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Multiple Integration by a Modified Clenshaw-Curtis Quadrature whose Number of Sample Points is Increased with Arithmetic Progression

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1. Introduction

In this paper we consider the following multiple integral

$$I = \int_{\phi_1}^{\psi_1} dx_1 \int_{\phi_2}^{\psi_2} dx_2 \cdots \int_{\phi_m}^{\psi_m} dx_m\, f(x_1, x_2, \cdots, x_m),$$

where the lower boundaries of integration $\phi_1, \phi_2, \cdots, \phi_m$ and the upper boundaries of integration $\psi_1, \psi_2, \cdots, \psi_m$ are given by

$$\begin{aligned} \phi_1 &= a, & \psi_1 &= b, \\ \phi_2 &= \phi_2(x_1), & \psi_2 &= \psi_2(x_1), \\ &\vdots & &\vdots \\ \phi_m &= \phi_m(x_1, x_2, \cdots, x_{m-1}), & \psi_m &= \psi_m(x_1, x_2, \cdots, x_{m-1}) \end{aligned}$$

Our automatic multiple integration scheme gives the approximation S to the integral I satisfying the following inequality

$$|I - S| \leqq max\,(\varepsilon_a, \varepsilon_r |I|)$$

with the required absolute (or relative) tolerance ε_a (or ε_r). Although a number of programs for the multiple integration have been published [1], it seems that no practical program exists for the automatic multiple integration scheme.

There are two approaches to numerically carry out the multiple integration.

1) Probabilistic approach; Monte Carlo methods are used for the higher dimensional integration with low accuracy.

2) The second approach is the product rule in which we repeatedly apply the one-dimensional rule to the each axis of integration to carry out the multiple integration. The product rule is suitable for the automatic integration but cannot be used as higher dimensional multiple integrator because tremendous number of points are necessary for successful approximation as the dimension becomes higher.

We made the multiple integration scheme of product type with at most three dimentions by using the modified Clenshaw-Curtis rule whose number of sample points is increased with arithmetic progression. This scheme is efficiently works for the integration of the oscillatory functions as well as the well-behaved ones.

2. Modified Clenshaw-Curtis Quadrature whose Number of Sample Points is Increased with Arithmetic Progression

One-dimensional automatic quadrature scheme has three main factors. First, what kind of the quadrature rules are to be chosen from a point of view of the stability and the convergency of the rule? Secondly, reliability of the scheme depends on the method of estimating the errors incurred in the quadrature scheme. The third factor which governs the efficiency of the automatic scheme is a strategy deciding how to choose the sequence of the sample points to obtain the approximation with prescribed accuracy.

The Clenshaw-Curtis quadrature [2] is known to be one of the most efficient rules for the integration of the well-behaved functions. In the Clenshaw-Curtis rule the number of points is increased with geometric progression, 2^n, so that the actual error is often much smaller than the requested error. This means that the amount of the redundant works wasted in this process would be larger if such one-dimensional quadrature rule is to be applied repeatedly to the multiple integration. To overcome this drawback of the Clenshaw-Curtis quadrature we will propose the modified Clenshaw-Curtis method (exactly speaking, Filippi type [3]) whose number of sample points is increased with arithmetic progression and show how to estimate precisely the error incurred.

2-1) The choice of the sample points

We consider the following integral

$$I = \int_{-1}^{1} f(x)\,dx \qquad (1)$$

The sequence of the sample points $\{x_K\} = \{cos 2\pi\alpha_K\}$ used to approximate (1) is generated in the following way. The sequence $\{\alpha_K\}$, the modification of the so-called Van der Corput sequence [4] which is known to be the best uniformly distributed on the interval (o,1), is defined by the recurrence relation

$$\alpha_{2K} = \alpha_K/2\ ,\quad \alpha_{2K+1} = \alpha_{2K} + 1/2\ ,\qquad k=1,2,\cdots$$

with the starting value $\alpha_1 = 1/4$. The sequence of Chebyshev distribution which constructs the generalized Chebyshev interpolation formula is obtained with projection over the range [-1,1] of the uniformly distributed sequence on the unit circle in the complex plane. The sequence $\{e^{2\pi i K^*}\}$ obtained from k^* shown in Table 1 is distributed symmetrically with respect to the real axis.

Table 1

Table of the sequence $\{\alpha_K\}$

k		binary form					α_K
				k^*			
1	=	1.	$\rightarrow$	.1	$\rightarrow$	$.\underline{01}$ =	1/4
2		10.		.01		$.0\underline{01}$	1/8
3		11.		.11		$.1\underline{01}$	5/8
4		100.		.001		$.00\underline{01}$	1/16
5		101.		.101		$.10\underline{01}$	9/16
6		110.		.011		$.01\underline{01}$	5/16
7		111.		.111		$.11\underline{01}$	13/16

There exist some of the elements of $\{cos 2\pi k^*\}$ which coincide with each other. On the other hand, all the elements of the sequence $\{cos 2\pi\alpha_K\}$ are different, so we use the sequence $\{cos 2\pi\alpha_K\}$ as the sample points. Some properties of the sequence $\{cos 2\pi\alpha_K\}$ are described below [5].

Since the sequence $\{cos 2\pi\alpha_K\}$ does not include both ends of the interval [-1,1], we have the open type quadrature formula. Next, our sequence $\{cos 2\pi\alpha_K\}$ $(1 \leqq k \leqq 2^j - 1)$ is the zeros $\{cos \frac{\pi K}{2^j}\}$ of $U_{2^j-1}(x)$ which are the abscissae used by Filippi [3], where $U_{2^j-1}(x)$ is the 2nd kind Chebyshev polynimial defined by

$$U_{2^j-1}(x) = sin(2^j \Theta)/sin\Theta , \qquad x = cos\Theta .$$

For any positive integer ℓ, the following relations hold,

$$\{cos 2\pi\alpha_{8\ell}, \cdots, cos 2\pi\alpha_{8\ell+7}\} = \text{zeros of } \{T_8(x) - cos 2\pi\alpha_\ell\}$$
$$= cos\frac{2\pi}{8}(j+\alpha_\ell), \quad (j=0,1,\cdots,7)$$

where $T_K(x)$ is the first kind Chebyshev polynomial defined by

$$T_K(x) = cos k\Theta , \qquad x = cos\Theta .$$

2-2) Interpolation

Now let us construct an interpolation polynomial for $f(x)$ on the sample points $\{x_K\} = \{cos 2\pi\alpha_K\}$. By using the first $7+8\ell$ abscissae from $\{cos 2\pi\alpha_k\}$, we have the interpolation polynomial $P_\ell(x)$ for $f(x)$ as follows

$$P_\ell(x) = P_{\ell-1}(x) + U_7(x)\omega_{\ell-1}(T_8(x)) \sum_{p=0}^{7}{}' A_{\ell,p} T_p(x), \quad (\ell \geqq 1) \qquad (2)$$

where $\sum'$ denotes a sum whose first term is halved and

$$P_0(x) = \sum_{p=1}^{7} A_{0,p} U_{p-1}(x)$$

$$\mathcal{W}_0(x) = 1$$
$$\mathcal{W}_i(x) = 2^i \sum_{j=1}^{i} (x - \cos 2\pi\alpha_j).$$

When we express ℓ in the form

$$\ell = \ell_1 + \ell_2 2^1 + \cdots + \ell_n 2^{n-1} \quad , \qquad (\ell_n = 1), \; \ell_\nu = 0 \text{ or } 1$$

the polynomial $\mathcal{W}_\ell(x)$ can be written by

$$\mathcal{W}_\ell(x) = 2^{\sum_{\nu=1}^{n-1} \ell_\nu} \times U_{2^n-1}(x) \prod_{\substack{\nu=1 \\ (\ell_\nu \neq 0)}}^{n-1} \left(T_{2^{\nu-1}}(x) + T_{2^{\nu-1}}(x_\ell) \right) . \tag{3}$$

By the transformation $x = \cos\Theta$, we get

$$P_\ell(\cos\Theta)\sin\Theta = \sum_{p=1}^{7} A_{0,p} \sin p\Theta + \sin 8\Theta \sum_{i=1}^{\ell} \mathcal{W}_{i-1}(\cos 8\Theta) \times \sum_{p=0}^{7}{}' A_{i,p} \sin p\Theta , \tag{4}$$

from eq.(2), where the coefficients $A_{i,p}$ are determined from the interpolation condition which we will described below.

The coefficients $A_{0,p}$ of first stage are to satisfy the condition

$$f(\cos\tfrac{\pi j}{8})\sin\tfrac{\pi j}{8} = \sum_{p=0}^{7} A_{0,p} \sin\tfrac{\pi p j}{8} . \quad (j = 1, 2, \cdots, 7) \tag{5}$$

From the above eq.(5), $A_{0,p}$ are written by

$$A_{0,p} = \frac{2}{8} \sum_{j=1}^{7} f(\cos\tfrac{\pi j}{8}) \sin\tfrac{\pi j}{8} \sin\tfrac{\pi p j}{8} .$$

Next, the coefficients $A_{\ell,p}$ $(\ell \geqq 1)$ of ℓ-th stage have to satisfy the following condition

$$f(\cos\Theta_j^\ell)\sin\Theta_j^\ell = \sum_{p=1}^{7} A_{0,p} \sin p\Theta_j^\ell + \sin 8\Theta_j^\ell \sum_{i=1}^{\ell} \mathcal{W}_{i-1}(\cos 2\pi\alpha_\ell) \times \sum_{p=0}^{7}{}' A_{i,p} \cos p\Theta_j^\ell , \tag{6}$$

where $\Theta_j^\ell = 2\pi(j + \alpha_\ell)/8$. The left hand side of eq.(6) can be expanded as follows

$$f(\cos\theta_j^{\ell})\sin\theta_j^{\ell} = \sum_{p=0}^{7}{}' b_p \cos p\theta_j^{\ell} \quad , \qquad (7)$$

where coefficients b_p can be computed by FFT. From eqs.(6) and (7), we have the following equation with respect to $A_{\ell,p}$,

$$b_p - (A_{0,8-p} - \cos 2\pi\alpha_\ell \cdot A_{0,p}) / \sin 2\pi\alpha_\ell = \sin 2\pi\alpha_\ell \sum_{i=1}^{\ell} \mathcal{W}_{i-1}(\cos 2\pi\alpha_\ell) A_{i,p} \qquad (8)$$

where we defined $A_{0,8} = A_{0,0} = 0$ for convernience.
With the notation $\ell = \ell' + 2^n$ $(0 \leqq \ell < 2^n)$, we have

$$\sin 2\pi\alpha_\ell \mathcal{W}_{2^n-1}(\cos 2\pi\alpha_\ell) = \sin[2\pi\cdot(2^n\alpha_\ell)] = 1 \quad . \qquad (9)$$

From eqs.(8) and (9), the coefficients $A_{\ell,p}$ are computed by the following recurrence relation

$$B_{\ell'+1} = b_p - (A_{0,8-p} - \cos 2\pi\alpha_\ell \cdot A_{0,p})/\sin 2\pi\alpha_\ell - \sin 2\pi\alpha_\ell \cdot \sum_{i=1}^{2^n-1} \mathcal{W}_{i-1}(\cos 2\pi\alpha_\ell) A_{i,p} \quad ,$$

$$B_{\ell'-i} = (B_{\ell'+1-i} - A_{2^n+1})/(\cos 2\pi\alpha_\ell - \cos 2\pi\alpha_{2^n+i})$$

$$i = 0, 1, \cdots, \ell' - 1, \qquad (10)$$

Hence

$$A_{\ell,p} = B_1 \quad ,$$

where each B_i depends on p.

The stability of the recurrence relation (10) is a little better than that of the Newton divided difference formula because the number of divisions in this formula , $\ell + 1$, is reduced to $\ell' + 1$ in eq.(10), although the sum of number of divisions and multiplications is not varied. Thus, we have obtained the interpolation polynomials with the abscissae $x_1, x_2, \cdots, x_{8\ell+7}$.

2-3) Integration scheme

The remainder $R_\ell(x)$ of interpolation polynomial $P_\ell(x)$ for

$f(x)$ is written by

$$R_\ell(x) = f(x) - P_\ell(x)$$
$$= U_7(x)\mathcal{W}_\ell(T_8(x))2^{-(8\ell+7)} f[x, x_1, \cdots, x_{8\ell+7}] , \qquad (11)$$

with the divided difference of order $8\ell + 7$, $f[x, x_1, \cdots, x_{8+7}]$.

By integrations of both sides of eq.(11) on [-1,1], we have

$$\int_{-1}^{1} R_\ell(x)dx = \int_{-1}^{1} f(x)dx - \int_{-1}^{1} P_\ell(x)dx = I - I_{8\ell+7} \qquad (12)$$

The approximation $I_{8\ell+7}$ of the integration with $8\ell + 7$ sample points is given by

$$I_{8\ell+7} = \sum_{p=1}^{7} A_{0,p}W_{0,p} + \sum_{i=1}^{\ell}\sum_{p=0}^{7}{}' A_{i,p}W_{i,p} \quad , (p=odd) \qquad (13)$$

where the weights $W_{i,p}$ are defined by

$$W_{0,p} = \int_0^\pi sinp\Theta d\Theta = 2/p \quad ,$$
$$W_{i,p} = \int_0^\pi sin8\Theta\, \mathcal{W}_{i-1}(cos8\Theta)sin\Theta d\Theta \ , \ (i\geqq 1) \ , (p=odd) . \qquad (14)$$

Now we show the recurrence relation for $W_{i,p}$ which needs only about $\frac{N}{2}(log_2 N-2)+4$ multiplications to compute $N(=2^{m+2})$ elements of $\{W_{i,p}\}$. The weights $W_{2^n,p}$ are computed by

$$W_{2^n,p} = \int_0^\pi sin2^{n+3}\Theta cosp\Theta d\Theta = \frac{2^{n+4}}{4^{n+3} - p^2}$$
$$n = 0, 1, \cdots, \quad 0\leqq p\leqq 2^n \quad (p=odd) \ ,$$

where we have used the following identity,

$$sin8\Theta\,\mathcal{W}_{2^n-1}(cos8\Theta) = sin2^{n+3}\Theta \ .$$

With this $W_{2^n,p}$ the starting value, the weights $W_{i,p}(1\leqq i\leqq 2^m)$, $(0\leqq p\leqq 7,\ p=odd)$ can be computed by using the following recurrence relation

$$
\begin{aligned}
W_{2^n+2^{n-j+1}s+2^{n-j},\,p} &= W_{2^n+2^{n-j+1}s,\,2^{n-j}\cdot 8+p} \\
&+ W_{2^n+2^{n-j+1}s,\,2^{n-j}\cdot 8-p} \\
&- 2\cos 2\pi\alpha_{2^j+2s}\cdot W_{2^n+2^{n-j+1}s,\,p} \quad , \qquad (16) \\
n &= 1,2,\cdots, m-1, \quad 1\leqq j\leqq n \ , \quad 0\leqq s\leqq 2^{j-1}-1, \\
& \qquad 0\leqq p\leqq 2^{n-j}\cdot 8-1 \quad , \quad (p=odd) \quad .
\end{aligned}
$$

The weights $W_{i,p}$ are to be computed and tabulated in the memory of computer before implementing this integration scheme in the computer. Numerical experiment shows that the $W_{i,p}$ $(1\leqq i\leqq 63)$, $(0\leqq p\leqq 7)$ are positive and bounded by $3.6\cdots$. Our result I_{2^n-1} in eq.(13) is identical with that of the Filippi's [3].

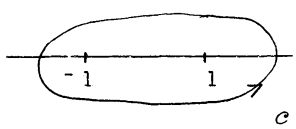
2-4) Truncation error

The truncation error $E_{8\ell+7}$ of the integral I in eq.(12) is written by

$$
E_{8\ell+7} = \int_{-1}^{1} R_\ell(x)\,dx = \int_{-1}^{1} U_7(x)\,\omega_\ell(T_8(x))\,2^{-(8\ell+7)} f[x,x_1,\cdots,x_{8\ell+7}], \qquad (17)
$$

Let $f(z)$ be analytic in and on contour C which encloses the segment [-1,1] on real axis. We make the Chebyshev expansion of the divided difference in eq.(17) which is also expressed in the contour integral,

$$
\begin{aligned}
2^{-(8\ell+7)} f\left[x,x_1,\cdots,x_{8\ell+7}\right] &= \frac{1}{2\pi i}\oint_C \frac{f(z)\,dz}{(z-x)\,U_7(z)\,\omega_\ell(T_8(z))} \\
&= \sum_{p=0}^{\infty}{}' \tilde{A}_{\ell,p} T_p(x) \quad ,
\end{aligned}
$$



where the coefficients $\tilde{A}_{\ell,p}$ are given by

$$
\tilde{A}_{\ell,p} = \frac{1}{2\pi i}\,\frac{2}{\pi}\oint_C \frac{\tilde{U}_p(z)\,f(z)\,dz}{U_7(z)\,\omega_\ell(T_8(z))} \quad . \qquad (19)
$$

The 2nd kind Chebyshev function $\widetilde{U}_p(z)$ in eq.(19) is defined by

$$\widetilde{U}_p(z) = \int_{-1}^{1} \frac{T_p(x)\,dx}{(z-x)\sqrt{1-x^2}} = \pi\, \omega^{-p}/\sqrt{z^2-1}\,, \quad \omega = z + \sqrt{z^2-1}\,, \quad (20)$$

From eqs.(17) and (18), we have

$$E_{8\ell+7} = \sum_{p=0}^{\infty}{}' \widetilde{A}_{\ell,p} \; W_{\ell,p} \quad , \quad (p=odd) \qquad (21)$$

where $W_{\ell,p}$ is given by eq.(14)

Suppose that $f(z)$ is a meromorphic function which has M simple poles at the points z_m $(m=1,2,\cdots, M)$ with residues $Res\ f(z_m)$. From the contour integral in eq.(19), we have

$$\widetilde{A}_{\ell,p} = -\frac{2}{\pi} \sum_{m=1}^{M} \frac{\widetilde{U}_p(z_m)\cdot Res\ f(z_m)}{U_7(z_m)\,\omega_\ell(T_8(z_m))} \quad .$$

Since formulas (3) and (20) give asymptotically

$$|\,\widetilde{A}_{\ell,p}\,| \propto r_m^{-(8\ell+7+p)}\,, \quad (r_m>1) \qquad r_m \equiv |\,z_m + \sqrt{z_m^2-1}\,|\,,$$

the following inequality holds

$$|\,\widetilde{A}_{\ell,1}\,| > |\,\widetilde{A}_{\ell,p}\,| \quad , \quad (p\geqq 3) \quad , \qquad (22)$$

unless the points z_m are not so close to the range [-1,1]. From eqs.(21) and (22), we have the estimate of the truncation error

$$|E_{8\ell+7}| \sim |\widetilde{A}_{\ell,1}|\cdot|W_{\ell,1}| \leqq (\,|A_{\ell-1,7}| + |A_{\ell-1,5}|\,)\,|W_{\ell,1}| \equiv e_{8\ell+7}\,, \qquad (23)$$

where we have used $|A_{\ell-1,7}|$ and $|A_{\ell-1,5}|$ instead of $|A_{\ell,1}|$ which cannot be evaluated actually.

If some points z_m are very close to [-1,1] or the p-th derivative $f^{(p)}(x), (p\geqq 1)$ is discontinuous on [-1,1], the error estimate $e_{8\ell+7}$ in eq.(23) does not holds. To guard against this failure we take the following check procefure. If $|A_{i,p}|$

decreases quickly, $|I_{2^n-1} - I_{2^{n-1}-1}|$ should be a good error estimate for $I_{2^{n-1}-1}$. Therefore, if the following inequality holds

$$e_{2^{n-1}-1} \geqq |I_{2^n-1} - I_{2^{n-1}-1}| \quad , \qquad (24)$$

we accept $e_{8\ell+7}$ in eq.(23) for $2^n-1 \leqq 8\ell+7 \leqq 2^{n+1}-1$, otherwise we take error estimation

$$e_{8\ell+7} \cdot |I_{2^n-1} - I_{2^{n-1}-1}|/e_{2^{n-1}-1} \quad , \qquad (25)$$

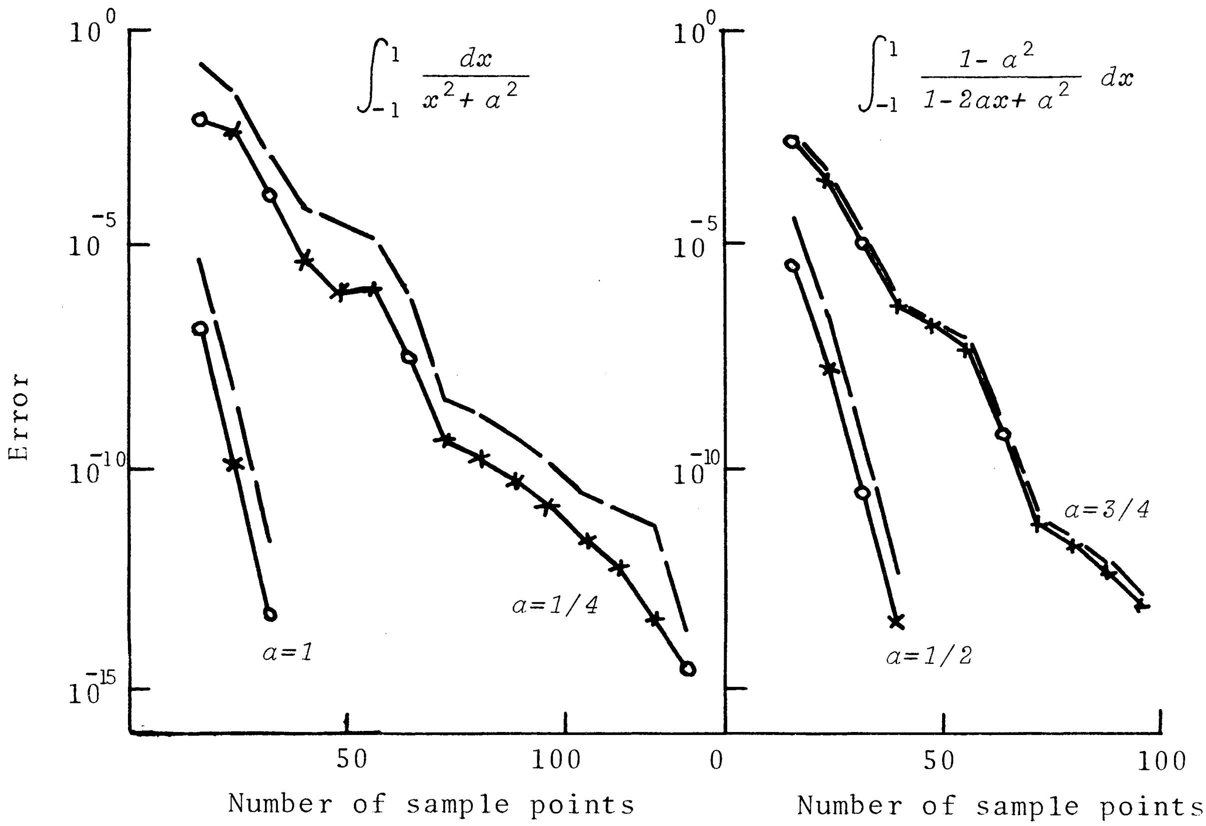


Fig. 1. Errors of integration

Solid curves are absolute errors and broken curves are estimated errors by eqs.(23) or (25). Circles mean 2^n-1 sample points.

In Fig.1, the absolute errors and the estimated errors based on eqs.(23) or (25) are shown for the 2 integrals. Fig.1 indicates that the estimated errors are very close to the absolute true errors.

2-5) Stability of the integration rule

The approximation $I_{8\ell+7}$ in eq.(13) can be rewritten as

$$I_{8\ell+7} = \sum_{j=1}^{8\ell+7} \mu_j^{\ell} f(x_j)$$

where the weights μ_j^{ℓ} are given by

$$\mu_j^{\ell} = \int_{-1}^{1} \frac{U_7(x) W_\ell(T_8(x))\,dx}{(x-x_j)\,[U_7(x) W_\ell(T_8(x))]'_{x=x_j}} \quad , \qquad (26)$$

and $[\ \cdots\cdot]'$ means the derivative with respect to x. It is difficult to evaluate theoretically μ_j^{ℓ} in eq.(26). Usually, the number of abscissae used to carry out the integration in the double precision arithmetic may be at most $511(=8\cdot 63+7)$. The results of the numerical experiments on μ_j^{ℓ} show that the norm $\|I_N\| = \sum_{j=1}^{8\ell+7} |\mu_j^{\ell}|$ satisfies the following

$$\|I_N\| = \sum_{j=1}^{8\ell+7} |\mu_j^{\ell}| \begin{cases} \leqq 3.0 & , (0 \leqq \ell \leqq 63,\ \ell \neq 30, 62) \\ = 4.45\cdots\cdot, & (\ell=30) \\ = 7.31\cdots\cdot, & (\ell=62) \end{cases} \qquad (27)$$

, which guarantees in effect the stability of the integration rule.

3. Multiple Integration

We consider the three-dimensional integral

$$I = \int_{\phi_1}^{\psi_1} dx_1 \int_{\phi_2}^{\psi_2} dx_2 \int_{\phi_3}^{\psi_3} dx_3 f(x_1, x_2, x_3) \quad . \qquad (28)$$

Table 2

The norm of the interpolatory numerical integration

Underlines mean (2^n-1) sample points used by Filippi [3].

Number of sample points	Norm	Number of sample points	Norm	Number of sample points	Norm	Number of sample points	Norm
7	2.0	135	2.0	263	2.0	391	2.003
15	2.0	143	2.0	271	2.0	399	2.006
23	2.0	151	2.004	279	2.001	407	2.005
31	2.0	159	2.0	287	2.0	415	2.01
39	2.0	167	2.005	295	2.001	423	2.007
47	2.0	175	2.01	303	2.002	431	2.01
55	2.28	183	2.09	311	2.01	439	2.05
63	2.0	191	2.0	319	2.0	447	2.03
71	2.0	199	2.01	327	2.001	455	2.03
79	2.0	207	2.02	335	2.002	463	2.07
87	2.02	215	2.03	343	2.004	471	2.04
95	2.0	223	2.07	351	2.006	479	2.26
103	2.05	231	2.13	359	2.01	487	2.29
111	2.14	239	2.51	367	2.04	495	3.24
119	3.0	247	4.45	375	2.20	503	7.31
127	2.0	255	2.0	383	2.0	511	2.0

By making the change of variables

$$x_1 = (\psi_1-\phi_1)x_1'/2 + (\psi_1+\phi_1)/2 = \alpha_1 x_1' + \beta_1 ,$$
$$x_2 = \alpha_2(x_1')x_2' + \beta_2(x_1') ,$$
$$x_3 = \alpha_3(x_1',x_2')x_3' + \beta_3(x_1',x_2') ,$$

integral I in eq.(28) can be transformed into the integral over cube $[-1,1]^3$,

$$\begin{aligned} I &= \alpha_1 \int_{-1}^{1} dx_1'\, h(x_1') , \\ h(x_1') &= \alpha_2(x_1') \int_{-1}^{1} dx_2'\, g(x_1', x_2') , \\ g(x_1',x_2') &= \alpha_3(x_1', x_2') \int_{-1}^{1} dx_3'\, f(\alpha_1 x_1' + \beta_1,\ \alpha_2(x_1')x_2' + \beta_2(x_1'), \\ &\qquad \alpha_3(x_1', x_2')x_3' + \beta_3(x_1', x_2')) . \end{aligned} \tag{29}$$

Applying the one-dimensional integration scheme in chap.2 to each integral in eq.(29), we have

$$g(x_1', x_2') = \alpha_3(x_1',x_2')[G_{8r+7}(x_1',x_2') + E_{8r+7}^{(3)}(x_1',x_2')] , \tag{30}$$

where $G_{8r+7}(x_1',x_2')$ is an approximation to the integral $\int_{-1}^{1} f(\cdots)dx_3'$ and $E_{8r+7}^{(3)}(x_1',x_2')$ is the truncation error.

From eqs.(29) and (30), we have

$$\begin{aligned} h(x_1') &= \alpha_2(x_1') \int_{-1}^{1} dx_2' \alpha_3(x_1',x_2') G_{8r+7}(x_1',x_2') \\ &+ \alpha_2(x_1') \int_{-1}^{1} dx_2' \alpha_3(x_1',x_2') E_{8r+7}^{(3)}(x_1',x_2') . \end{aligned} \tag{31}$$

Application of the integration scheme to the first term in the right hand side of eq.(31) yields the equation,

$$\int_{-1}^{1} dx_2' \alpha_3(x_1',x_2') G_{8r+7}(x_1',x_2') = H_{8s+7}(x_1') + E_{8s+7}^{(2)}(x_1') , \tag{32}$$

where $H_{8s+7}(x_1')$ is the approximation and $E_{8s+7}^{(2)}(x_1')$ is the truncation error. Furthermore, we have

$$\int_{-1}^{1}\alpha_2(x_1')H_{8s+7}(x_1')dx_1' = I_{8t+7} + E^{(1)}_{8t+7} \quad , \qquad (33)$$

where $\alpha_1 \cdot I_{8t+7}$ is just the approximation to the integral I in eq.(28). From eqs.(29)~(33), it follows that

$$I = \alpha_1 I_{8t+7} + \alpha_1 E^{(1)}_{8t+7} + \alpha_1\int_{-1}^{1}dx_1'\alpha_2(x_1')E^{(2)}_{8s+7}(x_1')$$
$$+ \alpha_1\int_{-1}^{1}dx_1'\alpha_2(x_1')\int_{-1}^{1}dx_2'\alpha_3(x_1',x_2')E^{(3)}_{8r+7}(x_1',x_2'). \qquad (34)$$

The 2nd to 4th terms in the right hand side of eq.(34) indicate the total truncation error E of the three dimensional integral.

Next, we investigate each truncation errors $E^{(i)}$ in eq.(34). Although there are many ways to determine $E^{(i)}$ in order to satisfy,

$$|E - \alpha_1 I_{8t+7}| \leqq \varepsilon_a \quad \text{(absolute tolerance)},$$

we equally distribute ε_a to integral of each axis

$$|\alpha_1 E^{(1)}_{8t+7}| \leqq \varepsilon_a/3 \quad ,$$
$$|\alpha_1\int_{-1}^{1}dx_1'\alpha_2(x_1')E^{(2)}_{8s+7}(x_1')| \leqq \varepsilon_a/3 \quad , \qquad (35)$$
$$|\alpha_1\int_{-1}^{1}dx_1'\alpha_2(x_1')\int_{-1}^{1}dx_2'\alpha_3(x_1',x_2')E^{3}_{8r+7}(x_1',x_2')| \leqq \varepsilon_a/3.$$

In order to satisfy (35), it is sufficient that the following inequalities hold,

$$|E^{(1)}_{8t+7}| \leqq \varepsilon_a/(3\alpha_1) \quad ,$$
$$|E^{(2)}_{8s+7}| \leqq \varepsilon_a/(6\alpha_1\alpha_2(x_1')) \quad , \qquad (36)$$
$$|E^{(3)}_{8r+7}| \leqq \varepsilon_a/(12\alpha_1\alpha_2(x_1')\alpha_3(x_1',x_2')) \quad .$$

It is obvious in (36) that the accuracy of integral along each axis of integration are dependent on the range of integration.

4. Numerical Examples

In Tables 3 and 4, our results for the following integrals are compared with the results by other methods.

$$A \; : \; \iiint_D \prod_{i=1}^{3} \frac{a}{x_i^2 + a^2} \, dx_1 dx_2 dx_3 \quad , \quad a = 1, \frac{1}{2}, \frac{1}{4} ,$$

$$B \; : \; \iiint_D \prod_{i=1}^{3} \frac{1 - a^2}{1 - 2ax_i + a^2} \, dx_1 dx_2 dx_3 \quad , \quad a = \frac{1}{4}, \frac{1}{2}, \frac{3}{4} ,$$

$$C \; : \; \iiint_D \prod_{i=1}^{3} a\cos ax_i \; dx_1 dx_2 dx_3 \quad , \quad a = 8, 16, 32 .$$

$$D = [-1, 1]^3$$

Problems A, B and C are integrals of the peaked type, the one with poles on real axis outside the range [-1,1] and the oscillatory type, respectively. In Tables 3 and 4, 'mod. $C.C.(\sqrt{2})$' means the results of the application to three-dimensional integral of the modified Clenshaw-Curtis method [6] with increasing sample points like $5, 7, 9, 13, 17, \cdots, 2^n+1, 1.5\times 2^n+1, \cdots,$ and 'adaptive Newton-Cotes' means the product rule based on the adaptive Newton-Cotes 9 points rule by Ninomiya [7]. 'NDIMRI' is the subroutine program to compute an approximation to the N dimensional integral $(N \leqq 9)$ over a parallelepiped using a Romberg type method based on a midpoint rule (see [1], p387). Tables 3 and 4 show that our method is more effective than other three methods for the test problems A, B and C. Not only our method has the same feature as Clenshaw-Curtis method that is effective for the well-behaved functions, but also the increase of sample points with arithmetic progression is more gradual than that of their method.

Table 3

Number of sample points $(\times 10^{3})$

$\varepsilon_a = 10^{-4}$, Double Precision FACOM-M200

	a	Present Method	Mod. C.C. with $\sqrt{2}$	Adaptive Newton-Cotes	NDIMRI
A	1	$3. \times 10^{3}$	5.	9.	8.
	1/2	29.	25.	40.	81.
	1/4	148.	179.	176.	256.
B	1/4	3.	2.	9.	3.
	1/2	12.	13.	28.	81.
	3/4	35.	101.	143.	(849.)
C	8	14.	16.	87.	256.
	16	46.	69.	(1000.)	849.
	32	216.	272.	(1000.)	(849.)

Table 4

Number of sample points $(\times 10^{3})$

$\varepsilon_a = 10^{-7}$, Double Precision FACOM-M200

	a	Present Method	Mod. C.C. with $\sqrt{2}$	Adaptive Newton-Cotes	NDIMRI
A	1	$12. \times 10^{3}$	16.	9.	81.
	1/2	59.	118.	(69.)	256.
	1/4	351.	862.	(335.)	(850.)
B	1/4	11.	5.	17.	26.
	1/2	30.	35.	110.	850.
	3/4	224.	497.	(670.)	(850.)
C	8	30.	33.	(800.)	850.
	16	65.	118.	(1000.)	(850.)
	32	272.	275.	(1000.)	(850.)

( ) means failure

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