique solution \(x^o(F) \overset{\text{def}}{=} (T_F)^{-1}(c)\), so that the set of optimal stopping levels \(x^o(F)\) (which must contain the point \(x^o(F)\)) of (1.3) is given by

(1.5)

\[x^o(F) = \{x \mid F(x) = F(x^o(F))\}.\]

Since the cdf \(F\) is right-continuous, this set is an interval of the form \([a, b]\).

We have the optimal expected reward \(\phi^o(F)\),

(1.3')

\[\phi^o(F) = x^o(F) = \phi(x^o(F), F),\]

where \(\phi(A, F)\) means \(\phi(y, F)\) for any \(y\) in a set \(A\).

**Lemma 2.** For any given cdf \(F\), the following stopping sets or stopping levels (i) (ii) (iii) are optimal, and the optimal expected reward is given by (1.3');

(i) the set \(\{X > a\}\) or level \(a\) where \(a = \min \{x \mid x \in x^o(F)\}\),
(ii) the set \(\{X \geq b\}\) or level \(b - 0\) where \(b = \sup \{x \mid x \in x^o(F)\}\),
(iii) the set \(\{X > x\}\) ( \(\{X \geq x\}\) ) or level \(x (x - 0)\) where \(\forall x \in (a, b)\).

First, we shall derive the maximal bound \(\phi^u\) for \(\phi(x, F)\) on \(R \times \mathcal{F}\)

(1.6)

\[\phi^u = \sup_{x \in R} \sup_{F \in \mathcal{F}} \phi(x, F) = \sup_{F \in \mathcal{F}} \phi^o(F)\]

\[= \phi(x^o(F^u), F^u) = \phi(x^u, F^u),\]

where \((x^u, F^u)\) is a joint maximizing point of \(\phi(x, F)\).

Second, we shall consider \(\phi(x, F)\) as a two-person zero-sum game in which the player 1 (gambler) decides his level \(x\) in \(R\) and the player 2 (nature) chooses her cdf \(F\) in \(\mathcal{F}\), before the observation of \(\{X_n; n \geq 1\}\). Then the minimax value \(\phi^*\) and the maximin value \(\phi_*\) on \(R \times \mathcal{F}\),

(1.7)

\[\phi^* = \inf_{F \in \mathcal{F}} \sup_{x \in R} \phi(x, F) = \inf_{F \in \mathcal{F}} \phi^o(F)\]

\[= \phi(x^o(F^*), F^*) = \phi(x^*, F^*),\]

(1.8)

\[\phi_* = \sup_{x \in R} \inf_{F \in \mathcal{F}} \phi(x, F) = \phi(x_*, F_*),\]
and the saddle value \( \phi^s \), the saddle point \((x^s, F^s)\) in \( R \times \mathcal{F} \),

\[
\phi^s = \text{value}_{x \in R, F \in \mathcal{F}} \phi(x, F) = \phi(x^s, F^s),
\]

will be derived for the following two classes \( \mathcal{F}(\mu, \sigma^2) \) and \( \mathcal{F}(\mu, \sigma^2, M) \) of cdf’s.

The class \( \mathcal{F}(\mu, \sigma^2, M) \) is the set of cdf’s whose mean \( \mu \), variance \( \sigma^2 \) and domain \([\mu - M, \mu + M]\) are assumed to be known.

\[
\mathcal{F}(\mu, \sigma^2, M) = \{F | \int_A dF(t) = 1, \int_A tdF(t) = \mu, \int_A t^2dF(t) = \mu^2 + \sigma^2 \text{ where } A = [\mu - M, \mu + M], M \geq \sigma\}
\]

The class \( \mathcal{F}(\mu, \sigma^2) \) is \( \mathcal{F}(\mu, \sigma^2, M) \) where \( M \) is arbitrary in \( R_{++} \), and \( \mathcal{F}(\mu) \) is \( \mathcal{F}(\mu, \sigma^2) \) where \( \sigma^2 \) is arbitrary in \( R_{++} \).

Let a random variable \( X \) has a mean \( \mu \) with a cdf \( F_\mu(t) \), then the new random variable \( X - \mu \) has the mean 0 with the cdf \( F_0(t) = F_\mu(t + \mu) \). The following Lemma 3 below holds immediately from the definition (1.1) of \( \phi(x, F) \).

2. Some Fundamental Lemmas

Lemma 3.

\[
(2.4) \quad \phi(x, F_\mu) = \mu + \phi(x - \mu, F_0) \quad \text{for any } x \in R.
\]

Therefore, we may assume without loss of generality that all the cdf’s in \( F \) have the mean 0. So that, we shall analyze the stopping problem in only two classes \( \mathcal{F}(0, \sigma^2) \) and \( \mathcal{F}(0, \sigma^2, M) \).

Lemma 4. For cdf’s \( F_i \) and non-negative numbers \( \lambda_i, i = 1, 2, \cdots, n \), such that \( \sum_{i=1}^{n} \lambda_i = 1 \), let \( F = \sum_{i=1}^{n} \lambda_i F_i \). Then

\[
(2.5) \quad \phi(x, F) = \sum_{j=1}^{n} \lambda_j(x)\phi(x, F_j) \quad \text{for any } x \in R, \text{ where}
\]

\[
\lambda_j(x) = \frac{\lambda_j F_j(x)}{\sum_{i=1}^{n} \lambda_i F_i(x)}.
\]

Let define \( G_n \) be a discrete cdf which has \( n \) probability masses \( p_i, p_i > 0 \), at \( n \) points \( t_i, \ i = 1, 2, \cdots, n \), respectively (\( \sum_{i=1}^{n} p_i = 1 \)), i.e., it is represented as

\[
(2.6) \quad G_n(t) = (\prec t_1, \cdots, t_n > < p_1, \cdots, p_n >),
\]

and \( G_n(\mu, \sigma^2) \) be all discrete cdf’s \( G_n \) in \( \mathcal{F}(\mu, \sigma^2) \). Let

\[
(2.7) \quad G_2(t; q) = (\prec -\frac{\sigma}{q}, \sigma q > < \frac{q^2}{1+q^2}, 1 >)
\]

3
for any $q, 0 < q < \infty$. Then $G_2(t; q)$ is the only two-point cdf which has the mean 0 and the variance $\sigma^2$.

**Lemma 5.** The class $\mathcal{G}_2(0, \sigma^2)$ of two-point cdf's is represented with a parameter $q$, $0 < q < \infty$, as follows,

$$\mathcal{G}_2(0, \sigma^2) = \{G_2(\cdot; q) \mid 0 < q < \infty\}.$$  

Let us define

$$T^u_F(x) = \sup_{F \in \mathcal{F}} T_F(x), \quad T^l_F(x) = \inf_{F \in \mathcal{F}} T_F(x).$$  

**Lemma 6.** Suppose $\mathcal{F} = \mathcal{F}(0)$ so that $\mu_F = 0$ for all $F \in \mathcal{F}$, then $T^u_F(x)$ and $T^l_F(x)$ have the same property as $T_F(x)$ in Lemma 1 with $\mu_F$ replaced by 0, except that $T^l_F(x)$ is not always convex.

From above Lemma 6, $T^u_F(x)$ and $T^l_F(x)$ have inverse functions $(T^u_F)^{-1}(c)$ and $(T^l_F)^{-1}(c)$ for all $c, c > 0$, respectively. Thus we have shown the existence of the values of $\phi^u$ and $\phi^*$:

$$\phi^u = \sup_{F \in \mathcal{F}} \{x \mid T_F(x) = c\} = (T^u_F)^{-1}(c),$$  

$$\phi^* = \inf_{F \in \mathcal{F}} \{x \mid T_F(x) = c\} = (T^l_F)^{-1}(c).$$

3. The Class $\mathcal{F}(\mu, \sigma^2)$

**Proposition 3.** [Feller p.151] If $F$ is an arbitrary cdf, then

$$\left( \int_A u(t)v(t)dF(t) \right)^2 \leq \left( \int_A u^2(t)dF(t) \right) \left( \int_A v^2(t)dF(t) \right)$$

for any set $A$ and any functions $u, v$ for which the integrals on the right exist. Furthermore, the equality sign holds if and only if

$$\int_A (au(t) + bv(t))^2dF(t) = 0 \text{ for some } a, b \in R.$$  

Note that if $u$ and $v$ are linearly dependent, i.e., for some $a, b \in R$, $au(t) + bv(t) = 0$, the condition (3.3) is satisfied for all $F \in \mathcal{F}$, and that if $u$ and $v$ are linearly independent, the condition (3.3) is satisfied only when the cdf $F$ is degenerated at one point in a set $A$.

We shall calculate $\phi^u$ and the maximizing point $(x^u, F^u)$ of the problem (2.9) by Proposition 3.

$$\left( \int_{(x, \infty)} (t-x)dF(t) \right)^2 \leq \left( \int_{(x, \infty)} dF(t) \right) \left( \int_{(x, \infty)} (t-x)^2dF(t) \right),$$
\[(3.4') \quad (\int_{(-\infty,x]}(t-x)dF(t))^2 \leq (\int_{(-\infty,x]}dF(t))\left(\int_{(-\infty,x]}(t-x)^2dF(t)\right).\]

Then, we obtain the maximal bound \(\phi^u\).

\[\phi^u = \sup_{F \in \mathcal{F}} \{x \mid T_F(x) = c\} = \frac{\sigma^2}{4c} - c.\]

Since the equality holds in two Schwartz inequalities (3.4) and (3.4'), from the remark of Proposition 3, the maximizing cdf \(F^u\) should be the two-point cdf. Then, we have

\[F^u(t) = G_2(t; \frac{\sigma}{2c}) = (-2c, \frac{\sigma^2}{2c} > < \frac{\sigma^2}{\sigma^2 + 4c^2} >).\]

\[x^u \in x^u = x^o(F^u) = [-2c, \frac{\sigma^2}{2c}).\]

**Theorem 1.** For a class \(\mathcal{F}(0, \sigma^2)\) of cdf's, the maximal bound \(\phi^u\) is \(\sigma^2/4c - c\) by (3.7) and the maximizing point \((x^u, F^u) \in x^u \times \{F^u\}\) is given by \(F^u(t) = G_2(t; \sigma/2c)\) in (3.8) and \(x^u = [-2c, \sigma^2/2c)\) in (3.9).

**Remark of Theorem 1.** From Lemma 2, the equation (3.9) means that the player may decide a stopping level \(x^u\) for some \(x^u \in [-2c, \sigma^2/2c)\) or \(\sigma^2/2c - 0\). If the player decides any of the above stopping levels, he stops the process whenever \(X_n = \sigma^2/2c\) is observed because the player 2 chooses only one cdf given by (3.8).

Second, we shall calculate the minimax value \(\phi^*\) of (2.10) and the minimax-mizing point \((x^*, F^*) \in (x^*, \mathcal{F}^*)\).

From Lemma 6, \(T_F(x) \geq (-x)^+\) for all \(x \in R\). Then it holds that \(T_{F^*}(x) = (-x)^+ \leq T_F(x)\) for \(x \in (-\infty, -c]\) if a cdf \(F^*\), which has all the mass on \([-c, \infty)\), is contained in \(\mathcal{F}\).

Since \(T_{F^*}(x) = (-x)^+\) is strictly decreasing on \((-\infty, -c]\), we have

\[\phi^* = \inf_{F \in \mathcal{F}} \{x \mid T_F(x) = c\} = \{x \mid T_{F^*}(x) = c\} = -c.\]

Such a class \(\mathcal{F}^*\) of cdf's \(F^*\) always exists in \(\mathcal{F}\) for all \(c, c > 0\).

\[\mathcal{F}^* = \{F \mid \int_{[-c, \infty)} dF(t) = 1, F \in \mathcal{F}\}.\]

In particular, we can find the class \(\mathcal{G}^*_2 = \mathcal{G}^*_2(0, \sigma^2)\) of two-point cdf's in \(\mathcal{F}^*\) from Lemma 5.

\[\mathcal{G}^*_2 = \{G_2(\cdot; q) \mid q \geq \frac{\sigma}{c}\}.\]

It is easily shown that for any \(F^* \in \mathcal{F}^*\) it is optimal for the player 1 to stop the process immediately. That is,

\[x^* = x^o(F^*) = (-\infty, -c)\text{ for all } F^* \in \mathcal{F}^*.\]
Theorem 2. For a class $\mathcal{F}(0, \sigma^2)$ of cdf's, the minimax value $\phi^*$ is $-c$ by (3.10) and the minimax-mizing point $(x^*, F^*) \in (x^*, \mathcal{F}^*)$ is given by (3.11) and (3.12). In particular, there exists the class $\mathcal{G}_2^*$ of two-point cdf's in $\mathcal{F}^*$ by (3.11').

Now, we shall derive the saddle value $\phi^s$ for $\phi(x, F)$ in $\mathcal{F} = \mathcal{F}(\mu, \sigma^2)$. We have a candidate $(x^* \mathcal{F}^*)$ for a set of saddle points $(x^s \mathcal{F}^s)$. Theorem 3. For a class $\mathcal{F}(0, \sigma^2)$ of cdf's, the saddle value $\phi^s$ is $-c$ and the saddle point $(x^s, F^s) \in (x^s \mathcal{F}^s)$ is given by $(x^s = x^*, F^s = \mathcal{F}^s$ and $\mathcal{G}_2^s = \mathcal{G}_2^* \subset \mathcal{F}^s$ defined in Theorem 2.

Theorem 3 says the class $\mathcal{F}(\mu, \sigma^2)$ is so rich for the player 2 that the player 1 must stop immediately. In this case, the information of the value $\sigma^2$ is useless for the player 1.

4. The Class $\mathcal{F}(\mu, \sigma^2, M)$

In this section, we shall derive the maximal bound $\phi^u$ and the saddle value $\phi^s$ in the more restrictive and interesting class $\mathcal{F} = \mathcal{F}(0, \sigma^2, M)$ (see (1.10)).

Theorem 4. For a class $\mathcal{F}(0, \sigma^2, M)$ of cdf's, $\sigma < M$, the maximal bound $\phi^u$ and the maximizing point $(x^u, F^u) \in (x^u \mathcal{F}^u)$ are as follows:

(i) When $0 \leq c \leq \sigma^2/2M$,

\[ \phi^u = M - c(1 + \frac{M^2}{\sigma^2}), \quad x^u = [-\frac{\sigma^2}{M}, M), \]

\[ F^u(t) = G_2(t; \frac{M}{\sigma}) = (\sigma^2/M, \sigma^2/M^2). \]

(ii) When $\sigma^2/2M \leq c \leq M/2$, the same result as Theorem 1 holds, i.e.,

\[ \phi^u = \frac{\sigma^2}{4c} - c, \quad x^u = [-2c, -\frac{\sigma^2}{2c}), \]

\[ F^u(t) = G_2(t; \frac{\sigma}{2c}) = (-2c, \frac{\sigma^2}{2c}). \]

(iii) When $M/2 \leq c \leq M$,

\[ \phi^u = \frac{\sigma^2}{M} - c(1 + \frac{\sigma^2}{M^2}), \quad x^u = [-M, \frac{\sigma^2}{M}), \]

\[ F^u(t) = G_2(t; \frac{\sigma}{M}) = (-M, \frac{\sigma^2}{M}). \]

Now, we shall derive the saddle value $\phi^s$.

We confine our consideration to the case:

\[ 0 < c < \sigma^2/M. \]
On the other hand, it holds that

\[(4.2') \quad \inf_{F \in \mathcal{F}} \phi(x, F) \leq \phi(x, G_2(\cdot; M/\sigma)) = -c \quad \text{for} \quad x \in [-M, -\sigma^2/M),\]

\[(4.3') \quad \inf_{F \in \mathcal{F}} \phi(x, F) \leq \phi(x, G_2(\cdot; \sigma/M)) = -\infty \quad \text{for} \quad x \in [\sigma^2/M, M],\]

because the player 1 stops immediately in the case of (4.2') or he cannot stop in the case of (4.3'). Then, the player 1 must decide his stopping level \(x\) in the interval

\[(4.5) \quad x^M \overset{\text{def}}{=} \left[-\frac{\sigma^2}{M}, \frac{\sigma^2}{M}\right),\]

in order not to make his reward \(\inf_{F \in \mathcal{F}} \phi(x, F) \leq -c\), where \(-c\) is the reward of immediately stopping or the saddle value \(\phi^s = -c\) in Section 3.

**Lemma 7.** For any strategy \((x, F), x \in x^M, F \in \mathcal{F},\) if \(F\) has a probability mass \(p\) at any point \(y\) in the interval \((x, M)\) and satisfies \(\phi(x, F) \geq -c\), then there exists a cdf \(F'' \in \mathcal{F}\) such that \(F''\) has no mass in the interval \((x, M),\) and it satisfies \(\phi(x - 0, F'') \leq \phi(x, F).\)

**Lemma 8.** For any strategy \(x \in x^M, F \in \mathcal{F},\) if \(F\) has probability mass \(p\) at any point \(y\) in the interval \((-M, x)\) and it satisfies \(\phi(x, F) \geq -c\), then there exists a cdf \(F'' \in \mathcal{F}\) such that \(F''\) has no mass in the interval \((-M, x),\) and it satisfies \(\phi(x, F'') \leq \phi(x, F).\)

Let us define for any \(x \in [-\sigma^2/M, \sigma^2/M],\) a three-point cdf \(G_3^{M}(\cdot; x) \in \mathcal{F}\) which has all the mass at three points \(-M, x, M\) with the mean 0 and the variance \(\sigma^2.\) This cdf is uniquely determined by

\[(4.11) \quad G_3^{M}(t; x) = (\langle -M, x, M > < \frac{Mx + \sigma^2}{2M(M + x)}, \frac{M^2 - \sigma^2}{M^2 - x^2}, \frac{\sigma^2 - Mx}{2M(M - x)} >),\]

and let \(G_3^{M} = \{G_3^{M}(t; x) \mid -\sigma^2/M \leq x \leq \sigma^2/M\}.\) Note that if \(x = \sigma^2/M\) or \(-\sigma^2/M,\)

\(G_3^{M}(t; x)\) becomes the two-point cdf \(G_2(t; \sigma/M)\) or \(G_2(t; M/\sigma)\) respectively.

The player 1 would decide a stopping level \(x\) in the following set

\[(4.13) \quad \{x \mid \phi(x, F) \geq -c \quad \text{for all} \quad F \in \mathcal{F}\} \cap x^M \overset{\text{def}}{=} x_{c}^M\]

This set is not empty because \(x = -c\) is contained in it.

If there exists a point \(x^s \in x_{c}^M\) such that

\[(4.15) \quad \phi(x^s, G_3^{M}(\cdot; x^s)) = (T_{G_3^{M}}^{t})^{-1}(c) = \phi(x^s - 0, G_3^{M}(\cdot; x^s)) \geq -c,\]

the strategy \((x^s, G_3^{M}(\cdot; x^s), \, x \in x_{c}^M, \, G_3^{M}(\cdot; x^s) \in G_3^{M} \subset \mathcal{F},\) is the saddle point and \(\phi^s = (T_{G_3^{M}}^{t})^{-1}(c)\) is the saddle value. Because, from (4.14), Proposition 1 and (2.10), the following relation is satisfied.

\[\phi(x^s - 0, G_3^{M}(\cdot; x^s)) \leq \sup_{x \in x^M} \inf_{F \in \mathcal{F}} \phi(x, F) \leq \inf_{F \in \mathcal{F}} \sup_{x \in x^M} \phi(x, F)\]
\[
\leq \inf_{F \in \mathcal{G}^{M}} \sup_{x \in x_{c}^{M}} \phi(x, F) = (T_{\mathcal{G}^{u}}^\ell)^{-1}(c) = \phi(x^{*}, G_{3}^{M}(\cdot; x^{*})).
\]

**Theorem 5.** For a class \( \mathcal{F}(0, \sigma^{2}, M) \) of cdf's, \( \sigma \leq M \), the saddle point \( (x^{*}, F^{*}) \in (x^{*}, \mathcal{F}^{*}) \) is as follows:

(i) When \( \sigma^{2}/M \leq c \leq M \), the same result as Theorem 3 holds, that is,

\[
\phi^{*} = -c, \quad x^{*} = [-M, -c] \text{ and } \mathcal{F}^{*} = \{F | \int_{[-c, M]} dF(t) = 1, F \in \mathcal{F}(0, \sigma^{2}, M)\}.
\]

(ii) When \( 0 < c < \sigma^{2}/M \),

\[
\phi^{*} = (\sigma^{2}/M - c)^{+} - c, \quad x^{*} = \{x^{*}\}, \quad x^{*} = (\sigma^{2}/M - c)^{+} - c \text{ and } \mathcal{F}^{*} = \{F^{*}\}, \quad F^{*}(t) = G_{3}^{M}(t; x^{*}) \text{ defined by (4.11)}.
\]

**References**

Figure 1. The maximal bound $\phi^\mu$ and the saddle value $\phi^S$ in the class $\mathcal{A}(0, \sigma^2, M)$ when $M = 12, \sigma = 8.$