

## EXACT PENALTY FUNCTIONS in $\varepsilon$ -PROGRAMMING PROBLEMS

新潟大学 自然科学 横山一憲 (Kazunori Yokoyama)

### 1. Introduction

In the theory of mathematical programming, the situation where the optimal solution is taken to be "exact" has been concerned. In contrast to this situation, the situation where the optimal solution is taken to be " $\varepsilon$ -approximate" has been studied recently in [2-5]. Such situation is of interest from the theoretical point of view as well as the computational one.

P.Loridan and J.Morgan [3] showed some results about such " $\varepsilon$ -approximate" optimal solutions for the nonlinear programming problem by using the classical penalty function and the exact penalty function. In the convex programming problem, Bertsekas [1] gave the conditions to yield the "exact" optimal solution by estimating the size of the penalty parameter of the exact penalty function in terms of the optimal Lagrange multipliers.

In this note we show several conditions to obtain " $\varepsilon$ -approximate" optimal solutions for the nonlinear programming problem by estimating the size of penalty parameter in terms of " $\varepsilon$ -approximate" optimal solutions for the dual problem. Also we study the relations among the " $\varepsilon$ -approximate" optimal solution for the convex programming problem, Lagrange  $\varepsilon$ -multipliers set referred to [5], and the penalty parameter.

### 2. Preliminaries

Let  $f, g_1, \dots, g_m : \mathbb{R}^n \rightarrow \mathbb{R}$  be real-valued functions defined on  $\mathbb{R}^n$ .  
 consider the nonlinear programming problem (Primal Problem):

$$\begin{aligned} & \text{minimize} && f(x) \\ & \text{subject to} && g_1(x) \leq 0, \dots, g_m(x) \leq 0. \end{aligned}$$

We denote the feasible set  $\{x \in \mathbb{R}^n \mid g_1(x) \leq 0, \dots, g_m(x) \leq 0\}$  by  $K$ .

We recall that the dual problem of (P) with respect to the constraints  $g_i(x) \leq 0$  is:

$$\begin{aligned} & \text{maximize} && \omega(\lambda) \\ & \text{where} && \lambda = (\lambda_1, \dots, \lambda_m) \in \mathbb{R}^m, \\ & && \omega(\lambda) = \inf_{x \in \mathbb{R}^n} L(x, \lambda), \\ & && L(x, \lambda) = \begin{cases} f(x) + \sum_{i=1}^m \lambda_i g_i(x) & \text{if } \lambda \geq 0 \\ -\infty & \text{if not.} \end{cases} \end{aligned}$$

The associated penalized problem is:

$$\begin{aligned} & \theta_\rho && \text{minimize } \theta(x, \rho) \\ & && \text{where } \theta(x, \rho) = f(x) + \rho \sum_{i=1}^m \max(0, g_i(x)), \\ & && \rho > 0. \end{aligned}$$

Throughout this note, we suppose that the following basic assumption is satisfied.

**Assumption.** The real number  $\rho$  is positive.

The objective function  $f$  is bounded from below.

The set  $\{\lambda \in \mathbb{R}^m \mid \omega(\lambda) > -\infty\}$  is nonempty.

The feasible set  $K$  is nonempty.

**Remark.** With this assumption, the "exact" optimal solution is not necessarily attained for the problem (P). However, there exists  $\bar{x} \in K$  such that  $f(\bar{x}) \leq \inf \{f(x) \mid x \in K\} + \varepsilon$ . So, we shall be interested in  $\varepsilon$ -optimal points. Furthermore, the "exact" solution for (D) (resp.  $(\theta_\rho)$ ) is not necessarily attained, but there exists  $\bar{\lambda} \geq 0$  such that  $\omega(\bar{\lambda}) \geq \sup_{\lambda \in \mathbb{R}^m} \omega(\lambda) - \varepsilon$  (resp.  $\bar{x} \in \mathbb{R}^n$  such that  $\theta(\bar{x}, \rho) \leq \inf_{x \in \mathbb{R}^n} \theta(x, \rho) + \varepsilon$ ).

By means of the above Remark, we define  $\varepsilon$ -solutions for (P), (D) and  $(\theta_\rho)$ , respectively.

**Definition.** If  $\bar{x} \in K$  satisfies  $f(\bar{x}) \leq \inf \{f(x) \mid x \in K\} + \varepsilon$ , we say that  $\bar{x}$  is an  $\varepsilon$ -solution for (P).

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**Definition.** If  $\bar{x} \in \mathbb{R}^n$  satisfies  $\theta(\bar{x}, \rho) \leq \inf_{x \in \mathbb{R}^n} \theta(x, \rho) + \varepsilon$ , we say that  $\bar{x}$  is an  $\varepsilon$ -solution for  $(\theta_\rho)$ .

We now define another solution concept for (P).

**Definition.** An element  $\bar{x}$  is said to be an almost  $\varepsilon$ -solution for (P) if

the following conditions are satisfied:

- (1)  $\bar{x} \in K_\varepsilon$  where  $K_\varepsilon = \{x \in \mathbb{R}^n \mid g_i(x) \leq \varepsilon, 1 \leq i \leq m\}$ ,
- (2)  $f(\bar{x}) \leq \inf \{f(x) \mid x \in K\} + \varepsilon$ .

### 3. Characterization of $\varepsilon$ -solutions

We characterize  $\varepsilon$ -solutions for (P) by estimating the size of the penalty parameter of the exact penalty function in terms of the  $\varepsilon$ -solution for (D). We assume neither the solvability of (P) nor (D).

Now we introduce two assumptions:

**Assumption (A1).** The functions  $f$  and  $g_i$  ( $1 \leq i \leq m$ ) are convex.

**Assumption (A2).** The set  $\{x \in \mathbb{R}^n \mid g_i(x) < 0 \text{ for any } i\}$  is nonempty.

**Remark.** The assumption (A2) is called the Slater constraint qualification. With (A1) and (A2), the duality gap  $\gamma (= \inf \{f(x) \mid x \in K\} - \sup_{\lambda \in \mathbb{R}^m} \omega(\lambda))$  is equal to zero and there exists an "exact" optimal solution for (D) which is called the optimal Lagrange multiplier.

The following proposition gives the necessary condition to obtain an  $\varepsilon$ -solution for (P).

**Proposition 3.1.** Let  $\bar{x}$  be an  $\varepsilon$ -solution for (P). Let  $\rho_0$  be defined by

$$\rho_0 = \|\bar{\lambda}\|_\infty,$$

re  $\bar{\lambda}$  is an  $\varepsilon$ -solution for (D),

$$\|\bar{\lambda}\|_{\infty} = \max_{1 \leq i \leq m} |\bar{\lambda}_i|.$$

then for all  $\rho \geq \rho_0$ ,  $\bar{x}$  is a  $(2\varepsilon + \gamma)$ -solution for  $(\theta_{\rho})$ .

**ollary 3.1.** Assume that (A1) and (A2) are satisfied. Let  $\bar{x}$  be an solution for (P) and  $\rho_0 = \|\bar{\lambda}\|_{\infty}$  for some optimal Lagrange multiplier

then for all  $\rho \geq \rho_0$ ,  $\bar{x}$  is an  $\varepsilon$ -solution for  $(\theta_{\rho})$ .

he following proposition gives the sufficient condition to obtain almost  $\varepsilon$ -solution for (P).

**osition 3.2.** We set the penalty parameter  $\rho$  as follows.

$$\rho \geq 3 + \|\bar{\lambda}\|_{\infty} + \frac{\gamma}{\varepsilon} \text{ where } \bar{\lambda} \text{ is an } \varepsilon\text{-solution for (D).}$$

f  $\bar{x}$  is an  $\varepsilon$ -solution for  $(\theta_{\rho})$ , then  $\bar{x}$  is an almost  $\varepsilon$ -solution for

**ollary 3.2.** Assume that (A1) and (A2) are satisfied. Let the alty parameter  $\rho$  be defined by

$$\rho \geq 2 + \|\bar{\lambda}\|_{\infty} \text{ for some optimal Lagrange multiplier } \bar{\lambda}.$$

f  $\bar{x}$  is an  $\varepsilon$ -solution for  $(\theta_{\rho})$ , then  $\bar{x}$  is an almost  $\varepsilon$ -solution for

hen we consider the convex programming problems, the following position is shown.

**osition 3.3.** Assume that (A1) is satisfied. If any  $\varepsilon$ -solution for

$(\theta_\rho)$  is also an  $\varepsilon$ -solution for (P), then

the origin in  $\mathbb{R}^m$  is an  $\varepsilon$ -solution for (D).

**Remark.** Proposition 3.3 says that unless some  $\varepsilon$ -solution for (D) is zero (and the problem is essentially unconstrained) we cannot obtain  $\varepsilon$ -solutions for (P) by solving  $(\theta_\rho)$ .

Now we show the relations among the  $\varepsilon$ -solution for (P),  $\varepsilon$ -Lagrange multipliers set due to J.J.Strodiot et al.[5], and the penalty parameter in the convex programming problems.

**Definition[5].** A vector  $\lambda \in \mathbb{R}^m$  is said to be a Lagrange  $\varepsilon$ -multiplier for (P) at  $x$  if the following conditions are satisfied:

$$(1) \lambda_i \geq 0 \quad (i = 1, \dots, m),$$

$$(2) \text{ there exist scalars } \varepsilon_i \geq 0 \quad (i = 0, \dots, m) \text{ such that}$$

$$(2-1) \quad 0 \in \partial_{\varepsilon_0} f(x) + \sum_{i=1}^m \partial_{\varepsilon_i} (\lambda_i g_i)(x)$$

$$(2-2) \quad \sum_{i=0}^m \varepsilon_i - \varepsilon \leq \sum_{i=1}^m \lambda_i g_i(x) \leq 0.$$

We denote the set of all Lagrange  $\varepsilon$ -multipliers for (P) at  $x$  which is called the Lagrange  $\varepsilon$ -multipliers set for (P) at  $x$  by  $L_\varepsilon(x)$ .

**Remark.** From [5, Theorem 3.2.], if (A1) and (A2) are satisfied and  $x$  is an  $\varepsilon$ -solution for (P), every element of  $L_\varepsilon(x)$  is an  $\varepsilon$ -solution for (D).

**Theorem 3.1.** Assume that (A1) is satisfied. Consider the following three conditions:

(a) there exists  $\rho_0$  such that for any  $\rho \geq \rho_0$ ,  $\bar{x}$  is an solution for  $(\theta_\rho)$ ,

(b)  $\bar{x}$  is an  $\varepsilon$ -solution for (P),

(c) the set  $L_\varepsilon(\bar{x})$  is nonempty, and  $\bar{x} \in K$ .

Condition (a) implies (b) and (c).

**Corollary 3.3.** Assume that (A1) and (A2) are satisfied. Then, the three conditions of Theorem 3.1 are equivalent.

#### References

- 1] D.P. Bertsekas, "Necessary and sufficient conditions for a penalty methods to be exact", Mathematical Programming 9(1975)87-99.
- 2] P.Loridan, "Necessary conditions for  $\varepsilon$ -optimality", Mathematical Programming Study 19(1982)140-152.
- 3] P.Loridan and J.Morgan, "Penalty functions in  $\varepsilon$ -programming and  $\varepsilon$ -minimax problems", Mathematical programming 26(1983)213-231.
- 4] E.Rosenberg, " $\varepsilon$ -Subgradient optimization techniques in convex programming and Lagrange duality", Opsearch 23(1986)71-88.
- 5] J.J.Strodiot, V.H.Nguyen and N.Heukemes, " $\varepsilon$ -Optimal solutions in nondifferentiable convex programming and some related questions" Mathematical Programming 25(1983)307-328.