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Kyoto University
Abelian varieties attached to Hilbert modular surfaces
Takayuki ODA

§ Introductory speculations.

In this note, I would like to consider the Hodge realization of the following speculations via motives (cf. Deligne [1]).

Let $F$ be a real quadratic field over the rational number field $Q$. Suppose that a Hilbert modular cusp form $f$ is given, which is a common eigenfunction of all Hecke operators. Let us call such a form $f$ ($\#$0) primitive. And let $K_f$ be the subspace of the complex number field $C$, generated by the eigenvalues of all Hecke operators. Then we have the following conjecture.

Tensor Product Conjecture. (0.1).

(i) We expect that there exists a motif $M(f)$ defined over $Q$ attached to $f$, on which $K_f$ acts as endomorphisms on $M(f) \times_{\text{Spec}(Q)} \text{Spec}(F)$. Moreover the rank of $M(f)$ over $K_f$ should be $2^2=4$.

(ii) For any $f$ which is given above, we expect that there exist two motives $M_{f,1}$ and $M_{f,2}$ defined over $F$ with $K_f$ actions on them, such that $M(f) \times_{\text{Spec}(Q)} \text{Spec}(F) \cong M_{f,1} \otimes_{K_f} M_{f,2}$ (an isomorphism as $K_f$-motives). Moreover these two motives $M_{f,1}$ and $M_{f,2}$, which will be of rank two over $K_f$, should be conjugate with respect to the extension $F/Q$.

(iii) Let $f^\sigma$ be a primitive Hilbert cusp form obtained from $f$, applying an automorphism $\sigma$ of $C$ over $Q$ to the Fourier coefficients of $f$. Then $M_{f,1} \cong M_{f,1}^\sigma$ and $M_{f,2} \cong M_{f,2}^\sigma$, or $M_{f,1} \cong M_{f,2}$ and $M_{f,2} \cong M_{f,1}^\sigma$. 

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Let us apply the above conjecture to the problem of counting the number of algebraic 2-cycles on Hilbert modular surfaces. Let \( M^2(S) \) be the 2-motif defined over \( \mathcal{O} \) attached to a Hilbert modular surface \( S \), and let \( W_2 M^2(S) \) be the pure part of \( M^2(S) \). Let \( \Phi \) be a representative system of equivalence classes of primitive Hilbert modular cusp forms of weight 2 with respect to the following equivalence relation:

Two primitive forms \( f \) and \( g \) are equivalent, if and only if \( f = c g \) for some \( c \in \text{Aut}(\mathcal{O}/\mathcal{O}) \) and \( c \in \mathcal{O} \).

Then we should have a decomposition of the motif \( W_2 M^2(S) \)

\[
W_2 M^2(S) \times \frac{\text{Spec}(F)}{\text{Spec}(\mathcal{O})} \cong \bigoplus_{f \in \Phi} F[-1] \otimes F[-1] \otimes \bigoplus_{\mathcal{O} M(f)},
\]

where \( F[-1] \) the Tate motif of "poids" 2 over \( F \).

By Tensor Product Conjecture (0.1), there exist two abelian varieties \( A_{f,1} \) and \( A_{f,2} \) defined over \( F \) of dimension \( d=[K_f:Q] \) with endomorphism ring \( K_f \), such that

\[
M(f) \cong A_{f,1} \otimes_{K_f} A_{f,2},
\]

where we regard these abelian varieties as 1-motives, i.e., objects twisted up to \( K_f \)-invariance.

Therefore,

\[
W_2 M^2(S) \times \frac{\text{Spec}(F)}{\text{Spec}(\mathcal{O})} \cong \bigoplus_{f \in \Phi} F[-1] \otimes \bigoplus_{\mathcal{O} M(f)},
\]

Now let us consider the Tate twist of \( W_2 M^2(S) \times \frac{\text{Spec}(F)}{\text{Spec}(\mathcal{O})} \)

\[
(W_2 M^2(S) \times \frac{\text{Spec}(F)}{\text{Spec}(\mathcal{O})})[1] \cong F[0] \otimes \bigoplus_{\mathcal{O} M(f)} A_{f,1} \otimes K_f A_{f,2}.
\]

Since \( A_{f,1} \) is an abelian variety, there exists a polarization \( A_{f,1} \otimes K_{f,1} \rightarrow K_f[-1] \).

Hence \( A_{f,1}[1] \) is isomorphic to the dual motif \( A_{f,1}^\vee \) of \( A_{f,1} \).
Therefore, we have

\[(\mathcal{W}_2 \mathcal{M}^2(S) \times \text{Spec}(\mathbb{F}))_{\text{Spec}(\mathbb{Q})} \cong \mathbb{F}[\mathfrak{a}] \oplus \mathbb{F}[\mathfrak{a}] \oplus \bigoplus_{f \in \mathbb{Q}} \text{Hom}_{K_f}(A_{f,1}, A_{f,2})\]  

where \(\text{Hom}_{K_f}(A_{f,1}, A_{f,2})\) is the Hom-object in the category of \(K_f\)-motives.

Let \((\mathcal{W}_2 \mathcal{M}^2(S) \times \text{Spec}(\mathbb{F}))_{\text{Spec}(\mathbb{Q})}\) be the submotif of \((\mathcal{W}_2 \mathcal{M}^2(S) \times \text{Spec}(\mathbb{Q}))_{\text{Spec}(\mathbb{Q})}\), generated by algebraic 2-cycles.

Then Hodge-Tate conjecture for motives implies that

\[(\mathcal{W}_2 \mathcal{M}^2(S) \times \text{Spec}(\mathbb{F}))_{\text{Spec}(\mathbb{Q})} \cong \bigoplus_{f \in \mathbb{Q}} \text{Hom}_{K_f}(A_{f,1}, A_{f,2})\]  

Here \(\text{Hom}_{K_f}(A_{f,1}, A_{f,2})\) is the \(K_f\)-module of \(K_f\)-linear homomorphism of \(K_f\)-isogeny class of abelian varieties defined by \(A_{f,1}'\), to \(K_f\)-isogeny class of abelian varieties defined by \(A_{f,2}'\).

So, if (\#) is true, we can reduce the problem of determination of algebraic cycles, to determination of endomorphisms of abelian varieties.

Here we consider the Hodge realization of the above argument. In this case, the validity of (\#) is guaranteed by the Lefschetz criterion on algebraic cycles on surfaces:

A 2-cycle on an algebraic surface is algebraic, if and only if it is rational and of (1,1)-type.
§0. Main results.

Let $F$ be a real quadratic field with discriminant $D$ and with class number 1. And suppose that $F$ has a unit with negative norm. Let $f$ be a primitive form of weight 2 with respect to $\Gamma = \text{SL}_2(O_F)$.

For any primitive form $f$ of weight 2, we can attach, in general, two abelian varieties $A_{f,1}$ and $A_{f,2}$ of dimension $d=[K_f:Q]$, which have $K_f$ as an subring of $\text{End}(A_{f,i}) \otimes Q$ ($i=1,2$).

Definition. (0.1) A primitive form $f(z_1, z_2)$ is called self-conjugate, iff $f(z_1, z_2) = f(z_2, z_1)$.

Remark. We avoid the term "symmetric", because the associated 2-form $\Omega_f = (2\pi i)^{-d} f(z_1, z_2) dz_1 \wedge dz_2$ is not symmetric with respect to the involution of Hilbert modular surfaces obtained from the mapping

$$\quad (z_1, z_2) \mapsto (z_2, z_1)$$

on passing to the quotients. There is also a theoretical reason to call these forms self-conjugate.

For a self-conjugate form $f$, it is known that there exists a reellen Neben type of cuspidal form $\varphi$ of weight 2 with respect to the elliptic curve $\Gamma_0(D)$ with multiplicity defined by Jacobi symbol \( \left( \frac{D}{\rho} \right) \), such that $f$ is the lifting of $\varphi$ (cf. Doi-Naganuma [11], Naganuma [12], Zagier [13], etc).

Let us recall the results of Shimura [20]. Let $K_{\varphi}$ be the field generated by the eigenvalues of Hecke operator over $Q$. And let $k_\varphi$ be the totally real subfield contained in $K_{\varphi}$ with $[K_{\varphi}:k_\varphi] = 2$. Then Shimura [20] naturally attached abelian variety
Let $f$ be a self-conjugate form of weight 2 obtained from a reellen Neben typus elliptic modular cusp form $\varphi$ of weight $2$. Then $k_{\varphi} = k_f$, and the abelian varieties $A_{f,1}$ and $A_{f,2}$ are both isogenous to $B_{\varphi}$, as $k$-abelian varieties over the complex number $\mathbb{C}$.

In order to state our next theorem, we need more terminology. Let $p_{sc}$ be the dimension of the subspace of cusp forms of weight 2, generated by self-conjugate forms, and let $p_{nsc}$ be $p_{g} - p_{sc}$ i.e. $p_{nsc}$ is the dimension of the non-self-conjugate forms.

Let $\lambda$ be $b_2(\tilde{S}) - \tilde{\rho}(\tilde{S})$, where $\tilde{S}$ is a smooth proper model of $S$, and $b_2(\tilde{S})$ and $\tilde{\rho}(\tilde{S})$ are the second Betti number and the Picard number of $\tilde{S}$, respectively. Since $\lambda$ is a birational invariant, it does not depend on the choice of the smooth proper model $\tilde{S}$.

Theorem B. The following two statements are equivalent.

(i) $\lambda = 3p_{sc} + 4p_{nsc}$.

(ii) For any non-self-conjugate primitive form $f$, the abelian varieties $A_{f,1}$ and $A_{f,2}$ are not isogenous as $K_f$-abelian varieties. For any self-conjugate primitive form $f$, $A_{f,1}$ (or equivalently $A_{f,2}$) is not of CM-type.

Corollary. If $p_{nsc} = 0$, then $\lambda = 3p_{sc}$. (More generally, $3p_{sc} + 2p_{nsc} \leq \lambda$).
Remark. The equality $\lambda \leq 3p_{sc} + 4p_{nsc}$ is proved by Hirzebruch [67].

Let us explain the outline of our proofs. In the first four sections, we construct abelian variety $A(f)$ of dimension $4d$ defined over $\mathbb{C}$, attached to a primitive form $f$, where $d = [K_f : \mathbb{Q}]$. Here we use an idea of Satake [97], Kuga-Satake [81], and Deligne [27]. More precisely speaking, we attach a polarized Hodge structure $H(f)$ of "poids" 2 with $K_f$ action to each primitive form ($\S 2$). And next, following Kuga-Satake [81], we construct abelian varieties, by using the Clifford algebras attached to polarization forms on $H(f)$.

In section 5, we consider two topological involutions on Hilbert modular surfaces. And by means of these involutions, we show that our Clifford algebras are isomorphic to two $2 \times 2$ matrix algebra over $K_f$. By construction, our Clifford algebras act on $A(f)$. Accordingly, we have a decomposition up to isogenie $A(f) \sim A_{f,1} \times A_{f,2}$.

In section 6, we calculate explicitly the period lattices of these two abelian varieties. In this calculation, the period relation of Riemann-Hodge plays a key role.

The rest of this note discusses the application of these results to the problem of counting the number of transcendental cycles on Hilbert modular surfaces. The sources are following: Manin's idea [81] to represent the period integrals of modular forms by the twisted $L$-functions of these modular forms; liftings of Doi-Naganuma [14], and Naganuma [21]; Shimura's results on period integrals of elliptic modular forms; the determination of endomorphisms rings of abelian varieties attached to elliptic cusp forms by Ribet [88] and Momose [97].
§ 1. Polarized Hodge structures attached to Hilbert modular surfaces.

(1.0) Definitions.

We fix a real quadratic field \( F = \mathbb{Q}(\sqrt{d}) \) with discriminant \( d \geq 0 \), and assume that \( F \) has a unit \( \epsilon \) with negative norm \( N\epsilon = -1 \).

Let \( \mathbb{H} \) be the complex upper half plane \( \{ z \in \mathbb{C} \mid \text{Im } z > 0 \} \), and let \( \text{SL}_2(\mathbb{R}) \) be the special linear group of degree 2 with entries in the real number field \( \mathbb{R} \). As usual \( \text{SL}_2(\mathbb{R}) \) acts on \( \mathbb{H} \) via

\[
g(z) = \frac{az + b}{cz + d} \quad \text{for } g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}_2(\mathbb{R}), \text{ and } z \in \mathbb{H}.
\]

The product group \( \text{SL}_2(\mathbb{R}) \times \text{SL}_2(\mathbb{R}) \) acts on the product \( \mathbb{H} \times \mathbb{H} \) factorwise. Let \( \alpha_1 \) and \( \alpha_2 \) be two embeddings of an element \( \alpha \) of \( F \) into \( \mathbb{R} \). Then the mapping

\[
(\alpha \beta, \delta) \mapsto (\alpha_1 \beta_1, \alpha_2 \beta_2)
\]

defines an injective homomorphism of groups \( \text{SL}_2(F) \rightarrow \text{SL}_2(\mathbb{R}) \times \text{SL}_2(\mathbb{R}) \).

By means of this injection, the group \( \text{SL}_2(F) \) acts on \( \mathbb{H} \times \mathbb{H} \), and accordingly its subgroup \( \Gamma = \text{SL}_2(\mathbb{Z}_F) \) also acts on \( \mathbb{H} \times \mathbb{H} \). Here \( \mathbb{Z}_F \) is the ring of integers of \( F \). \( \Gamma \) acts properly discontinuously on \( \mathbb{H} \times \mathbb{H} \), and the quotient analytic space \( S = \Gamma \backslash \mathbb{H} \times \mathbb{H} \) has a natural structure of quasi-projective algebraic surface (cf. Baily-Borel [3d] for example), which is smooth except finite number of quotient singularities corresponding to elliptic fixed points on \( \mathbb{H} \times \mathbb{H} \) with respect to \( \Gamma \).

The surface \( S \) is called the Hilbert modular surface attached to \( \Gamma \), or to \( F \).

Let \( \overline{S} \) be the standard compactification, which is a union of \( S \) and \( \text{SL}_2(\mathbb{Z}_F) \backslash \mathbb{P}^1(F) \) as a set. Here \( \mathbb{P}^1(F) = \mathbb{P} \cup \{ \infty \} \) is the 1-dimensional projective line over \( F \). As is well known, the cardinality of the finite set \( \text{SL}_2(\mathbb{Z}_F) \backslash \mathbb{P}^1(F) \) is equal to the class number \( h_F \) of
F. An element of $SL_2(O_F) \backslash P^1(F)$ is called an equivalence class of cusps, or by an abuse of languages, simply called a cusp. $\tilde{S}$ has singularities at these cusps, whose resolution is precisely studied by Hirzebruch [5].

(1.1) Cohomology groups of Hilbert modular surfaces.

Let us recall some basic facts on the homology and cohomology groups of $S$ and $\tilde{S}$. Let $\pi: \tilde{S} \rightarrow S$ be a resolution of the quotient and cusp singularities of $\tilde{S}$. Then the cohomology group $H^2(S, \mathbb{Q})$ is a direct sum of the image of $\pi^*: H^2(S, \mathbb{Q}) \rightarrow H^2(\tilde{S}, \mathbb{Q})$, and the subspace of $H^2(S, \mathbb{Q})$ generated by algebraic cycles which are obtained as irreducible components of the inverse image loci of singularities by $\pi$. Restricting the intersection form on $H^2(S, \mathbb{Q})$ to $\pi^*H^2(S, \mathbb{Q})$, we can define a nondegenerate pairing on $\pi^*H^2(S, \mathbb{Q})$.

Lemma. Let $(\tilde{S}', \pi')$ be another desingularization of $\tilde{S}$. Then $\pi^*H^2(S, \mathbb{Q})$ and $\pi'^*H^2(S', \mathbb{Q})$ are canonically isomorphic as vector spaces with inner products defined by intersection forms.

Proof. Routine. Q.E.D.

By this lemma we can define a unique natural intersection form on the coimage of $\pi^*: H^2(S, \mathbb{Q}) \rightarrow H^2(S, \mathbb{Q})$. Now let us investigate the kernel of $\pi^*$. Let $r_1, r_2, \ldots, r_k$ (resp. $c_1, c_2, \ldots, c_h$) be the set of quotient (resp. cusp) singularities on $S$. Put $R_i = \pi^{-1}(r_i)$ and $C_i = \pi^{-1}(c_i)$. Then, considering the exact sequences of the relative cohomology groups, we have

$$
\begin{array}{ccc}
0 & \longrightarrow & H^2(S \text{ mod } \{r_1, \ldots, r_1 \cup c_1, \ldots, c_h\}, \mathbb{Q}) \longrightarrow H^2(\tilde{S}, \mathbb{Q}) \longrightarrow 0 \\
\downarrow & & \downarrow \\
H^1(\bigsqcup_{i=1}^k R_i \cup \bigsqcup_{i=1}^h C_i, \mathbb{Q}) & \longrightarrow & H^2(S \text{ mod } \bigsqcup_{i=1}^k R_i \cup \bigsqcup_{i=1}^h C_i, \mathbb{Q}) \longrightarrow H^2(S, \mathbb{Q}) \end{array}
$$

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Here the isomorphism \( \# \) is obtained by the excision theorem.

By means of this diagram, we have \( \text{Image} \chi = \text{Ker} \chi \). Since \( r_1 \) (1 \leq i \leq k) are quotient singularities, \( H^1(R_1, \mathbb{Q}) = 0 \). Therefore, we have the following proposition.

**Proposition.** The sequence

\[
\bigoplus_{i=1}^k H^1(C_i, \mathbb{Q}) \xrightarrow{j_1 - j_2} H^2(S, \mathbb{Q}) \xrightarrow{\partial} H^2(S, \mathbb{Q})
\]

is exact.

We apply this exact sequence for calculation of the mixed Hodge structure of \( H^2(S, \mathbb{Q}) \). From now on we consult with Deligne [4].

(1.2) The mixed Hodge structures of Hilbert modular surfaces.

Let us consider the mixed Hodge structure of \( H^2(S, \mathbb{Q}), H^2_c(S, \mathbb{Q}) \) and \( H^2(S, \mathbb{Q}) \). By means of the previous proposition, we have an exact sequence of cohomological descent (cf. Saint-Donat [4]), we can calculate the mixed Hodge structure of \( H^2(S, \mathbb{Q}) \) by this (Deligne [4]). By results of Hirzebruch [5], we can take as \( C_1 \) a stable curve of genus 1, which is not an elliptic curve (the so-called Neron's N-polygon). Therefore, \( H^1(C_1, \mathbb{Q}) = \mathbb{Q} \), and the weight filtration of the mixed Hodge structure of \( H^1(C_1, \mathbb{Q}) \) is given by

\[
W_k H^1(C_1, \mathbb{Q}) = 0 \quad (k < 0), \quad W_0 H^1(C_1, \mathbb{Q}) = \mathbb{Q} \quad (k = 0), \quad W_k H^1(C_1, \mathbb{Q}) = H^1(C_1, \mathbb{Q}) \quad (k \geq 0).
\]

Therefore the weight filtration of \( H^2(S, \mathbb{Q}) \) is given by

\[
W_k H^2(S, \mathbb{Q}) = 0 \quad (k < 0),
\]
\[
W_0 H^2(S, \mathbb{Q}) = \mathbb{Q} \oplus \mathbb{Q} \quad (k = 0),
\]
\[
W_1 H^2(S, \mathbb{Q}) = W_0 H^2(S, \mathbb{Q}) \quad (k = 1),
\]
\[
W_2 H^2(S, \mathbb{Q}) = H^2(S, \mathbb{Q}) \quad (k \geq 2).
\]

Moreover

\[
Gr^2_2 H^2(S, \mathbb{Q}) = W_2 H^2(S, \mathbb{Q}) / W_1 H^2(S, \mathbb{Q})
\]

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is a polarized homogeneous sub-Hodge structure of the polarized Hodge structure of $H^2(S,\mathbb{Q})$.

By means of the trivial isomorphism $H^2_c(S,\mathbb{Q}) \cong H^2(\overline{S},\mathbb{Q})$, we can transcribe the mixed Hodge structure on $H^2(\overline{S},\mathbb{Q})$ to $H^2_c(S,\mathbb{Q})$.

Since $S$ is rationally smooth, we have the Poincare duality

$$H^2(S,\mathbb{Q}) \times H^2_c(S,\mathbb{Q}) \rightarrow \mathbb{Q}[{-2}] .$$

Therefore the weight filtration of the mixed Hodge structure on $H^2(S,\mathbb{Q})$ is given by

$$W_k H^2(S,\mathbb{Q}) = 0 \quad (k \leq 1),$$
$$W_2 H^2(S,\mathbb{Q}) = W_3 H^2(S,\mathbb{Q}),$$
$$W_k H^2(S,\mathbb{Q}) = H^2(S,\mathbb{Q}) \quad (k \geq 4),$$

and $W_2 H^2(S,\mathbb{Q})$ is a homogeneous polarized Hodge structure of "poids" 2, and $Gr_2^{WH} H^2(S,\mathbb{Q}) \cong \mathbb{Q}[-2]$.

(1.3) The Hodge decomposition of the pure part $W_2 H^2(S,\mathbb{Q})$ of $H^2(S,\mathbb{Q})$.

From now on, for simplicity, we assume that the class number $h_F$ of $F$ is 1. Let $\mathcal{H} = \mathcal{H}(\Gamma, \Delta)$ be the Hecke algebra $\Gamma = \text{SL}_2(0_F)$ with respect to the commensurator

$$\Delta = \{ \alpha \in \text{Mat}_2(0_F) \mid \text{det} \alpha \in \mathcal{O}_F - \{0\} \} .$$

Since we can naturally regard any element of $\mathcal{H}$ as an algebraic correspondence of $S$, the Hecke algebra $\mathcal{H}$ acts on $H^2(S,\mathbb{Q})$, $H^2_c(S,\mathbb{Q})$, and $H^2(\overline{S},\mathbb{Q})$.

Remark. We can write the cohomology groups $H^2(S,\mathbb{Q})$, and

$H^2_c(S,\mathbb{Q}) \cong H^2(\overline{S},\mathbb{Q}) \cong H^2(S \text{ mod cusps},\mathbb{Q})$ as relative cohomology groups of the discrete group $\Gamma = \text{SL}_2(0_F)$. Therefore these cohomology groups are given as cohomology groups of certain complex of $\Gamma$-equivariant cochains on $\mathbb{H} \times \mathbb{H}$. Therefore, the commensurator $\Delta$ acts naturally on these cohomology groups.
The Hecke algebra $\mathcal{H}$ acts also on the space $S_2(\Gamma)$ of holomorphic cusp forms on $\mathbb{H} \backslash \mathbb{H}$ with respect to $\Gamma$. By means of the mapping

$$ f(z_1, z_2) \in S_2(\Gamma) \mapsto (2\pi i)^2 f(z_1, z_2) dz_1 \wedge dz_2, $$

we can regard any element of $S_2(\Gamma)$ as a holomorphic 2-form on $\mathbb{H}$. And it is known that these 2-forms are prolonged to holomorphic 2-forms on a smooth model $\tilde{S}$, and moreover by this mapping $S_2(\Gamma)$ is canonically isomorphic to $\Gamma(\widetilde{S} \Omega^2_2)$ (cf. Freitag [31]).

Especially, we have

$$ \dim_{\mathbb{C}} S_2(\Gamma) = p_g(\tilde{S}). $$

Now the Hodge decomposition of $\mathcal{W}^2_2(S, \Omega)$ is given by the following theorem.

**Theorem (1.1).** (Hirzebruch [67]).

Let $F$ be a real quadratic field with a norm. Let $\xi$ be such a unit, whose two embeddings into $\mathbb{R}$ satisfies $\xi_1 > 0$, $\xi_2 < 0$, $\xi_1 \xi_2 = -1$. Then as $\Gamma$-modules, we have a natural isomorphism

$$ \mathcal{W}^2_2(S, \Omega) \otimes \mathbb{C} \cong \left\{ (2\pi i)^2 f(z_1, z_2) dz_1 \wedge dz_2 \mid f \in S_2(\Gamma) \right\} $$

$$ \oplus \left( (2\pi i)^2 f(\xi z_1, \xi z_2) dz_1 \wedge d\bar{z}_2 \mid f \in S_2(\Gamma) \right) $$

$$ \oplus \left( (2\pi i)^2 f(-\bar{z}_1, \xi z_2) d\bar{z}_1 \wedge dz_2 \mid f \in S_2(\Gamma) \right) $$

$$ \oplus \left( (2\pi i)^2 f(-z_1, -\bar{z}_2) d\bar{z}_1 \wedge dz_2 \mid f \in S_2(\Gamma) \right) $$

$$ \oplus \mathbb{C} \frac{dz_1 \wedge d\bar{z}_1}{y_1^2} \oplus \mathbb{C} \frac{dz_2 \wedge d\bar{z}_2}{y_2^2}. $$

Here $z_i = x_i + \sqrt{-1} y_i$ (i=1,2).
(1.4) The primitive part $W_2^2(S,\mathbb{Q})_{\text{pr}}$ of $W_2^2(S,\mathbb{Q})$.

Let $L_1$ and $L_2$ be the invertible sheaves corresponding to the automorphic factors $j(z,\bar{z})=c_i z_1 + d_i (i=1,2)$, respectively. Here

$$z=(z_1, z_2) \in \mathbb{H} \times \mathbb{H}, \quad \text{and} \quad g=\left(\begin{array}{cc} a_1 & b_1 \\ c_1 & d_1 \end{array}\right), \left(\begin{array}{cc} a_2 & b_2 \\ c_2 & d_2 \end{array}\right) \in SL_2(\mathbb{R}) \times SL_2(\mathbb{R}).$$

We can prolong these two sheaves to two invertible sheaves $L_1$ and $L_2$ on a smooth proper model of $S$ by the results of Ueno-Van der Geer [98]. The Chern classes of these invertible sheaves define two elements of $H^2(S,\mathbb{Q})$, and accordingly two elements of $W_2^2(S,\mathbb{Q})$.

Let $W_2^2(S,\mathbb{Q})_{\text{pr}}$ be the orthogonal complement of the space spanned by these two elements, with respect to the intersection form. Then the Hodge decomposition of $W_2^2(S,\mathbb{Q})_{\text{pr}}$ is given by the right hand side of the isomorphism of Theorem (1.1), without the last two direct factors

$$\frac{dz_1}{y_1^2} \wedge \frac{dz_2}{y_2^2}.$$

Because the Chern forms of $L_1$ and $L_2$ are given the above two $(1,1)$-type 2-forms.
§ 2. Polarized Hodge structures attached to primitive cusp forms of weight 2.

Definition. A Hilbert modular cusp form $f$ ($\neq 0$) of weight 2 with respect to $\Gamma$ is called primitive, if it is a common eigenfunction of all Hecke operators: $T_{\mathfrak{a}} f = a_{\mathfrak{a}} f$ ($\mathfrak{a} \subset O_\mathbb{F}$).

We denote by $K_f$ a subfield of $\mathbb{C}$, generated by eigenvalues $a$ over $\mathbb{Q}$.

Lemma. $K_f$ is contained in $\mathbb{R}$.

Proof. Well-known. Recall that in our case all Hecke operators $T_\mathfrak{a}$ are self-adjoint with respect to the Petersson metric $( , )$ on $S_2(\mathfrak{a})$. Q.E.D.

Let $\phi_f$ be the homomorphism of $\mathcal{H}$ into $K_f$, defined by the mapping $\phi_f : T_{\mathfrak{a}} \longrightarrow a_{\mathfrak{a}}$.

Put $\tilde{\mathbb{H}}_2(S, \mathbb{Q})_{pr} = \text{Coimage}(j_\ast : \mathbb{H}_2(S, \mathbb{Q}) \longrightarrow \mathbb{H}_2(\tilde{S}, \mathbb{Q})) / \{\text{the space generated by the homology classes corresponding to the line bundles } L_1 \text{ and } L_2 \}$, and define $H_2(f)$ by

$$H_2(f) = \tilde{\mathbb{H}}_2(S, \mathbb{Q})_{pr} \otimes_{\mathbb{Q}} K_f.$$ 

Then the dual space of $H_2(f)$ over $K_f$ is canonically isomorphic to

$$H^2(f) = \{ \eta \in \mathbb{W}_2^2(S, \mathbb{Q})_{pr} \otimes_{\mathbb{Q}} K_f \mid T_{\mathfrak{a}} \eta = a_{\mathfrak{a}} \eta \}.$$ 

Theorem (2.1). Let $f$ be a primitive form of weight 2.

(i) The restriction $\psi_f : H^2(f) \times H^2(f) \longrightarrow K_f$ of the extension of scalars

$$\psi \otimes_{\mathbb{Q}} K_f : \mathbb{W}_2^2(S, \mathbb{Q})_{pr} \otimes_{\mathbb{Q}} K_f \times \mathbb{W}_2^2(S, \mathbb{Q})_{pr} \otimes_{\mathbb{Q}} K_f \longrightarrow K_f$$

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of the intersection form $\psi$, is a nondegenerate symmetric bilinear form.

(ii) The extension of scalars of $\psi_f$ with respect to the injection $K_f \subset \mathbb{R}$

$$\psi_f \otimes \mathbb{R} : H^2(f) \otimes \mathbb{R} \times H^2(f) \otimes \mathbb{R} \rightarrow \mathbb{R}$$

is of signature $(2+2-)$, and a polarization of a homogeneous Hodge structure of "poids" $\nu$ given by (iii)

(iii) $H^2(f) \otimes \mathbb{C}$ has a Hodge decomposition:

$$H^2(f) \otimes \mathbb{C} \cong \mathbb{C}(2\pi i)^2 f(\bar{z}_1, z_2) dz_1 \wedge dz_2 \oplus \mathbb{C}(2\pi i)^2 f(\xi_1 \bar{z}_1, \xi_2 \bar{z}_2) dz_1 \wedge \bar{dz}_2 \oplus \mathbb{C}(2\pi i)^2 f(\bar{z}_1 z_2, \bar{z}_2 z_1) dz_1 \wedge \bar{dz}_2 .$$

Especially $\dim_{K_f} H^2(f) = 4$.

Proof. We first recall the following basic fact.

Multiplicity One Theorem. Let $f$ and $g$ be two primitive forms of weight 2. If $T_{nf} = a_n f$ and $T_{ng} = a_n g$ for all integral ideal of $O_F$. Then $f$ is a constant multiple of $g$.

Evidently, we can canonically identify $H^2(f) \otimes \mathbb{C}$ with

$$\{ \gamma \in \bigoplus_{p \in \mathbb{P}} H^2(S, \mathbb{Q}) \otimes \mathbb{C} \mid T_p \gamma = a_p \gamma \}.$$

By Multiplicity One Theorem and Theorem (1.1), this space is isomorphic to the right hand side of the statement (iii). So (iii) is proved.

Let $f_1, f_2, \ldots, f_g$ be a basis of $S_2(F)$ consisting of primitive forms. Then we have

$$\bigoplus_{i=1}^g H^2(f_i) \otimes \mathbb{C} = \bigoplus_{i=1}^g H^2(f_i) \otimes \mathbb{C}.$$
Hence \( W_2^2(S, \mathbb{Q}) \otimes_{\mathbb{Q}} K_f \cong \bigotimes_{i=1}^{\mathfrak{f}} H^2(f_i) \otimes_{K_{f_i}} \mathbb{R} \).

Let us consider \( \psi \otimes \mathbb{R} \). Let \( f_1 \) and \( f_2 \) be two primitive forms such that \( f_1 \not\equiv c f_2 \) for any \( c \in \mathbb{C} \), and denote by \( a_{\eta,1} \) and \( a_{\eta,2} \) the eigenvalues of \( T_{\eta} \) for \( f_1 \) and \( f_2 \), respectively. Then, for \( \gamma_1 \in H^2(f_1) \otimes_{K_f} \mathbb{R} \) and \( \gamma_2 \in H^2(f_2) \otimes_{K_f} \mathbb{R} \), we have

\[
\langle a_{\eta,1}, \gamma_1, \gamma_2 \rangle = \langle T_{\eta_1}, \gamma_1, \gamma_2 \rangle = \langle \eta_1, \eta_2, \gamma_1 \rangle = \langle \eta_1, a_{\eta,2} \gamma_2 \rangle = \langle \eta_1, \gamma_2 \rangle
\]

where \( \langle , \rangle \) is the intersection form \( \psi \otimes \mathbb{R} \).

Since \( f_1 \not\equiv c f_2 \) for any \( c \in \mathbb{C} \), by Multiplicity One Theorem, there exist some ideals \( \mathfrak{m} \) such that \( a_{\eta,1} \not\equiv a_{\eta,2} \). Hence \( \langle \eta_1, \gamma_2 \rangle = 0 \).

Therefore the bilinear form \( \psi \otimes_{\mathbb{Q}} \mathbb{R} \) is a direct sum of \( \psi_{f_i} \otimes \mathbb{R} \) \((i=1, \ldots, p_g)\), and consequently each \( \psi_{f_i} \) is nondegenerate. Thus (i) is proved.

Because of the index theorem of Hodge, the bilinear form \( \psi \otimes_{\mathbb{Q}} \mathbb{R} \) has signature \( (2p_g^+, 2p_g^-) \), where \( p_g = \dim_{\mathbb{C}} S_g(\Gamma) \). For we took the "primitive part" of \( W_2^2(S, \mathbb{Q}) \). The Hodge decomposition of \( H^2(f) \otimes_{K_f} \mathbb{C} \) and the fact that Hecke operator commute with the C operator of Weil imply (ii). Q.E.D.
3. Q-basis theorem.

Let \( S_2(\Gamma; \mathbb{Q}) \) be the space of Hilbert cusp forms, whose Fourier coefficients are rational numbers. Then the following theorem is known.

**Q-Basis Theorem (3.1).** (Shimura [14], or Rapoport [10].)

(i) \( S_2(\Gamma; \mathbb{Q}) \) spans \( S_2(\Gamma) \). In other words, there is a natural isomorphism \( S_2(\Gamma; \mathbb{Q}) \cong S_2(\Gamma) \).

(ii) Let \( f(z_1, z_2) \) be an element of \( S_2(\Gamma) \) with Fourier expansion

\[
f(z_1, z_2) = \sum_{\gamma \in \mathcal{D}^{-1}_{+}} a(\gamma) \exp[2\pi i (\gamma_1 z_1 + \gamma_2 z_2)],
\]

where \( \mathcal{D}_{+} \) is the intersection of the codifferent and the set of totally positive elements.

For any element \( \xi \in \text{Aut}(\mathbb{C}/\mathbb{Q}) \), we define \( f^\xi(z_1, z_2) \) by a formal Fourier expansion

\[
f^\xi(z_1, z_2) = \sum_{\gamma \in \mathcal{D}^{-1}_{+}} a(\gamma)^\xi \exp[2\pi i (\gamma_1 z_1 + \gamma_2 z_2)].
\]

Then \( f^\xi(z_1, z_2) \) belongs to \( S_2(\Gamma) \).

**Corollary (3.2).** Let \( f \) be a primitive form of weight 2 such that \( T_{a_f} f = a_f f \). And let \( \{ \xi_{1}, \xi_{2}, \ldots, \xi_{d} \} \) be the set of all embeddings of \( K_f \) into \( \mathbb{R} \), where \( d = [K_f : \mathbb{Q}] \). Then there exist primitive forms in \( S_2(\Gamma) \), such that

\[
T_{a_{f_i}} f_i = \xi_i(a_{f_i}) f_i \quad (i=2,3,\ldots,d)
\]

for all integral ideal \( \mathfrak{a} \). Especially we have

\[
K_{f_i} = \xi_i(K_f)
\]

(1 \( \xi_i \) \( d \)), and \( K_f \) is a totally real field.
§ 4. Clifford algebras and abelian varieties.

Let us recall the data given in the previous section.

Data (4.1)

$K_f$: a totally real algebraic number field of degree $d$ over $\mathbb{Q}$.

$\{\xi_1, \xi_2, \ldots, \xi_d\}$ be the set of all embeddings of $K_f$ to $\mathbb{R}$.

$H(f)$: a $K_f$-module of rank 4 on which a symmetric non-degenerate bilinear form $\psi_f$ with values in $K_f$ is defined, which satisfies the following condition:

(i) For any $\xi_i$ ($i=1, \ldots, d$), $H(f) \otimes_{K_f, \xi_i} \mathbb{R}$ is equipped with a homogeneous Hodge structure of "poids" 2, and $\psi_f \otimes_{K_f, \xi_i} \mathbb{R}$ gives a polarization with respect to this Hodge structure.

Let start from this data, and construct an abelian variety $A$ attached to this data. First we choose an integral structure.

From now on, until the last part of this section, we omit the subscript $f$ to simplify the notation. Now let $O_K$ be an order of $K$, and let $H_Z$ be an $O_K$-module of rank 4 in $H$. And moreover we can choose $H_Z$ such that the values of $\psi_f$ on $H_Z \otimes H_Z$ is contained in $O_K$.

Let $C^+(H_Z)$ be the even Clifford algebra attached to over $O_K$. Thus this algebra is of rank $2^3=8$ over $O_K$. There is a sophisticated description to attach abelian variety by Deligne [2]. We recall here more naive definition of $C^+(H_Z)$.

First, we consider a real torus of dimension $8d$

$$C^+(H_Z) \otimes_{\mathbb{Z}} \mathbb{R}/C^+(H_Z).$$
Clearly on this real torus, the Clifford algebra $C^+(H_Z)$ acts by left multiplication. Next, we define a complex structure on this real torus.

Let us note the natural isomorphism

$$C^+(H_Z) \otimes_R \cong \bigoplus_{i=1}^d C^+(H_Z) \otimes_{O_K, C_i} R \cong C^+(H_Z) \otimes_{O_K, C_i} R.$$ 

Let us consider the intersection

$$H_{Z, O_K, C_i} \cap (H^{2,0} \oplus H^{0,2}),$$

for each $i (i=1,2,\ldots,d)$. Choose an orthogonal basis $e^+_i, e^-_i$ of this intersection subspace with respect to $\otimes_R$. Then,

$$J_i = e^+_i e^-_i \ (i=1,2,\ldots,d)$$
defines a complex structure on each factor $C^+(H_Z) \otimes_{O_K, C_i} R$ by means of right multiplication. Therefore the direct sum $J = \bigoplus J_i$ of $J_i$ defines a complex structure on our real torus by right multiplication.

Theorem (4.2) \textbf{This complex torus has a structure of abelian variety.}

Proof. It is well-known that a complex torus with sufficient many endomorphisms becomes automatically an abelian variety, as in our case. See also Kuga-Satake\cite{[8]} or Deligne\cite{[9]} for $K=Q$. Q.E.D.

Remark. The dimension of this abelian variety is equal to $4d$. The isogenous class of this abelian variety does not depend upon the choice of the integral structure $H_Z$ in $H$. 

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§ 5. Involutions of Hilbert modular surfaces and decomposition of abelian varieties.

Let $\epsilon$ be a unit such that $\epsilon_1 > 0$ and $\epsilon_2 < 0$. Let $G_\infty$, $H_\infty$, and $F_\infty$ be involutions on $H \times H$ defined by

\[ G_\infty: (z_1, z_2) \in H \times H \mapsto (\epsilon_1 z_1, \epsilon_2 z_2) \in H \times H, \]
\[ H_\infty: (z_1, z_2) \in H \times H \mapsto (\epsilon_2 z_1, \epsilon_1 z_2) \in H \times H, \]
\[ F_\infty: (z_1, z_2) \in H \times H \mapsto (-z_1, -z_2) \in H \times H. \]

Clearly, we have $G_\infty H_\infty = H_\infty G_\infty = F_\infty$.

On passing to the quotient $S$, we obtain involutions on $H \times H$, which we denote by the same symbol

\[ G_\infty: S \rightarrow S, \ H_\infty: S \rightarrow S, \ F_\infty = H_\infty G_\infty = G_\infty H_\infty: S \mapsto S. \]

Evidently these involutions act on the homology and cohomology groups $H_2(S, \mathbb{Q})$, $H^2(S, \mathbb{Q})$, $W_2 H^2(S, \mathbb{Q})$ etc.

Note here that $G_\infty$ and $H_\infty$ changes the orientation. To check this apply $G_\infty$ and $H_\infty$ to $dz_1 \wedge d\bar{z}_1 \wedge dz_2 \wedge d\bar{z}_2$.

Remark. $F_\infty$ is the Frobenius at infinity. Namely let $S_\mathbb{R}$ be a canonical model of $S$ defined over the real number field $\mathbb{R}$. Then $F_\infty$ coincides with the action of the nontrivial element of $\text{Gal}(\mathbb{C}/\mathbb{R})$ on $\mathbb{C}$-valued points $S$ of $S_\mathbb{R}$.

Definition.- Proposition (5.1) Put

\[ H_{++}(f) = \{ \omega \in H(f) \mid G_\infty \omega = \omega, \ H_\infty \omega = \omega \}, \]
\[ H_{+-}(f) = \{ \omega \in H(f) \mid G_\infty \omega = \omega, \ H_\infty \omega = -\omega \}, \]
\[ H_{-+}(f) = \{ \omega \in H(f) \mid G_\infty \omega = -\omega, \ H_\infty \omega = \omega \}, \]
\[ H_{--}(f) = \{ \omega \in H(f) \mid G_\infty \omega = -\omega, \ H_\infty \omega = -\omega \}. \]

Then we have a direct sum decomposition

\[ H(f) = H_{++}(f) \oplus H_{+-}(f) \oplus H_{-+}(f) \oplus H_{--}(f). \]
Moreover \( H_{++}(f), H_{+-}(f), \ldots, H_{--}(f) \) are all of rank one over \( K_f \).

Proof. Note that the Hecke operators commutes with \( G_{\omega}, H_{\omega} \), and \( F_{\omega} \). And use the Hodge decomposition (Theorem (1.1)).

Noting that \( G_{\omega} \) and \( H_{\omega} \) change the orientation of \( S \), we have the following proposition.

Proposition (5.2). For the intersection form \( \psi_f \), we have
\[
\psi_f(H_{++}(f), H_{++}(f)) = \psi_f(H_{+-}(f), H_{+-}(f)) = \psi_f(H_{--}(f), H_{--}(f)) = \psi_f(H_{--}(f), H_{--}(f)) = 0.
\]

And \( H_{++}(f) \) and \( H_{--}(f) \), and \( H_{+-}(f) \) and \( H_{-+}(f) \) are mutually dual with respect to \( \psi_f \) each other.

Corollary (5.3). \( \psi_f \) is a kernel form.

Corollary (5.4) \( c^+(H(f)) \) the even Clifford algebra over \( H(f) \) with respect to \( \psi_f \) is a direct product of two copies of \( 2 \times 2 \) matrices with entries in \( K_f \); \( c^+(H(f)) \cong M_2(K_f) \oplus M_2(K_f) \).

Corollary (5.4) Let \( A(f) \) be an abelian variety constructed in the previous section. Then we have a decomposition upto isogeny:
\[
A(f) \cong \bigoplus_{i=1}^2 A_{f,i} \times A_{f,i} \times A_{f,i} \times A_{f,i}.
\]

Here \( A_{f,i} \) \((i=1,2)\) are both of dimension \( d=[K_f:Q] \), with endomorphism rings \( K_f \cong \text{End}_C(A_{f,i}) \otimes \mathbb{Q} \).

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6. **Explicit calculation of period lattices of** $A_{f,1}$ **and** $A_{f,2}$.

In this section, we give the period lattices of the isogeny classes of abelian varieties represented by $A_{f,1}$ and $A_{f,2}$.

Let $f$ be a primitive form of weight 2, and let $K_f$ be the field of the eigenvalues. Let $\mathcal{E}_1=\text{id}_{K_f}$, $\mathcal{E}_2$, $\ldots$, $\mathcal{E}_d$ be the set of embeddings of $K_f$ into $R$, where $d=[K_f:Q]$. Let $f_{\mathcal{E}_i}$ be the primitive forms obtained from $f$ by applying $\mathcal{E}_i$ to the Fourier coefficients of $f$ (We normalize $f$). Let $H_2(f_{\mathcal{E}_i})$ be the homology group attached to $f_{\mathcal{E}_i}$ ($i=1,2,\ldots,d$). Let consider the action of $G_\infty$ and $H_2(f_{\mathcal{E}_i})$ on $H_2(f_{\mathcal{E}_i})$. And define $H_2^{++}(f)$ etc. similarly as the case of cohomology groups. We normalize $f$ such that the "first" Fourier coefficient of $f$ is a rational number.

Put

$$\omega_f = (2\pi i)^2 f(z_1, z_2) dz_1 \wedge dz_2$$

for such form $f$. Then we have the following lemma.

**Lemma (6.1)**

If $\gamma^{++} \in H_2^{++}(f)$, and $\gamma^{--} \in H_2^{--}(f)$. Then the period integrals

$$\int_{\gamma^{++}} \omega_f \quad \text{and} \quad \int_{\gamma^{--}} \omega_f$$

are real numbers, and if $\gamma^{+-} \in H_2^{+-}(f)$, and $\gamma^{-+} \in H_2^{-+}(f)$, then the period integrals

$$\int_{\gamma^{+\cdot}} \omega_f \quad \text{and} \quad \int_{\gamma^{-\cdot}} \omega_f$$

are purely imaginary numbers.

**Definition.** Fix the above four cycles $\gamma^{++}$, $\gamma^{+-}$, $\gamma^{-+}$, $\gamma^{--}$ in $H_2(f)=H_2(S,Q) \otimes K_f$. And let $\gamma_1^{++}$, etc be the element of $H_2(f_{\mathcal{E}_1})$. 

obtained from $\gamma^{++}$ etc. with respect to conjugation of $K_f$ over $Q$.

Then we put

$$W_{++}(f^i) = \int_{\gamma^{++}_i} \omega_{f^i} \epsilon_{f^i}, \quad W_{-+}(f^i) = \ldots$$

Theorem (6.2) The period lattices of $A_{f,1}$ and $A_{f,2}$ are given by

$$L_1 = \left\{ v = (v_1, v_2, v_3, \ldots, v_d) \in \mathbb{C}^d \mid v_i = \epsilon_{f_i}(\alpha) + \epsilon_{f_i}(\beta) W_{++}(f^i)/W_{++}(f^i) \quad (i=1,2,\ldots,d), \right. \right.$$

$$\left. \quad \text{for some } \alpha, \beta \text{ in } K_f. \right\}$$

and

$$L_2 = \left\{ v = (v_1, v_2, \ldots, v_d) \in \mathbb{C}^d \mid v_i = \epsilon_{f_i}(\alpha) + \epsilon_{f_i}(\beta) W_{-+}(f^i)/W_{-+}(f^i) \quad (i=1,2,\ldots,d), \right. \right.$$

$$\left. \quad \text{for some } \alpha, \beta \text{ in } K_f. \right\}$$

In the calculation of these periods, we need the following theorem.

Theorem (6.3) (The period relation of Riemann-Hodge).

If $\hat{\Psi}_f (\gamma^{++}, \gamma^{--}) = \hat{\Psi}_f (\gamma^{+-}, \gamma^{-+})$ for the intersection form $\hat{\Psi}_f$ of $H_2(f)$, we have

$$W_{++}(f)W_{-+}(f) = W_{-+}(f)W_{++}(f).$$

Proof of Theorem (6.2). A very easy computation of Clifford algebras. We omit it here, because of a short of time.
以下、緒切時期を過ぎているので日本語で。

前節までで、Abel多様体の構成に関係、general nonsenseはおしまいである。

注。さて京都での研究集会のときに質問された、Shimura [21],
Moriyō [22], Hida [23] 等によって定義されている、アーベル多様体と、上に構成したものとの関係についてであるが、これについては若干のように考えている： 上の数文は、本質的には、
natural polarized Hodge structureが、あるべき、それよりアーベル多様体が構成できるというものである（この条件は Moriyō [22]に書かれている）。最初にとる、natural polarized Hodge structureの“poids”を考えるととき、これの構成法はある意味で、これを、
natural polarized Hodge structure of “poids”と見做すことになる。役立て
者になれば、このアーベル多様体は、“自然な”ものとは思われ
ない。少なくとも、Grothendieck—Deligne の motins の考え方では
説明不可能である。

さて、特に、やはり Hida moduli curve forme にアーベル多様体を attach した Hida [23]との関係で考え、この場合“poids” n =
[F=0]（但しことは考えている数列内）で、n+1 である。こ
れで納得できない人は、前節定理 (6.2) の周期と [21] での周期
と同じてみてみれば、この二つは全くちが、たとえることも
容易に check できるはずである。

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§7. Self-conjugate forms に付随するアーベル多様体.

本文に著める定理 A を証明するがこの節の目的である。この証明は容易であるが長い。概略のみ示す。

定理 (7.1) （定理 A）

\( \Phi \) を realer Neben-type （elliptic modular）cusp form of weight 2, \( \Phi \)
は positive とする。\( \Phi \) はその lifting と Hilbert modular forms とする。
\( \Phi \) と Shimura [20] のアーベル多様体, \( A_{\Phi} \sim B_{\Phi} \times \mathbb{P}^{1} \) はの方
解とする。\( B_{\Phi} \sim B' \) （isogee）は知られている。

さてこのとき, \( \Phi \) 上の \( B_{\Phi} \) と \( A_{\Phi,1}, A_{\Phi,2} \) は isogee である。

（証明の概略）定理 (6.2) により pencell lattice の計算による。
\( W_{+}(\Phi)/W_{-}(\Phi) \subset K_{\Phi} \) はすくにわかる。さて \( \Phi_{1}, \Phi_{2} \) と
は \( K_{\Phi} \) -isogeneous. 問題は \( W_{+}(\Phi), W_{-}(\Phi) \) の計算が
となる。\( B_{\Phi}=K_{\Phi} \) はすくに知られている。すなわちこれかわからない。

命题 (7.2) \( W_{+}(\Phi)/W_{-}(\Phi)^{2} \subset K_{\Phi}, \ W_{-}(\Phi)/W_{-}(\Phi)^{2} \subset K_{\Phi} \)。

（証明の概略） \( W_{+}(\Phi), W_{-}(\Phi) \) は Shimura [18, 19] で定義されるも
とす。この命题は \( \Phi/\chi_{\Phi} \) の本来の定義 (Poi-Nogamune [11],
Nogame [12]) から容易に出てくる。
問題は $W_+\langle f \rangle, W_-\langle f \rangle$ の計算法である。$W_+\langle f \rangle = W_-\langle f \rangle$ と即
おき、Riemann-Hodge と period relation (定理 (6.7)) より

$$\frac{\langle f \rangle}{\langle W_+(f)W_-\langle f \rangle \rangle} = 1$$

よって

$$\frac{\langle W_+\langle f \rangle \rangle}{\langle W_+(f)W_-\langle f \rangle \rangle} \in K_f$$

これはすなわち $W_+(f)/W_+(\langle f \rangle)W_-\langle f \rangle \in K_f$ は明らかで、$W_-\langle f \rangle/W_+(\langle f \rangle)W_-\langle f \rangle \in K_f$ は別の方向で証明する。

基本的な考え方とは、Oda [3] の定理 1 を用いて、Poin-
caré map の adjoint map を考える。すなわち周期積分と Fourier
係数も含む、so-called heat kernel elliptic map 形式で表される。

(Hirzebruch-Zagier はこれの周期積分を交点数と考え、定
式) typcal な Fourier 係数をとること。$W_+\langle f \rangle/c_{-1} \in K_f$ は自明に
はない。これは、$c_{-1}/W_+(\langle f \rangle)W_-\langle f \rangle$ を取ることである。

Poincaré notation と (1) ellipt な site で示すことに
する。（適当な normalization が必要である）

定理 (7.3) (Oda [2] など)

$$c_{(\phi, \eta)} \in K_f$$

命題 (7.4)

$$(\eta, \phi) \in K_f = K_f$$

$$(\phi, \eta) \in K_f = K_f$$
(7, 5) \quad \epsilon \frac{W_{+}(f) W_{-}(f)}{W_{+}(f) W_{-}(f)} \in K\tilde{f}

\text{すなわち} \quad \frac{W_{+}(f)}{W_{+}(f) W_{-}(f)} \in K\tilde{f} \Rightarrow \frac{W_{-}(f)}{W_{-}(f) W_{+}(f)} \in K\tilde{f}.

\text{したがって} \quad \frac{W_{-}(f)}{W_{+}(f) W_{-}(f)} \in K\tilde{f}

よって定理 (7.1) は証明された。

§8. 定理 B の証明。

これも全く同様的応用期の計算である。但し定理 B の
系の証明では、Shimura [15] [16] の結果を少し一般化する必要
がある（Hecke型 elliptic modular の場合に）。ある種の同期の萌え
ないことをいう必要がある。ここでも再び [25] の Theorem 1 を使
う。そして次の命題を得る。

命題 (8.1) \quad f \text{ は self-conjugate primitive form of weight } 2 \text{ である}

Hecke型 elliptic cusp form であり抬起が得られる。

このとき \quad W_{+}(f) = W_{+}(f)^{2}, \quad W_{-}(f) = W_{-}(f) = \frac{W_{+}(f) W_{-}(f)}{W_{+}(f) W_{-}(f)}

\text{は } K\tilde{f} \text{ 上線形独立な数である。}
§89. $l$-adic realization of the tensor product conjecture

Tensor product conjecture の $l$-adic realization は $S$ が $K3$
曲面と対角線相等しいときは、Deligne [27] の結果によって、非常に
ありふれた形であるため OK である。$S$ 自身が平面Symmetric Hilbert
modular surface のようなことが同様にありえる。

問題. $A_{f,1}$, $A_{f,2}$ の定義体を代数体にまで下げることが
予想されていることを期待している。Level が 1 でない
Hilbert modular surface については、これは容易に解ける場合が
多いと思うておりそうである。Level が 1 の場合は難しく見える。
Deligne [27] の証明の論理をさらに一般化しようとすれば,
Hilbert modular surface の変形を考える必要がある。

$F = \mathbb{Q}(\sqrt{13}), \mathbb{Q}(\sqrt{17}), \mathbb{Q}(\sqrt{19})$ と 3 は elliptic fibre space と
図 p' と双曲理同値を 、これに対しては Aka の結果がある。

$F = \mathbb{Q}(\sqrt{13}), \mathbb{Q}(\sqrt{17})$ と 3 は $B_2 = \mathbb{Q}, \quad \Delta = 2, \quad c_1 = 2, \quad c_2 = 2p - 4$ で
general type の曲面と双曲理同値である。これらの結果から出発して少し特別な場合に、定義体を下げることと、
Tensor product conjecture の $l$-adic realization の形での証明は
可能なもの知られていない。しかしその証明にはいくつかの step はま
ら無の中である。いずれにせよ、定理 A によって $\mathcal{M}_{f_{ij}}$ の
Hilbert modular forms $f$ に対しては $A_{f,1} \sim A_{f,2}$ の定義体は、これ
が $B_2$ と isogenous なことから自然に下げている。

$A_{f,1}, A_{f,2}$ の定義は一般化された Birch–Swinnerton-Dyer conjecture と
も compatible であることは容易にわかる。ときに $A_{f,1}, A_{f,2}$
の定義で自然なものであることはまず性質しなければならない。

Tensor product conjecture の $l$-adic realization を考えること
という問題がある。
References.


3. ________.: Théorie de Hodge. II.


27. Horikawa, E.: Algebraic surfaces of general type with small $c_1^2$, I

28. Manin, Y.: Pervel's forms and...

29. Faltings, G.: Preprint

注意: 文献表における項目は、preprint "未定" に変更されており、
詳細は次後参照。