数理解析研究所講究録
400巻 1980年 24-32

RAMIFICATION AND SINGULARITIES

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0. INTRODUCTION.

Let $\pi: X \to Y$ be a morphism of locally noetherian schemes, finite, separable and surjective on all the components of X. Assume X is reduced, Y is normal and let $\mathfrak{D}_{X/Y}$ (resp. Y_D) be the discriminant sheaf (resp. scheme) of π (see section 1).

The problem we are interested in is to study the singularities of X in terms of $\mathfrak{D}_{X/Y}$ and/or Y_D.

In section 1 we give the definitions of $\mathfrak{D}_{X/Y}$ and Y_D, in sections 2 and 3 we study respectively the number of isolated singularities of a projective surface, and the singularities in codimension 1 of a scheme, and finally in section 4 we give two applications.

1. THE DISCRIMINANT SHEAF (SCHEME) OF π.

Assume that $\deg \pi = n$ and that X and Y are integral; then for every affine open subset $U = \operatorname{spec} A_U$ of Y put $\Gamma(U, \mathfrak{D}_{X/Y}) =$ $= D_{B'_U / A_U}$, where $\operatorname{spec} B'_U = \pi^{-1}(U)$ and $D_{B'_U / A_U}$ denotes the discriminant of B'_U over A_U.

$\mathfrak{D}_{X/Y}$ is a coherent sheaf of $\mathcal{O}_Y$ - ideals (see [5], 3.1), and it is called the discriminant sheaf of π : the corresponding closed subscheme Y_D of Y is called the discriminant scheme of π.

2. ISOLATED SINGULARITIES.

Let X be an irreducible surface of order n in the complex projective 3-space (more generally: over an algebraically closed field of characteristic zero).

Assume that X has only d conical double points as singularities, and look for an upper-bound for d.

In 1906-1907 Basset (see [1],[2]) proved the following two limitations for d:

I) $d \leq (2/3)n(n-1)(n-2)$

II) $d \leq (1/2)[n(n-1)^2-5-\sqrt{n(n-1)(3n-14)+25}\,]$ $(n \geq 5)$.

The tecnique he used was the following: consider the projection $\pi: X \to Y$ from a generic point P of $\mathbb{P}^3$ over a projective plane Y, and look at the singularities of the discriminant scheme Y_D of π. Taking for granted that Y_D is an irreducible curve with only plückerian singularities, Basset deduced I) and II) by applying the Plücker 's formulas to the characters of Y_D.

Basset 's proof was not correct, as it is not known whether Y_D has only plückerian singularities, which seems rather unlikely.

Neverthless the limitations I) and II) given by Basset are correct and actually they are the best upper-bound so far obtained for d.

They were recently proved by Stagnaro in [14].

Considering the projection $\pi : X \to Y$ as before, Stagnaro proved that:

i) Y_D is an irreducible curve, of order $n(n-1)$ and class $n(n-1)^2-2d$,

ii) the non-linear branches of Y_D have order ν and class 1, and are centered at the points $A' = \pi(A)$, where $i(A; \mathcal{P}'_P \cap \mathcal{P}''_P \cap X) = \nu - 1 > 0$. ($\mathcal{P}'_P$ and $\mathcal{P}''_P$ are respectively the first and the second polar of P with respect to X).

Then he proved I) and II) by applying the generalized Plücker 's formulas to the characters of Y_D.

For the sake of completness we list now the upper-bounds for d given by I) and II), and the results so far obtained in order to construct algebraic projective surfaces of order n having the maximum number of conical double points.

	n =	3	4	5	6	7	8	..
I)	$d \leq$	4	16	40	80	140	224	..
II)	$d \leq$	-	-	34	66	114	181	..

(For $n \geq 5$ II) gives better limitations than I)).

The best examples of algebraic projective surfaces of order n having a high number d(n) of conical double points are the following:

d(3)=4 ([4]) ; d(4)=16 ([11]) ; d(5)=31* ([15]) ;

d(6)=64 ([3],[13]) ; d(7)=90 ([13]) ; d(8)=160 ([9],[10]) ;

for $n \geq 9$ see [8],[10].

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* I have recently been told that in a preprint by Beauville it is proved that for n=5 d cannot exceed 31.

3. SINGULARITIES IN CODIMENSION 1.

Let $\pi : X \to Y$ be a morphism as in section 1. Let $y \in Y$ be a point of codimension 1, and put:

$A = \mathcal{O}_{Y,y}$, k its residue field, $B' = \mathcal{O}_{X,\pi^{-1}(y)}$,

$B = \overline{B}'$, $f = \ell_A(B/\mathrm{rad}\ B)$, $g = \ell_A(B'/\mathrm{rad}\ B')$, $D_{B'/A}$ the discriminant of B' over A (remark that $D_{B'/A} = (\mathfrak{D}_{X/Y})_y$) , v the valuation associated with A.

Assume that $k(m)/k$ is separable for all $m \in \mathrm{Max}\ B$.

We want to study the singularities of B', that is the singularities of X in codimension 1; in particular we want to study the normality and the seminormality of B', which is equivalent to study the normality and the seminormality of the whole X, if we assume X to be S_2.

THEOREM 1. (Characterization of normality). ([7], 1.3).

i) $v(D_{B'/A}) \geq n-g$.

ii) $v(D_{B'/A}) = n-g$ <u>iff</u> B' <u>is normal and tamely ramified over</u> A.

The proof of this theorem relies on the following facts:

a) B is tamely ramified over A iff the different $\delta_{B/A}$ of B over A is equal $\prod_i m_i^{e_i-1}$ where $m_i \in \mathrm{Max}\ B$ and e_i is its ramification index for all i (see [12], prop.13, p.67);

b) $D_{B/A} = N(\delta_{B/A})$, N denoting the norm (see [12], prop.6, p.60);

c) $v(D_{B'/A}) = 2\ell_A(B/B') + v(D_{B/A})$ ([7], 1.1).

THEOREM 2. (A sufficient condition for normality). ([7], 1.8).

If B' *is normal and tamely ramified over* A, *then* $v(D_{B'/A}) \leq n-1$.

The converse holds if either:

i) $n=2$, *or*

ii) B' *is local, or*

iii) *there exists a finite group* G *of automorphisms of* B' *such that* $B'^G = A$.

THEOREM 3. (Characterization of seminormality). ([7], 2.3).

i) $v(D_{B'/A}) \geq n+f-2g$.

ii) $v(D_{B'/A}) = n+f-2g$ *iff* B' *is seminormal and* B *is tamely ramified over* A.

The proof of this theorem relies on the following facts:

a) B' is seminormal iff $\ell_A(B/B') = f-g$ ([7], 2.1) ;

b) $v(D_{B'/A}) = 2\ell_A(B/B') + v(D_{B/A})$;

c) theorem 1.

THEOREM 4. (A sufficient condition for seminormality). ([7], 2.8 and 3.1).

Assume that B *is tamely ramified over* A. *If* B' *is seminormal (resp. seminormal and Gorenstein), then* $v(D_{B'/A}) \leq n+f-1$ (*resp.* $v(D_{B'/A}) \leq n$).

The converse holds if either:

i) $n=2$, *or*

ii) B' *is local, or*

iii) *there exists a finite group* G *of automorphisms of* B' *such that* $B'^G = A$.

THEOREM 5. (The monogenic case). ([6], and [7] section 3).
Suppose $B'=A[x]$ and let X^n-a ($a \in A$, $n \geq 3$) be the caracteristic polynomial of x : assume that either char $k=0$ or char $k>n$.
Then the following are equivalent:

i) B' is seminormal.

ii) B' is normal.

iii) $v(D_{B'/A}) \leq n$.

iv) $v(a) \leq 1$.

4. TWO APPLICATIONS.

(In the following examples, for the sake of simplicity, we shall frequently denote by the same symbol a surface and its equation).

EXAMPLE 1.

Let X be an irreducible surface (not a cone) of order n in the projective 3-space over a field k algebraically closed, of characteristic $\neq 2$.
Assume that X has equation $X_0^2\, a + 2X_0 b + c = 0$, where $a,b,c \in k[X_1,X_2,X_3]$ are forms of degree $n-2, n-1, n$ respectively, and (X_0,X_1,X_2,X_3) are the coordinates in $\mathbb{P}^3(k)$.
The point $P(1,0,0,0)$ is $(n-2)$-fold for X and $a=0$ is the tangent cone to X at P: assume that it has no multiple generatrices, and let Δ be the curve of the plane $X_0=0$ given by $b^2-ac=0$.

We have: X <u>is normal</u> (resp. <u>seminormal</u>) <u>iff</u> Δ <u>does not have multiple components</u> (resp. Δ <u>has at most double components</u>).
Indeed: put $V=X-(X\cap a)$, $W=Y-(Y\cap a)$ (where Y denotes the plane $X_0=0$) and let $\pi:V\to W$ be the projection from P; clearly π is a finite, separable, surjective morphism of degree 2, having $W_D = \Delta - (\Delta\cap a)$ as discriminant scheme. Therefore, from theorem 2 (resp. theorem 4) it follows that V is normal (resp. seminormal) iff W_D has no multiple components (resp. W_D has at most double components).
Moreover it can be proved that, under our assumptions, $X-V$ has only normal points and that $\Delta - W_D$ has no multiple components, and from this our claim follows.

EXAMPLE 2.
Let X be an irreducible hypersurface of order $n \geq 3$ in $\mathbb{P}^r(k)$, where k is an algebraically closed field of characteristic either 0 or $> n$.
Assume that X has equation $X_0^n = h(X_1,..,X_r)$, where $h \in k[X_1,..,X_r]$ is a form of degree n and $(X_0,..,X_r)$ are the coordinates in $\mathbb{P}^r(k)$.
We have: X <u>is normal iff</u> X <u>is seminormal iff the polynomial</u> h <u>has no multiple factors</u>.
X is normal (resp. seminormal) iff X is normal (resp. seminormal) on all the charts of an affine covering; therefore we may assume X affine. Let $Z^n = h(V_1,..,V_{r-1})$ be its equation, and consider

the projection $\pi: X \to Y$ from the point $P(1,0,..,0)$ on the hyperplane Y having equation $Z=0$: π is the finite, separable, surjective morphism of degree n, which corresponds to the canonical ring homomorphism $R=k[V_1,..,V_{r-1}] \to \{R[Z]/(Z^n-h)\} \cong R[z]$.

In codimension 1 we have the following situation: $A=R_p$, $B'=R_p[z]$ (ht p = 1) ,and therefore, by applying theorem 5 we can prove our claim.

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