数理解析研究所講究録
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Notes on effective usage of double exponential formulas for numerical integration.

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1. Introduction

We shall describe the results of numerical evaluation of an integral

(1-1) $$I = \int_a^b f(x)\,dx$$

using the double exponential formulas (DE formulas), and empirical evidences derived from these results.

The DE formulas have been developed as a class of numerical quadrature formulas by Takahashi and Mori [1], and they offer one of the most powerful methods to evaluate an improper integral on $[a,b]$, in which the integrand is singular at one endpoint or both. Moreover, we shall show a method to evaluate difficult integrals, where the integrand is highly oscillating.

2. Comparison of results by some numerical integration formulas.

As a numerical comparison of formulas among DE, IMT, CADRE and QUAD we show in Table 2-1 the number of integrand evaluations and the relative error estimated from the following convergence criterion (2-1) by choosing 15 test cases of Patterson [3] and Ichida and Kiyono [4].

A convergence criterion of our program is satisfied when

(2-1) $$|S_{i+1} - S_i| \leq \varepsilon\,|S_{i+1}|$$

where S_i is a value of DE formula with mesh size h_i, and S_{i+1} is a value of DE formula with mesh size $h_{i+1} = h_i/2$, and ε is some preassigned tolerance (e.g., 10^{-9}).

2.1 DE formula and IMT formula

Table 2.1-1

DE	IMT
$I = \int_a^b f(x)\,dx$	$I = \int_0^1 f(x)\,dx$
$x = \phi(u)^*$	$x = \psi(u) = \frac{1}{Q}\int_0^u \exp(-\frac{1}{t} - \frac{1}{1-t})\,dt$ $Q = \int_0^1 \exp(-\frac{1}{t} - \frac{1}{t-1})\,dt$
$I = \int_{-\infty}^{\infty} f(\phi(u))\,\phi'(u)\,du$ $\approx I_h = h\sum_{n=-\infty}^{\infty} f(\phi(nh))\,\phi'(nh)$	$I = \int_0^1 f(\psi(u))\,\psi'(u)\,du$ $\approx S_n = \frac{1}{N}\sum_{n=1}^{N-1} f(\psi(\frac{n}{N}))\,\psi'(\frac{n}{N})$
based on the fact that the trapezoidal rule gives the best result for $\int_{-\infty}^{\infty} g(u)du$ with mesh size h .	based on Euler-Maclaurin summation formula.

* where

(2.1-1) $x = \phi(u) = \tanh(\frac{\pi}{2}\sinh u)$ if $I = \int_{-1}^{1} f(x)\,dx$

(2.1-2) $x = \phi(u) = \exp(\frac{\pi}{2}\sinh u)$ if $I = \int_0^{\infty} f(x)\,dx$

(2.1-3) $x = \phi(u) = \exp(u - \exp(-u))$ if $I = \int_0^{\infty} \exp(-x)\,f(x)\,dx$

(2.1-4) $x = \phi(u) = \sinh(\frac{\pi}{2}\sinh u)$ if $I = \int_{-\infty}^{\infty} f(x)\,dx$

Table 2-1 Comparison of results by DE, IMT, CADRE and QUAD

	D E		IMT		CADRE		QUAD	
P_1 $\int_0^1 \sqrt{x}\, dx = \frac{2}{3}$	44	$3.3_{10}{-12}$	127	$4.9_{10}{-13}$	129	$1.2_{10}{-10}^*$	361	$2.2_{10}{-10}^*$
P_2 $\int_{-1}^1 (0.92 \cosh x - \cos x)\, dx = 1.84 \sinh 1 - 2 \sin 1$	96	$1.7_{10}{-12}$	127	$2.4_{10}{-12}$	33	$1.3_{10}{-13}^*$	37	$2.1_{10}{-14}^*$
P_3 $\int_{-1}^1 \frac{dx}{x^4 + x^2 + 0.9} \doteqdot 1.5822\ 32964$	92	$3.1_{10}{-11}$	127	$2.8_{10}{-13}$	129	$4.2_{10}{-12}^*$	73	$1.5_{10}{-12}^*$
P_4 $\int_0^1 x^{\frac{3}{2}}\, dx = \frac{2}{5}$	40	$1.0_{10}{-11}$	127	$8.1_{10}{-13}$	529	$2.7_{10}{-10}^*$	163	$5.0_{10}{-12}^*$
P_5 $\int_0^1 \frac{dx}{1 + x^4} = \frac{1}{4\sqrt{2}}\left\{\log(3 + 2\sqrt{2}) + \pi\right\}$	92	$2.3_{10}{-12}$	127	$5.6_{10}{-13}$	65	$2.1_{10}{-10}^*$	73	0^*
P_6 $\int_0^1 \frac{dx}{2 + \sin 10\pi x} = \frac{2}{\sqrt{3}}$	724	$1.2_{10}{-10}$	509	$2.2_{10}{-12}$	785	$8.4_{10}{-12}^*$	757	$3.3_{10}{-13}^*$

P_7	$\int_0^1 \frac{x}{e^x - 1}\, dx \doteqdot 0.77750\ 46341$	48	$3.4_{10}-12$	127	$6.6_{10}-13$	17	$2.2_{10}-12^*$	37	$2.1_{10}-14^*$
P_8	$\int_{0.1}^1 \frac{\sin 100\pi x}{\pi x}\, dx$	620	$2.5_{10}-13$	509	$4.1_{10}-16$	3505	$3.2_{10}-12^*$	2773	$5.6_{10}-13^*$
P_9	$\int_0^{10} \frac{50}{\pi(2500 x^2 + 1)}\, dx = \frac{1}{\pi}\tan^{-1}500$	180	$2.2_{10}-10$	255	$1.5_{10}-11$	337	$1.3_{10}-11^*$	343	$3.3_{10}-12^*$
P_{10}	$\int_0^\pi \cos(\cos x + 3\sin x + 2\cos 2x + 3\sin 2x + 3\cos 3x)\, dx$	186	$8.6_{10}-12$	127	$1.2_{10}-12$	417	$1.5\ -12^*$	343	$1.0_{10}-12^*$
P_{11}	$\int_0^1 \log x\, dx = -1$	44	$3.9_{10}-13$	127	$2.0_{10}-11$	369	$4.5\ -9^*$	415	$5.7_{10}-7^*$

The results with superscript * are due to Ninomiya's paper.

$K_1 \quad \int_{-1}^{1} \frac{2^{-\alpha}}{4^{-\alpha} + x^2}\, dx$	$\alpha = 1$	202 $\quad 2.4_{10}-13$	127 $\quad 2.3_{10}-13$	49 $\quad 1.2_{10}-7^{**}$	63 $\quad 1.6_{10}-8^{**}$
	K_1'	103 $\quad 1.7_{10}-14$	127 $\quad 7.0_{10}-13$		
$= 2 \tan^{-1} 2^{\alpha}$	$\alpha = 8$	24450 $\quad 2.5_{10}-12$	....	1440 $\quad 3.4_{10}-8^{**}$	567 $\quad 3.0_{10}-8^{**}$
$\alpha = 1, 2, \ldots, 31$	K_1'	204 $\quad 2.0_{10}-12$	255 $\quad 2.3_{10}-13$		
$\left(K_1' \quad 2\int_{0}^{1} \frac{2^{-\alpha}}{4^{-\alpha} + x^2}\, dx \right)$	$\alpha = 31$	....	....	....	....
	K_1'	786 $\quad 5.9_{10}-12$	509 $\quad 5.4_{10}-3$		
$K_2 \quad \int_{0}^{1} x^{\alpha}\, dx = \frac{1}{\alpha + 1}$	$\alpha = -.1$	53 $\quad 2.1_{10}-14$	127 $\quad 4.5_{10}-12$	2452 $\quad 1.0_{10}-7^{**}$	357 $\quad 1.4_{10}-6^{**}$
	K_1'	89 $\quad 7.3_{10}-10$			
$\alpha = -.1, -.2, \ldots, -.9, -.99$	$\alpha = -.8$	64 $\quad 7.3_{10}-16$	509 $\quad 5.3_{10}-10$	3043 $\quad 2.1_{10}-6^{**}$	3297 $\quad 4.7_{10}-6^{**}$
	K_2'	185 $\quad 1.4_{10}-13$			
$\left(K_2' \quad \int_{0}^{\infty} e^{-(1+\alpha)x}\, dx \right)$	$\alpha = -.9$	2013 $\quad 6.2_{10}-10$	509 $\quad 1.7_{10}-7$	3532 $\quad 9.5_{10}-2^{**}$	6699 $\quad 1.2_{10}-4^{**}$
	K_2'	189 $\quad 4.8_{10}-12$			
	$\alpha = -.99$	....			
	K_2'	394 $\quad 2.1_{10}-16$			
$H_1 \quad \int_{0}^{1} \log \lvert \log x \rvert\, dx = -\gamma$		48 $\quad 2.2_{10}-11$	121 $\quad 3.1_{10}-11$		

The results with superscript ** are due to Ichida and Kiyono [4] , $\varepsilon = 10^{-7}$

P6

$$f(x) = \frac{2}{2 + \sin 10\pi x}$$

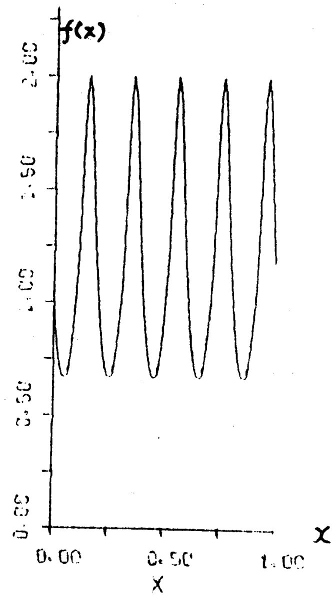


Figure 2-1

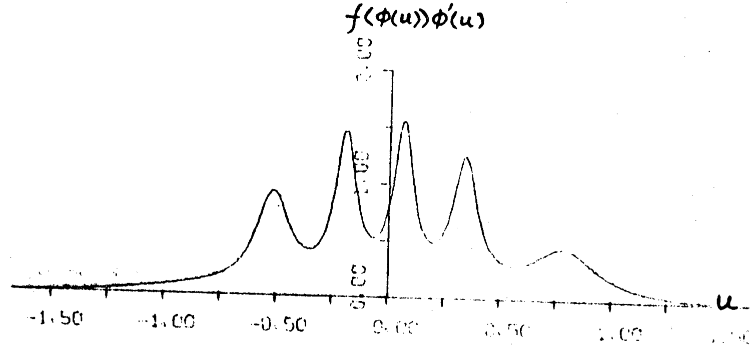


Figure 2-2

P7

$$f(x) = \frac{x}{e^x - 1}$$

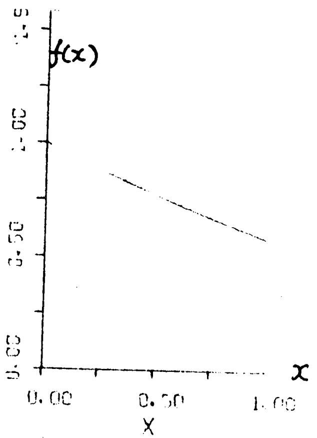


Figure 2-3

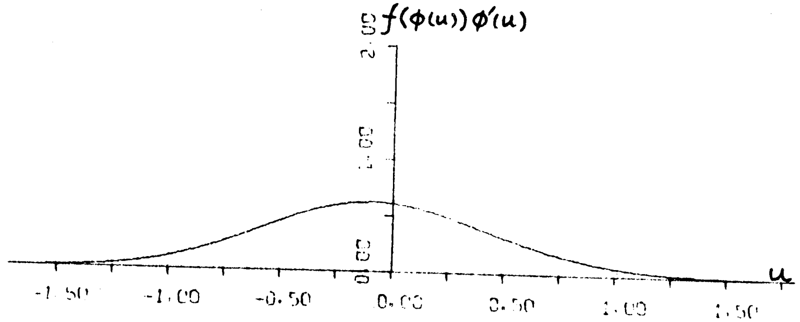


Figure 2-4

P 8

$$f(x)=\frac{\sin 100\pi x}{\pi x}$$

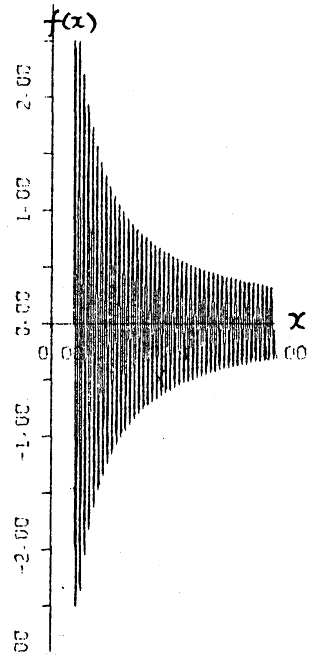


Figure 2-5

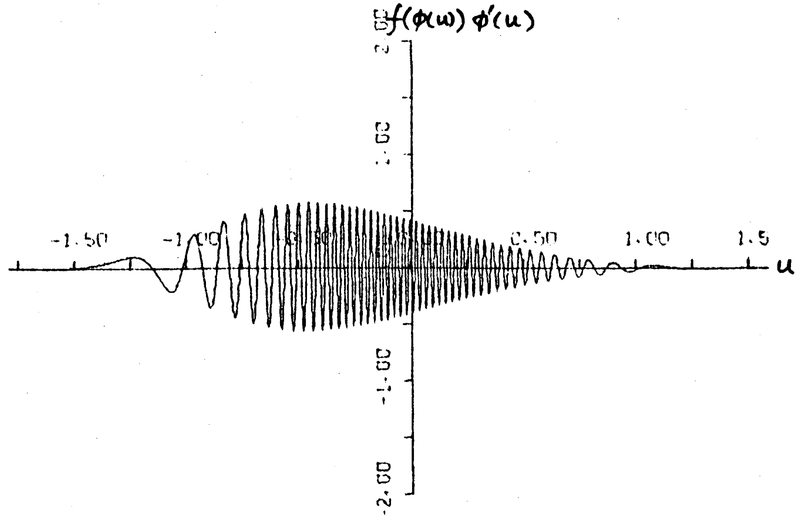


Figure 2-6

P 10

$$f(x)=\cos(\cos x+3\sin x$$
$$+2\cos 2x+3\sin 2x+3\cos 3x)$$

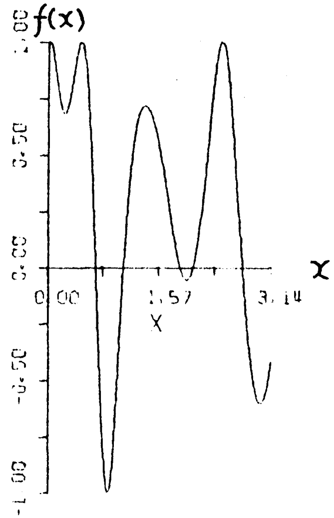


Figure 2-7

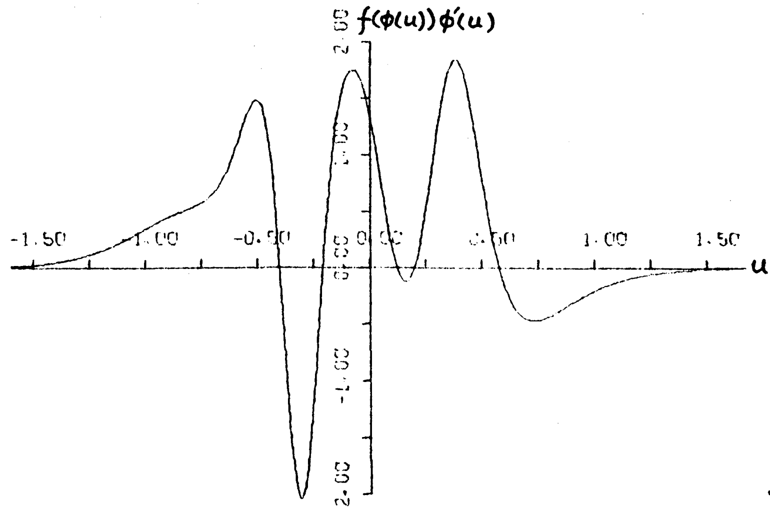


Figure 2-8

K 1

$$f(x) = \frac{2^{-\alpha}}{4^{-\alpha} + x^2}$$

$\alpha = 2, 4, 6, 8, 10$

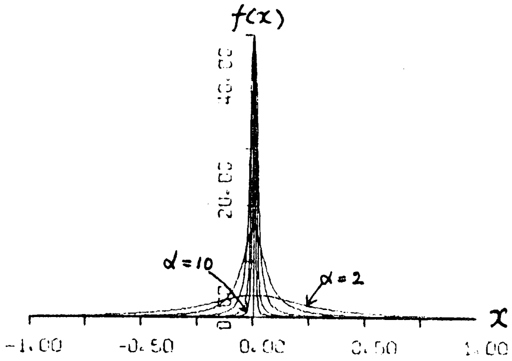


Figure 2-9

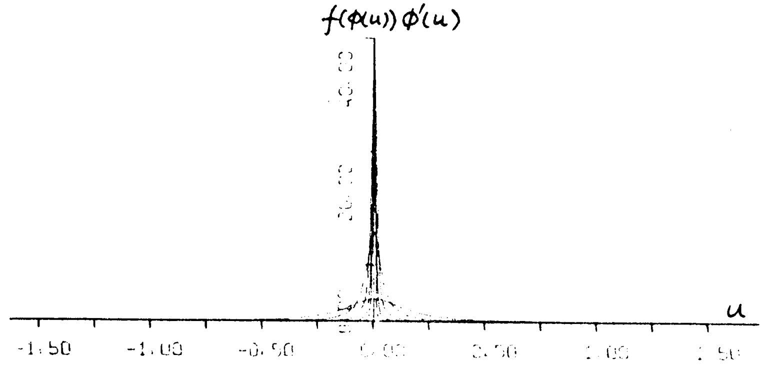


Figure 2-10-1

K 1'

$$\int_{-1}^{1} f(x)\,dx = 2\int_{0}^{1} f(x)\,dx = \int_{-1}^{1} f\left(\frac{x+1}{2}\right) dx$$

$$= \int_{-1}^{1} g(x)\,dx$$

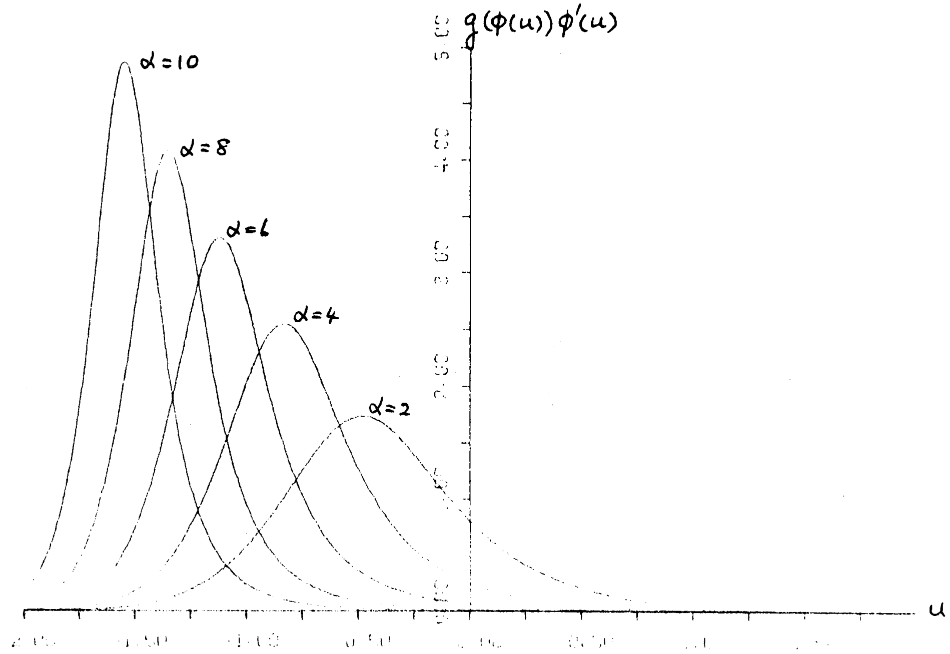


Figure 2-10-2

K 2

$f(x) = x^{\alpha}$, $\alpha = -0.1$, -0.9

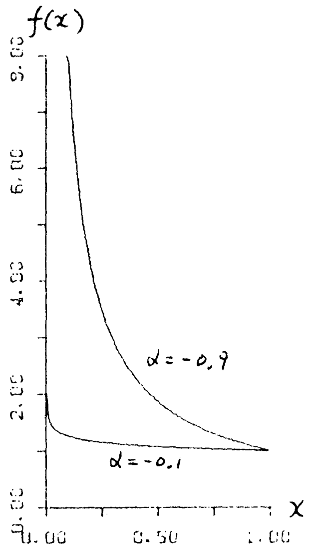


Figure 2-11

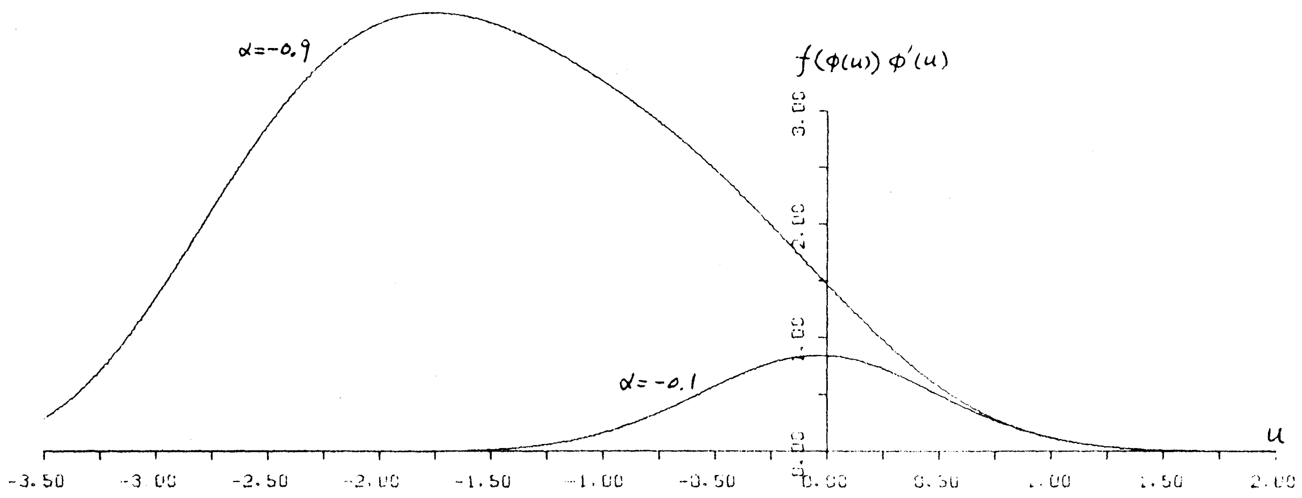


Figure 2-12

H 1

$f(x) = \log|\log x|$

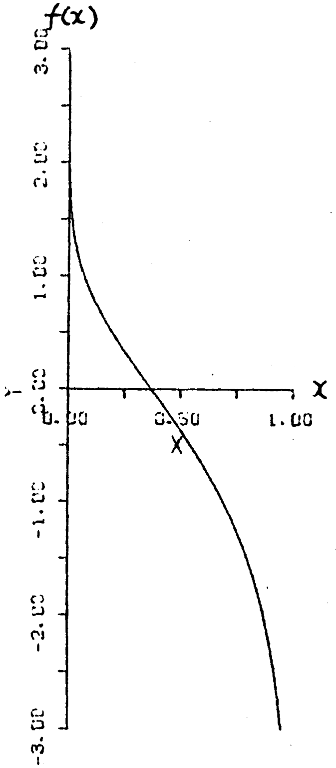


Figure 2-13

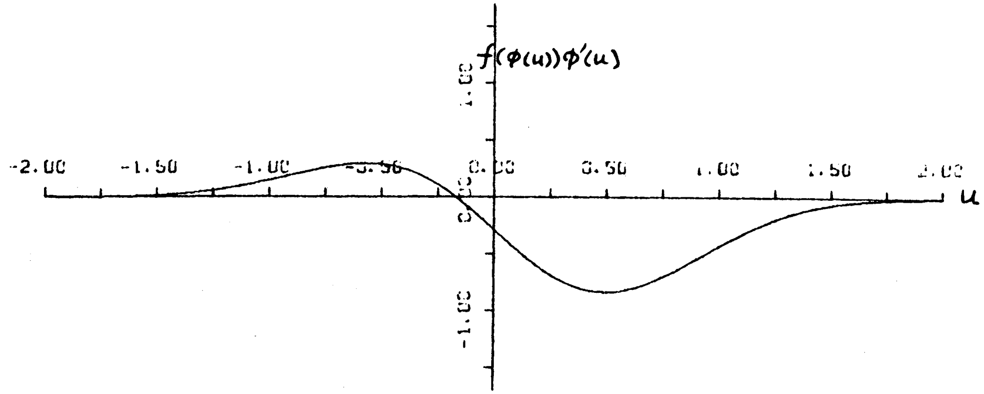


Figure 2-14-1

IMT quadrature

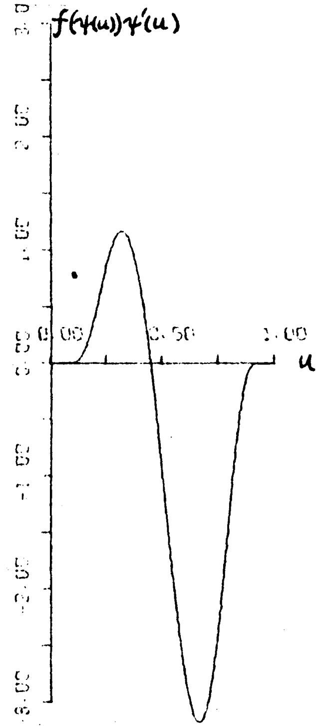


Figure 2-14-2

H1′

$$\int_0^1 \log|\log x|\,dx = \int_0^\infty e^{-x}\log x\,dx = \int_0^\infty f(x)\,dx$$

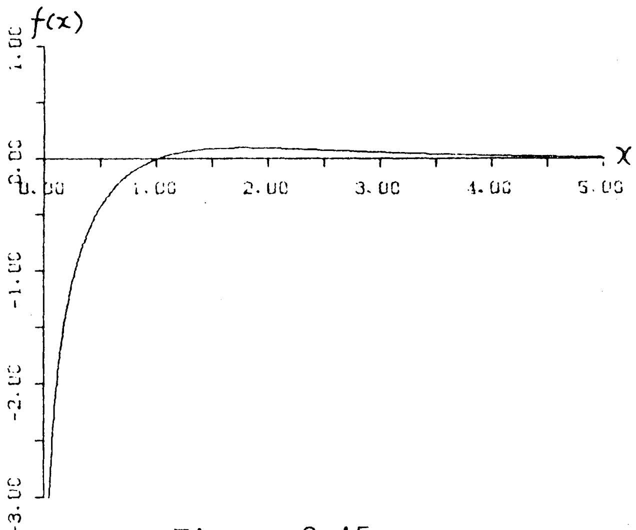


Figure 2-15

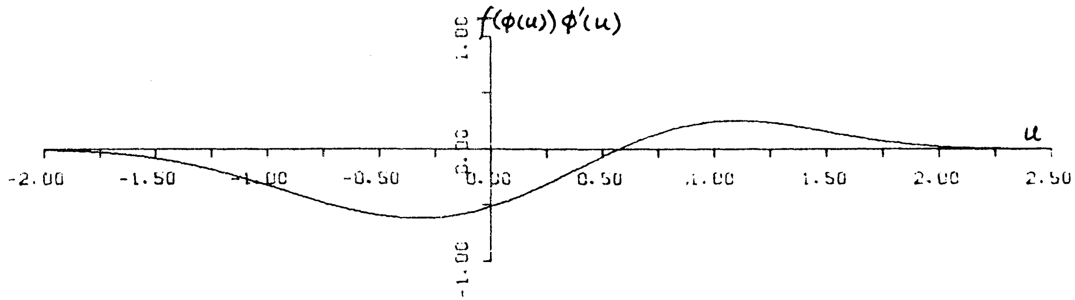


Figure 2-16

2.2 Choosing a quadrature formula

Table 2-1 supports the following statements:

i) For integrals over finite intervals, if the integrand has no singularity through the interval, CADRE or QUAD gets an advantage over DE (P2, P5, P7).

ii) For improper ingegrals over finite intervals with singularities at the endpoints of the interval, DE gets the best advantage and IMT gets the second best (P1, P4, P11).

iii) For integrals over infinite and semi-infinite intervals, if the integrand is not regular at infinity, DE is better than all the other formulas (e.g., $\int_0^\infty \exp(-x)\cdot\log x\, dx$). However, if the integrand is regular at infinity, it can be reduced to the case i) by suitable change of variable (e.g., $\int_0^\infty \frac{1}{1+x^4}dx = \int_0^{\frac{\pi}{2}} \frac{\cos^2\theta}{\cos^4\theta + \sin^4\theta}d\theta$). But even if DE may be used to evaluate the original integral, it will give slightly worse results than CADRE or QUAD.

iv) For integrals with singularity close to the real axis in the interior of the interval (e.g., K1), it is hard to evaluate the integral with desired accuracy using any of all the methods presented here, and DE is the most unsuccessful among them. But by splitting the integral into two integrals at the real part of the singular point, it can be reduced to case ii) .

3. Notes on usage of DE formula on digital computer.

Some notes must be necessary to effective use of DE formula.

3.1 A numerical technique by Takahashi and Mori to avoid subtractive cancellation by the factor $(1+x)^{\alpha}(1-x)^{\beta}$, $\alpha, \beta < 0$.

In the case of integral $\int_{-1}^{1} f(x)\, dx$, $f(x)$ having a factor $(1+x)^{\alpha}(1-x)^{\beta}$, $\alpha,\beta<0$ (e.g., $f(x) = 1/\sqrt{1-x}$, $1/\sqrt{\tan((\pi/2)\cdot(1+x))}$), the number of its significant digits decreases as $x \to \pm 1$. Thus the relative error in the $f(x)$ will be greately increased. In such a case, following Takahashi and Mori [1] p.735, we must make the change of variable to avoid the cancellation; splitting the integral into two integrals, we have

$$(3.1\text{-}1) \qquad I = \int_{-1}^{1} f(x)\, dx = \int_{-1}^{0} f(x)\, dx + \int_{0}^{1} f(x)\, dx = I_1 + I_2$$

and we make direct substitution of $1 \pm x$ as follows:

$$(3.1\text{-}2) \qquad -t = 1+x = 1 + \tanh(\tfrac{\pi}{2} \sinh u) = \frac{\exp(\frac{\pi}{2} \sinh u)}{\cosh(\frac{\pi}{2} \sinh u)} \qquad t \in [-1,0]$$

$$(3.1\text{-}3) \qquad t = 1-x = 1 - \tanh(\tfrac{\pi}{2} \sinh u) = \frac{\exp(-\frac{\pi}{2} \sinh u)}{\cosh(\frac{\pi}{2} \sinh u)} \qquad t \in [0,1]$$

Then we obtain

$$(3.1\text{-}4) \qquad I_1 = \int_{-1}^{0} f(-1-t)\, dt = \int_{-\infty}^{0} f(-1-t(u)) \cdot \frac{\frac{\pi}{2} \cosh u}{\cosh^2(\frac{\pi}{2} \sinh u)}\, du$$

$$(3.1\text{-}5) \qquad I_2 = \int_{0}^{1} f(1-t)\, dt = \int_{0}^{\infty} f(1-t(u)) \cdot \frac{\frac{\pi}{2} \cosh u}{\cosh^2(\frac{\pi}{2} \sinh u)}\, du$$

3.2 Note on mechanizing DE formula on digital computer.

The essence of the technique described above is to substitute $\mp \exp(\pm\frac{\pi}{2}\sinh u) / \cosh(\frac{\pi}{2}\sinh u)$ directly into $t = \mp 1-x$. Thus, it will be of no use to avoid cancellation to make the change of variable $x = -1-t$ for $x \in [-1,0]$ and $x = 1-t$ for $x \in [0,1]$ in library programs and to write $f(x)$ in its original form. Since the term $1+x$ will be evaluated not by $-t$ but by $1 + (-1-t)$, the loss of significance will be very large. Then we must transform $f(x)$ into the expression $g(t)$ of the independent variable $t = -1-x$ for $x \in [-1,0]$, and $t = 1-x$ for $x \in [0,1]$. For example,

$$f(x) = (1+x)^{-\frac{1}{2}}(1-x)^{-\frac{1}{3}}$$

we must make transformation as follows:

$$f(x) = g(t) = \begin{cases} (-t)^{-\frac{1}{2}}(2+t)^{-\frac{1}{3}} & t \in [-1,0] \\ (2-t)^{-\frac{1}{2}}t^{-\frac{1}{3}} & t \in [0,1] \end{cases}$$

The same situation holds for the integral $\int_a^b f(x)dx$, if $f(x)$ has a factor $(x-a)^{\alpha}(b-x)^{\beta}$, $\alpha, \beta < 0$. In this case we must transform $f(x)$ into the expression of the variable $t = -\frac{2}{b-a}(x-a)$ for $x \in [a, \frac{a+b}{2}]$ and $t = \frac{2}{b-a}(b-x)$ for $x \in [\frac{a+b}{2}, b]$. If this transformation is very tedious, we can transform the integral into an integral over $[0,\infty)$ by suitable change of variable (e.g., $x = \exp(-t)$), and then apply (2.1-2) or (2.1-3). For H1, by the change of variable $x = \exp(-t)$, we get

$$\text{H1'} \qquad I = \int_0^{\infty} \exp(-t) \log t \, dt$$

In this case, there is a problem how to choose the initial step size h_0 for H1. We shall consider this problem in the next section.

3.3 The initial step size h_0.

Our starting algorithm to find M, N such that $S_0 = \sum_{n=-M}^{N} f(\phi(nh_0))\cdot\phi'(nh_0)$, where S_0 is the first approximation of I, is as follows:

Input

ε (some preassigned tolerance)

h_0 (initial step size)

Algorithm

```
h ← h₀;  S₀ ← f(φ(0))·φ'(0);  N ← 0;  M ← 0
for n=1,2,... do
    if N=0  then  Tₙ ← f(φ(nh))·φ'(nh)
                  if |Tₙ| ≤ |S₀|*max(ε*10⁻³, 10⁻¹⁶)  then  N ← n
                                                      else  S₀ ← S₀ + Tₙ
                  endif
    endif
    if M=0  then  Uₙ ← f(φ(-nh))·φ'(-nh)
                  if |Uₙ| ≤ |S₀|*max(ε*10⁻³, 10⁻¹⁶)  then  M ← n
                                                      else  S₀ ← S₀ + Uₙ
                  endif
    endif
endfor
next:
```

Thus,

$$|T_n| \le |S_0| * \max(\varepsilon * 10^{-3}, 10^{-16}) \tag{3.3-1}$$
$$|U_n| \le |S_0| * \max(\varepsilon * 10^{-3}, 10^{-16})$$

must hold befor nh_0, $n=1,2,\ldots,$ exceeds a certain threshold value which depends on representation of number by computer.

For example, in the case of H1, we have

$$I = \int_0^1 \log \left|\log x\right| dx = \int_{-1}^1 \frac{1}{2} \log \left|\log \frac{1+x}{2}\right| dx$$

$$= \int_{-1}^0 \frac{1}{2} \log \left|\log(-\frac{t}{2})\right| dt + \int_0^1 \frac{1}{2} \log \left|\log \frac{2-t}{2}\right| dt$$

where

$$-t = \frac{\exp(\frac{\pi}{2} \sinh u)}{\cosh(\frac{\pi}{2} \sinh u)} \qquad t \in [-1,0]$$

$$t = \frac{\exp(-\frac{\pi}{2} \sinh u)}{\cosh(\frac{\pi}{2} \sinh u)} \qquad t \in [0,1]$$

Since $2-t$ and 2 are considered as the same values (additive no contribution) for $nh_0 > 3.45$ with double precision arithmetic in T-56, the second integrand $\frac{1}{2} \log \left|\log \frac{2-t}{2}\right|$ is regarded as $\log 0$ (error message "not allowed").

In the case of P7, we have

$$I = \int_0^1 \frac{x}{\exp x - 1} dx = \int_{-1}^1 \frac{1}{2} \cdot \frac{\frac{1+x}{2}}{\exp(\frac{1+x}{2}) - 1} dx$$

$$= \int_{-1}^0 \frac{1}{2} \cdot \frac{-\frac{t}{2}}{\exp(-\frac{t}{2}) - 1} dt + \int_0^1 \frac{1}{2} \cdot \frac{\frac{2-t}{2}}{\exp(\frac{2-t}{2}) - 1} dt$$

In the denominator of the first integral $\exp(-\frac{t}{2})$ is considered to be 1 for $nh_0 \geq 3.31$.

In both cases for large step size h_0, our starting algorithm results in failure by evaluating $f(\phi(nh_0)) \cdot \phi'(nh_0)$ or $f(\phi(-nh_0)) \cdot \phi'(-nh_0)$ at a point nh_0 beyond the limit 3.45 or 3.31 respectively. In contrast, it can be expected that by taking h_0 small enough depending on the tolerance ε (3.3-1) can hold before nh_0 exceeds that threshold value. This is illustrated in Fig. 3.3-1.

```
ERROR REQUIREMENT      0.1000E-14     HO =  0.50000          P7 - 1

 FT957W ERROR AT(020531)   DIVIDE CHECK

 0.50000               7               7  0.7774990274260581
 0.25000              13              13  0.7775046341101479     0.7211E-05
 0.12500              26              26  0.7775046341122483     0.2701E-11

 S =  0.7775046341122483        ER =  0.1314E-16

 ERROR REQUIREMENT      0.1000E-14     HO =  0.25000          P7 - 2

  0.25000              13              13  0.7775046341101479
  0.12500              26              26  0.7775046341122483     0.2701E-11

  S =  0.7775046341122483        ER =  0.1358E-16


ERROR REQUIREMENT      0.1000E-11     HO =  0.50000          H1 - 1

 FT911W ERROR AT(020621)   DLOG    ARG= 0.0          ( ARG.LE.0 )          RESULT=(MIN VALUE)
 FT911W ERROR AT(020621)   DLOG    ARG= 0.0          ( ARG.LE.0 )          RESULT=(MIN VALUE)
 0.50000                   8               7 -0.2026589570990376E+56

ERROR REQUIREMENT      0.1000E-11     HO =  0.25000          H1 - 2

 0.25000              13              13 -0.5772156649014750

S =-0.5772156649015324     ER = 0.9944E-13    TRUE VALUE -0.5772156649015328608          ERROR -0.8648E-15
```

Figure 3.3-1

3.4 On the criterion of convergence.

Since the error term for DE formula with mesh size h is roughly estimated (Ref. [5] p.265)

$$(3.4\text{-}1) \qquad |\Delta I_h| \simeq \exp(-\frac{A}{h})$$

where A is a constant, we have

$$(3.4\text{-}2) \qquad |\Delta I_{\frac{h}{2}}| \simeq |\Delta I_h|^2$$

So that if we halve the mesh size, then the number of significant digits are doubled. Thus we can usually get the desired accuracy when

$$(3.4\text{-}3) \qquad |S_{i+1} - S_i| \leq \sqrt{\varepsilon}\,|S_{i+1}|$$

holds, except in the case where the integrand oscillates (e.g., P8 and P10).

In the above-mentioned case, however, on the safe side, we should like to test whether both

$$|S_{i+1} - S_i| \leq \varepsilon\,|S_{i+1}|$$

and

$$|S_{i+2} - S_{i+1}| \leq \varepsilon\,|S_{i+2}|$$

hold as the criteria of convergence. From Fig. 3.4-1, we see that (3.4-2) does not hold untill mesh size h becomes small enough.

P6

ERROR REQUIREMENT 0.1000E-14 HO = 0.50000

0.50000	7	7	1.145065201836647	
0,25000	13	13	1,109398079973005	0,3215E-01
0,12500	26	26	1,101740303515750	0,6951E-02
0,62500E-01	52	52	1,157349192484848	0,4805E-01
0,31250E-01	103	103	1,154711000023246	0,2285E-02
0,15625E-01	205	205	1,154700538516411	0,9060E-05
0,78125E-02	409	409	1,154700538379251	0,1188E-09

S = 1,154700538379251 ER = 0.7079E-17 TRUE VALUE 1,154700538379251529 ERROR -0,5033E-16

P8

ERROR REQUIREMENT 0.1000E-14 HO = 0.50000

0.50000	6	6	0,2455391953890915	
0,25000	11	11	0,2312281657296014	0,6189E-01
0,12500	22	22	0,1126117247672289	0,1053E+01
0,62500E-01	44	44	0,6130785022273157E-01	0,8368E+00
0,31250E-01	87	88	0,3891818870620057E-01	0,5753E+00
0,15625E-01	173	175	0,9098637539166843E-02	0,3277E+01

S = 0.9098637539166843E-02 ER = 0,3978E-16

P10

ERROR REQUIREMENT 0.1000E-14 HO = 0.50000

0.50000	7	7	1,873635947230349	
0,25000	14	14	0,8168465615955236	0,1294E+01
0,12500	27	27	0,8396163176771946	0,2712E-01
0,62500E-01	53	53	0,8386763427025238	0,1121E-02
0,31250E-01	105	105	0,8386763426944296	0,9651E-11

S = 0,8386763426944296 ER = 0,3249E-17

Figure 3.4-1

4. Difficult problems of integration

4.1 The restriction imposed by the number of bits for exponent.

The number of bits for exponent is usually taken to be 7 or 8 . Then, for example, the integral to get the abscissas of IMT formula

$$I = \int_0^t \exp\left(-\frac{1}{x} - \frac{1}{1-x}\right) dx$$

$$= t\int_0^\infty \exp\left(-\frac{1}{t\,\exp(-x)}\right) \exp\left(-\frac{1}{1-t\,\exp(-x)}\right) \exp(-x)dx$$

with $t = 3/512,\ 2/512,\ 1/512$, has the value far smaller than 10^{-58}. Thus, if we wish to evaluate this integral using DE formula (2.1-3), our algorithm will fail, since (3.3-1) is not satisfied before $nh=5.5$, with this nh $x = \exp(nh-\exp(-nh))$ becomes 243.7 and $\exp(-x)$ vanishes.

Such a problem requires the use of multiple-precision arithmetic [6] .

5. Conclusion

We conclude from the experiment described above that DE formulas offer a very powerful method for numerical integration, except for the special cases described in the previous section. It is recommended to use DE formulas especially for integrals over infinite or semi-infinite intervals.

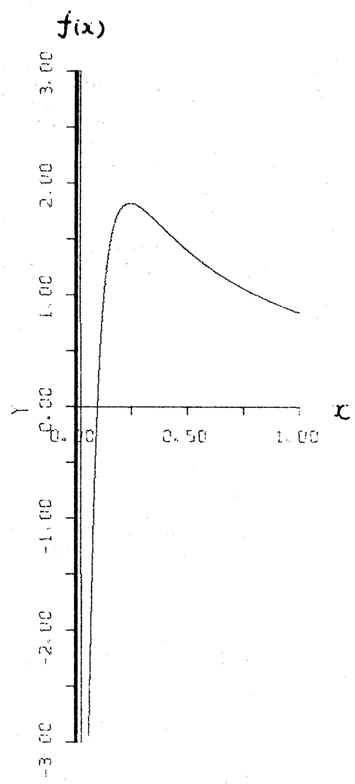
Appendix.

Dr. Rabinowitz presented the following problem when he visited ETL at Tsukuba on June 4,1980.

$$I = \int_0^1 f(x)dx = \int_0^1 \frac{\sin \frac{1}{\sqrt{x}}}{\sqrt{x}} dx \doteqdot 1.00813\ 41238\ 139$$

at x=0 value=0

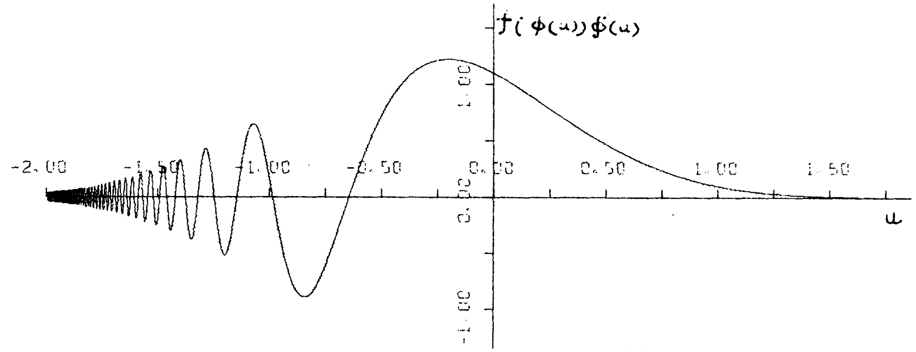
We will show the results obtained by DE formulas below.



```
LIST 10@@0170-0270

       DOUBLE PRECISION FUNCTION FM1P11(X)
       DOUBLE PRECISION X, T, G
       IF(X) 1,2,2
 1      T = -0.5D0*X
       GO TO 3
 2      T = 1.0D0 - 0.5*X
 3       CONTINUE
       G = DSIN(1.0D0/DSQRT(T)) /DSQRT(T)
       FM1P11 = G / 2.0D0
       RETURN
       END
```

$$I = \int_{-1}^{0} \frac{\frac{1}{2}\sin\frac{1}{\sqrt{-t/2}}}{\sqrt{-t/2}}\,dt + \int_{0}^{1} \frac{\frac{1}{2}\sin\left(\frac{1}{\sqrt{(2-t)/2}}\right)}{\sqrt{(2-t)/2}}\,dt$$

42

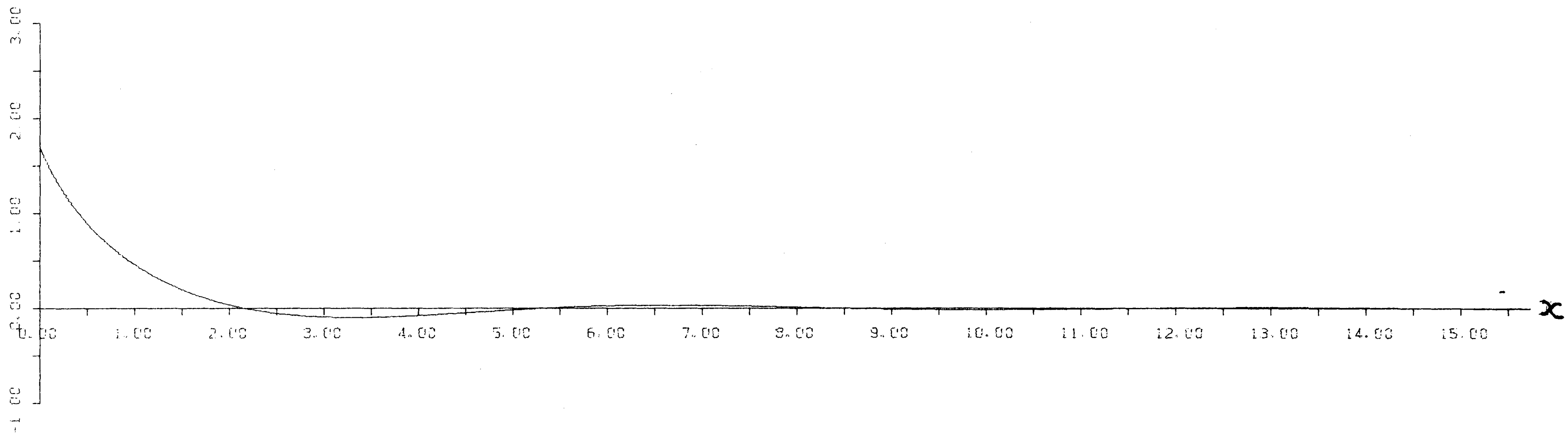



```
*RUN

PLEASE TYPE IN H0, EPS, LIST
=                  0.125D0                 1.0D-9   1
 0.12500             23         28  0.9963033435784173
 0.62500E-01         46         56  0.9812535060125632   0.1534E-01
 0.31250E-01         91        112  1.008041952064974    0.2657E-01
 0.15625E-01        181        224  1.003388731021478    0.4638E-02
 0.78125E-02        361        447  1.005907923892132    0.2504E-02
 0.39063E-02        721        893  1.008449386893941    0.2520E-02
 0.19531E-02       1441       1785  1.007959610813495    0.4859E-03
 0.97656E-03       2881       3569  1.008295296649887    0.3329E-03
 0.48828E-03       5761       7137  1.008103976816533    0.1898E-03
 0.24414E-03      11521      14273  1.008165123910790    0.6065E-04
 0.12207E-03      23041      28545  1.008143849296904    0.2110E-04
 0.61035E-04      46081      57089  1.008135239122684    0.8541E-05
 0.30518E-04      92161     114177  1.008135329025766    0.8918E-07
 0.15259E-04     184321     228353  1.008134649610200    0.6739E-06
 0.76294E-05     368641     456705  1.008134257201770    0.3892E-06
 0.38147E-05     737281     913409  1.008134050475504    0.2051E-06
 0.19073E-05    1474561    1826817  1.008134144078395    0.9285E-07
 0.95367E-06    2949121    3653633  1.008134131626789    0.1235E-07
*
```

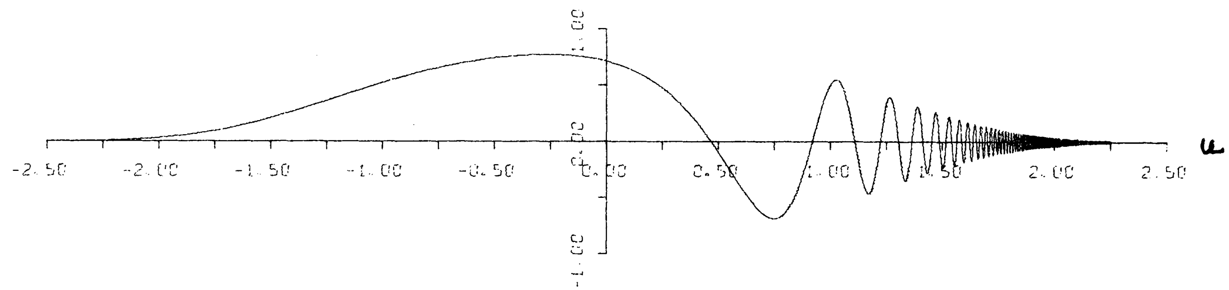
?

$$I = \int_0^1 \frac{\sin\left(\frac{1}{\sqrt{x}}\right)}{\sqrt{x}}\,dx = \int_0^\infty \frac{2\sin(1+x)}{(1+x)^2}\,dx$$ (by Dr. Rabinowitz's comment)

$f(x) = \frac{2\sin(1+x)}{(1+x)^2}$



$f(\phi(u))\,\phi'(u)$

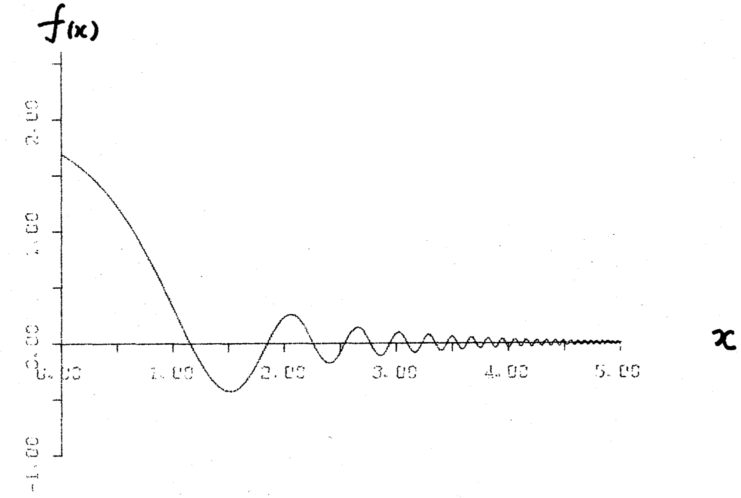


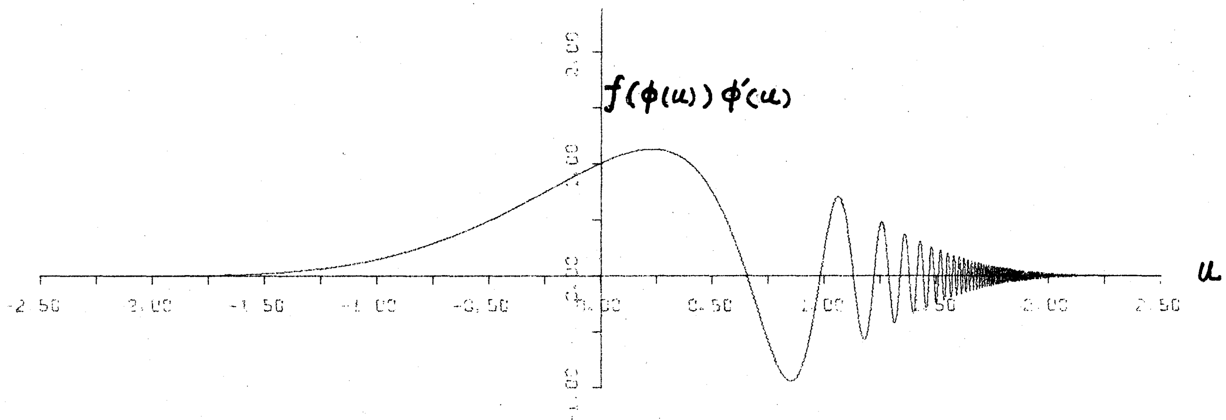
```
      LIST 180-1000

180        DOUBLE PRECISION FUNCTION F0INF(X)
190        DOUBLE PRECISION X
200         F0INF = 2.0D0*DSIN(1.0D0+X)/(1.0D0+X)**2
210        RETURN
220        END
```

$$I = 2\int_0^\infty \frac{\sin(1+x)}{(1+x)^2}\,dx$$

```
*RUN

          (C)  INTEGRATE(F0INF, 0, INF)

PLEASE TYPE IN  H0, EPS, LIST
=               0.03125D0                  1.0D-7   1
 0.31250E-01           100          105   1.000068727537460
 0.15625E-01           200          210   1.007688400674151      0.7562E-02
 0.78125E-02           400          419   1.006775973599763      0.9063E-03
 0.39063E-02           800          837   1.007513512037899      0.7320E-03
 0.19531E-02          1600         1673   1.007897695988269      0.3812E-03
 0.97656E-03          3199         3345   1.008203125717950      0.3029E-03
 0.48828E-03          6397         6689   1.008180234159500      0.2271E-04
 0.24414E-03         12793        13377   1.008170029883467      0.1012E-04
 0.12207E-03         25585        26753   1.008137971140978      0.3180E-04
 0.61035E-04         51169        53505   1.008139021682263      0.1042E-05
 0.30518E-04        102337       107009   1.008135015685097      0.3974E-05
 0.15259E-04        204673       214017   1.008133892035357      0.1115E-05
 0.76294E-05        409345       428033   1.008134156187340      0.2620E-06
 0.38147E-05        818689       856065   1.008134129054916      0.2691E-07

   1.008134129054916       0.2691E-07
                        ?
```

$$I = \int_0^1 \frac{\sin\left(\frac{1}{\sqrt{x}}\right)}{\sqrt{x}}\,dx = \int_0^\infty 2\,e^{-x}\sin(e^x)\,dx \equiv \int_0^\infty f(x)\,dx$$





$$I = \int_0^\infty 2\, e^{-x} \sin(e^{x})\, dx$$

```
 LIST 180-10000

180        DOUBLE PRECISION FUNCTION FEXP(X)
190         DOUBLE PRECISION X, DEXPF
200       FEXP= 2.0D0*DEXP(-X)*DSIN(DEXP(X))
210        RETURN
220        END


*RUN

           (D) INTEGRATE(FEXP, 0, INF)

PLEASE TYPE IN  H0, EPS, LIST
=                 0.125D0                 1.0D-9   1
 0.12500                27         26   1.071704042186671
 0.62500E-01            54         51   1.004933608340182   0.6644E-01
 0.31250E-01           107        101  0.996979779721295 0   0.7978E-02
 0.15625E-01           213        201   1.015120956996488   0.1787E-01
 0.78125E-02           425        401   1.009720373317079   0.5349E-02
 0.39063E-02           849        801   1.007896100079553   0.1810E-02
 0.19531E-02          1697       1601   1.008593032756302   0.6910E-03
 0.97656E-03          3393       3201   1.008131668867897   0.4576E-03
 0.48828E-03          6785       6401   1.008111677324365   0.1983E-04
 0.24414E-03         13569      12801   1.008157314514286   0.4527E-04
 0.12207E-03         27137      25601   1.008136230970505   0.2091E-04
 0.61035E-04         54273      51201   1.008135523133097   0.7021E-06
 0.30518E-04        108545     102401   1.008132923350206   0.2579E-05
 0.15259E-04        217089     204801   1.008134416324451   0.1481E-05
 0.76294E-05        434177     409601   1.008133909877286   0.5024E-06
 0.38147E-05        868353     819201   1.008134085382540   0.1741E-06
 0.19073E-05       1736705    1638401   1.008134103770320   0.1824E-07
 0.95367E-06       3473409    3276801   1.008134109726709   0.5908E-08
*                                                     ?
↑
BREAK
```

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