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Threefolds whose canonical bundles are not numerically effective

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Introduction. In this paper, we study the cone NE(X) of effective 1-cycles on a non-singular projective variety X and state how this result on the cone can be applied to the study of 3-folds.

If the first Chern class $c_1(X)$ is ample, then NE(X) is spanned by finitely many edges (Theorem 1.1). In general, NE(X) has edges (called extermal rays) in the set $S = \{Z \in NE(X) \mid (Z.c_1(X)) > 0\}$ if $S \neq \phi$ (Corollary 1.7). This extremal ray is spanned by a class represented by a rational curve (Theorem 1.3). From this, follows the assertion : the canonical bundle K_X is numerically effective if X contains no rational curves.

In Chapter 2, we give a geometric interpretation of extremal rays on a surface and explain how the result is generalized to the case of 3-folds : if X is a non-singular projective 3-fold with an extremal ray over an algebraically closed field of characteristic 0, we have either (1) X has an exceptional divisor which is described explicitly, (2) X is a conic bundle over a surface, (3) X has a morphism to a curve whose general fibers are Del Pezzo surfaces, or (4) X is a Fano 3-fold such that $\mathrm{Pic}\, X \cong \mathbb{Z}$.

Chapter 1. The cone of effective 1-cycles.

§1. Notation, definitions, and statements.

Let X be a non-singular projective variety of demension n defined over an algebraically closed field k of characteristic $p \geq 0$, with a very ample divisor H. We will keep these symbols throughout this paper.

By a 1-cycle on X, we understand an element of the free abelian group generated by all the irreducible reduced subvarieties of dimension 1 (or curves) of X. A 1-cycle $Z = \Sigma n_C C$ $(n_C \in \mathbb{Z})$ is called effective if $n_C \geq 0$ for all C. If two 1-cycles Z_1 and Z_2 are algebraically equivalent (resp. numerically equivalent) in the usual sense [2], we express it as $Z_1 \approx Z_2$ (resp. $Z_1 \approxeq Z_2$). Let

$$A(X) = (\{\text{1-cycles on } X\}/\approx) \otimes_{\mathbb{Z}} \mathbb{Q},$$

$$N(X) = (\{\text{1-cycles on } X\}/\approxeq) \otimes_{\mathbb{Z}} \mathbb{R},$$

and AE(X) (resp. NE(X)) the smallest convex cone in A(X) (resp. N(X)) containing all effective 1-cycles, closed under multiplication by $\mathbb{Q}_+ = \{q \in \mathbb{Q} \mid q \geq 0\}$ (resp. $\mathbb{R}_+ = \{r \in \mathbb{R} \mid r \geq 0\}$). Via the intersection pairing (.) of 1-cycles and divisors, N(X) is dual to $NS(X) \otimes_{\mathbb{Z}} \mathbb{R}$, where NS(X) is the Neron-Severi group, {divisors of X}/$\approxeq$. Thus N(X) is a real vector space of finite dimension $\rho(X)$, the rank of NS(X). Let $\| \ \|$ be any norm of N(X). Then $\overline{NE}(X)$, the closure of NE(X) for the metric topology, is dual to the pseudo-ample cone of X (cf. [2]) by Kleiman's criterion for ampleness : a divisor D on X is ample if and only if $(D.Z) > 0$ for all $Z \in \overline{NE}(X) \cap \{Z \in N(X) \mid \|Z\| = 1\}$.

This cone $NE(X)$, which is interesting from various viewpoints, is rational polyhedral if $c_1(X)$, the first Chern class of X, is ample.

Theorem 1.1. If $c_1(X)$ is ample, then X contains finitely many rational curves $\ell_1, \ell_2, \cdots, \ell_r$ such that $(\ell_i.c_1(X)) \leq n+1$ for all i,

a) $AE(X) = \mathbb{Q}_+[\ell_1] + \cdots + \mathbb{Q}_+[\ell_r]$ if $p > 0$, and

b) $NE(X) = \mathbb{R}_+[\ell_1] + \cdots + \mathbb{R}_+[\ell_r]$ if $p \geq 0$, where $[Z]$ denotes the class of 1-cycle Z.

To be precise, a rational curve means an irreducible reduced curve defined over k whose normalization is $\mathbb{P}^1_k$. This theorem enables us to improve our Theorem 3 in [5].

Corollary 1.2. If $c_1(X)$ is ample, then

a) a divisor D on X is ample if D is numerically positive,

b) $\rho(X) = 1$ if every numerically effective divisor is either numerically trivial or ample, and

c) $\rho(X) = 1$ if every non-zero effective divisor is ample and $p = 0$, where a divisor D is said numerically positive (resp. numerically effective, numerically trivial) if $(D.Z) > 0$ (resp. $(D.Z) \geq 0$, $(D.Z) = 0$) for all irreducible curves Z.

Indeed, (a) follows from $NE(X) = \overline{NE}(X)$ by virtue of Kleiman's criterion. If $\rho(X) > 1$, then we can take a divisor D such that $D > 0$ on the interior of $NE(X)$ and $D = 0$ on $\mathbb{R}_+[\ell_i]$ for some i as a real valued linear function on $N(X)$, which implies that D is numerically effective, and not numerically trivial, or ample. This shows (b), and (c) follows from (b) by Lemma 2, (2) in [5].

To study $\overline{NE}(X)$ for a general X, we need more definitions. For an arbitrary positive real number ε, let

$$A_\varepsilon(X, H) = \{Z \in A(X) \mid (Z \cdot c_1(X)) \le \varepsilon(Z \cdot H)\},$$

$$N_\varepsilon(X, H) = \{Z \in N(X) \mid (Z \cdot c_1(X)) \le \varepsilon(Z \cdot H)\},$$

$AE_\varepsilon(X, H) = AE(X) \cap A_\varepsilon(X,H)$, and $NE_\varepsilon(X, H) = NE(X) \cap N_\varepsilon(X, H)$.

If there is no danger of confusion, $A_\varepsilon(X, H)$, $N_\varepsilon(X, H)$, $AE_\varepsilon(X,H)$, $NE_\varepsilon(X, H)$ will be abbreviated to $A_\varepsilon(X)$, $N_\varepsilon(X)$, $AE_\varepsilon(X)$, $NE_\varepsilon(X)$, respectively.

<u>Theorem</u> 1.3. For an arbitrary positive ε, there exist a finite number r (≥ 0) of rational curves $\ell_1, \cdots, \ell_r$ in X such that $(\ell_i \cdot c_1(X)) \le n+1$ for all i,

a) $AE(X) = \mathbb{Q}_+[\ell_1] + \cdots + \mathbb{Q}_+[\ell_r] + AE_\varepsilon(X)$ if $p > 0$, and

b) $\overline{NE}(X) = \mathbb{R}_+[\ell_1] + \cdots + \mathbb{R}_+[\ell_r] + \overline{NE}_\varepsilon(X)$ if $p \ge 0$,

where $\overline{NE}_\varepsilon(X) = \overline{NE}(X) \cap N_\varepsilon(X)$.

Now Theorem 1.1 follows from Theorem 1.3. Indeed, if

$c_1(X)$ is ample, then $AE_\varepsilon(X) = \overline{NE}_\varepsilon(X) = \emptyset$, when $1/\varepsilon$ is a sufficiently large integer such that $(1/\varepsilon)c_1(X) - H$ is ample. Theorem 1.3 will be proved in the next section.

§2. Proof of Theorem 1.3.

We will begin by reformulating Theorems 4 and 5 in [3].

Theorem 1.4. For a non-singular projective curve C of genus g over k and a morphism $f : C \to X$, there exist a morphism $h : C \to X$ and an effective 1-cycle Z with the properties ; (a) $(h_*(C)\cdot c_1(X)) \leq ng$, (b) an arbitrary irreducible component Z' of Z is a rational curve such that $(Z'\cdot c_1(X)) \leq n+1$, and (c) $f_*(C) \approx h_*(C) + Z$.

Proof. In the statement, f_* is the cycle-theoretic direct image ; $f_*(C) = 0$ if $\dim f(C) = 0$, $[C : f(C)]\, f(C)$ if $\dim f(C) = 1$. We will treat two cases. First we assume $g = 0$. We use induction on $(f_*(C)\cdot H)$. If $(f_*(C)\cdot c_1(X)) \leq n+1$, then we can set h to be any constant map and $Z = f_*(C)$. If $(f_*(C)\cdot c_1(X)) > n+1$, Theorem 4 [3] implies that $f_*(C) \approx Z_1 + Z_2$, where Z_1 and Z_2 are non-zero effective 1-cycles whose components are rational curves. Since $(f_*(C)\cdot H) = (Z_1\cdot H) + (Z_2\cdot H)$, we can apply the induction hypothesis to Z_1 and Z_2, and the case $g = 0$ is done. We prove the case $g > 0$ again by induction on $(f_*(C)\cdot H)$. If $(f_*(C)\cdot c_1(X)) \leq ng$, we can set $h = f$ and $Z = 0$. If $(f_*(C)\cdot c_1(X)) > ng$, it follows from the proof of Theorem 5 [3] that $f_*(C) \approx f'_*(C) + U$, where $f' : C \to X$ and U is a non-zero effective 1-cycle whose components are rational curves. Since $U \neq 0$, $(f'_*(C)\cdot H) < (f_*(C)\cdot H)$. Now we have only to apply the induction hypothesis to f' and the result on the case $g = 0$ to each component of U. q.e.d.

Now Theorem 1.3, (a) is an easy corollary to Theorem 1.4.

Proof of Theorem 1.3,(a). Let us consider the set Φ of all the rational curves ℓ in X such that $(\ell\cdot c_1(X)) \leq n+1$ and $[\ell] \notin AE_\varepsilon(X)$. These curves ℓ form a bounded family, i.e. parametrized by a quasi-projective scheme [1, n°221, 4 , because $(\ell\cdot H) < (\ell\cdot c_1(X))/\varepsilon \leq (n+1)/\varepsilon$. Hence there exist finitely many rational curves $\ell_1, \cdots, \ell_r$ which form a complete set of representatives of $\Phi/\approx$. We will show that the convex cone $V = \mathbb{Q}_+[\ell_1] + \cdots + \mathbb{Q}_+[\ell_r] + AE_\varepsilon(X)$ is equal to $AE(X)$. We treat two cases. Let ℓ be a rational curve in X. By Thoerem 4, $\ell \approx Z$ for some effective 1-cycle Z whose components Z' are rational curves such that $(Z'\cdot c_1(X)) \leq n+1$. Thus for each component Z' of Z, we have either $Z' \in \Phi$ or $Z' \in AE_\varepsilon(X)$. Hence $[\ell] \in V$, and the rational curve case is done. Let C be a non-singular projective curve of genus $g > 0$ and $f : C \to X$ a morphism. Let C_i be the p^{-i}-th power of C and $\pi_i : C_i \to C_{i-1}$ the p-th power morphism. We then inductively find morphisms $f_i : C_i \to X$ and its image $D_i = f_{i*}(C_i)$ for $i \geq 0$ so that $f_0 = f$, $(D_{i+1}\cdot c_1(X)) \leq ng$, and $p[D_i]-[D_{i+1}] \varepsilon V$ for all $i \geq 0$. Indeed, if we apply Theorem 4 to $f_i \circ \pi_{i+1} : C_{i+1} \to X$, then we get $h = f_{i+1}$ and $h_*(C_{i+1}) = D_{i+1}$ such that $p[D_i] - [D_{i+1}]$ is equivalent to a sum of rational curves which belong to V as we have seen before. Now if $[D_a] \varepsilon V$ for some a, then $[D_0] \varepsilon V$ because

$$D_0 = \sum_{j=0}^{a-1} p^{-j-1}(pD_j - D_{j+1}) + p^{-a}D_a .$$

If $[D_i] \notin AE_\varepsilon(X)$ for all i, then $(D_i \cdot H) \leq (D_i\ c_1(X))/\varepsilon \leq ng/\varepsilon$ for all i. Since $(D_i \cdot H)$ is uniformly bounded, there are numbers a and b such that $D_a \approx D_b$ and $a < b$ [1, n°221]. Then

$$(p^{b-a} - 1)D_a \approx p^{b-a}D_a - D_b = \sum_{i=a}^{b-1} p^{b-1-i}(pD_i - D_{i+1})$$

implies that $[D_a] \varepsilon V$, from which follows $[D_0] \varepsilon V$. q.e.d.

To prove a result in characteristic 0, we prove a variant of Theorem 1.3, (b) which is actually equivalent to Theorem 1.3,(b).

Lemma 1.5. Let Z be an effective 1-cycle on X such that $(Z \cdot c_1(X)) > 0$, and M an arbitrary ample divisor on X. Then there exists a rational curve Z' such that

$$\frac{n+1}{(M \cdot Z')} \geq \frac{(c_1(X) \cdot Z')}{(M \cdot Z')} \geq \frac{(c_1(X) \cdot Z)}{(M \cdot Z)} .$$

Proof. If we can prove the lemma in characteristic $p > 0$, we can prove the lemma in characteristic 0 by using the arguments on schemes over Spec $\mathbb{Z}$ because the inequility in the theorem gives an upper bound of $(M \cdot Z')$; $(M \cdot Z') \leq (n+1)(M \cdot Z)/(c_1(X) \cdot Z)$ which is independent of p (see the proof of Theorem 6 in [3]). Hence assuming that $p > 0$, we can apply Theorem 1.3, (a).

We choose ε so that $1/\varepsilon$ is a natural number and $(1/\varepsilon)M - 2(M\cdot Z)H$ is ample. Then there exist non-negative rational numbers $a_1, \cdots, a_r$ and $Y \in NE_\varepsilon(X)$ such that $[Z] = \Sigma a_i[\ell_i] + Y$. Since $Y \in NE_\varepsilon(X)$ and $(M\cdot Y) \geq 2\varepsilon(M\cdot Z)(H\cdot Y)$, we see $(c_1(X)\cdot Y) \leq \varepsilon(H\ Y) \leq (M\cdot Y)/2(M\cdot Z)$. Thus

$$\frac{(c_1(X)\cdot Z)}{(M\cdot Z)} \leq \frac{\Sigma a_i(c_1(X)\cdot \ell_i)+(M\cdot Y)/2(M\cdot Z)}{\Sigma a_i(M\cdot \ell_i)+(M\cdot Y)}$$

and since $a_i \geq 0$ and $(M\cdot Y) \geq 0$, we have

$$\frac{(c_1(X)\cdot Z)}{(M\cdot Z)} \leq \text{Max}\ \{\text{Max}_i \frac{(c_1(X)\cdot \ell_i)}{(M\cdot \ell_i)},\ \frac{1}{2(M\cdot Z)}\}.$$

Since $(c_1(X)\cdot Z) \geq 1$, we can take $Z' = \ell_i$ for some i. q.e.d.

Let us prove Theorem 1.3, (b). As in the proof of Theorem 3, (a), the set Φ of rational curves ℓ in X such that $(\ell\cdot c_1(X)) \leq n+1$ and $[\ell] \notin NE_\varepsilon(X)$ is bounded, and $\Phi/\approx$ has a complete set of representatives $\ell_1, \cdots, \ell_r$. We claim

<u>Lemma</u> 1.6. The cone $V = \mathbb{R}_+[\ell_1] + \cdots + \mathbb{R}_+[\ell_r] + \overline{NE}_\varepsilon(X)$ is closed in $N(X) \simeq \mathbb{R}^{\rho(X)}$.

<u>Proof</u>. Let $Z \in N(X)$ be a limit of $Z(i) = a(i, 1)\ell_1 + \cdots + a(i, r)\ell_r + Y(i)$ $(i \geq 1)$, where $a(i, j) \in \mathbb{R}_+$ and $Y(i) \in \overline{NE}_\varepsilon(X)$. Then the sequence $(Z(i)\cdot H)$ is bounded because $(Z(i)\cdot H) \to (Z\cdot H)$ as $i \to \infty$. Since $a(i, j) \leq (Z(i)\cdot H)/(\ell_j\cdot H)$ and $(Y(i)\cdot H) \leq (Z(i)\cdot H)$, the numbers $a(i, j)$, $(Y(i)\cdot H)$, and hence $||Y(i)||$ have a uniform upper bound by Kleiman's criterion. Thus there exists a subsequence $Z(n_i)$ such that $a(n_i, j)$ and $Y(n_i)$ converge as $i \to \infty$, whence $Z \in V$. q.e.d.

Going back to the proof of Theorem 1.3, (b), we will assume that $V \neq \overline{NE}(X)$ and show that this leads to a contradiction. By the ampleness of H, $\overline{NE}(X) \cap \{Y \in N(X) \mid (Y \cdot H) = 1\}$ is compact. Hence by the separation theorem for convex sets, there is an element $M \in NS(X) \otimes_{\mathbb{Z}} \mathbb{R}$ with the properties, (a) $M \geq 0$ on $\overline{NE}(X)$ and $M(Z) = 0$ for some non-zero Z in $\overline{NE}(X)$, and (b) $M > 0$ on $V - \{0\}$ considered as a real valued function on $N(X)$. By the above compactness, there exist a sequence $\{M_j\}_{j \geq 0}$ of ample divisors and a sequence $\{m_j\}_{j \geq 0}$ of natural numbers such that M is the limit of M_j/m_j in $NS(X) \otimes_{\mathbb{Z}} \mathbb{R}$ as $j \to \infty$. Let Z (given in the condition (a)) be the limit of $[Z_j]/n_j$, where Z_j is an effective 1-cycle and n_j a natural number. Since $V_1 = V \cap \{Y \in N(X) \mid (Y \cdot H) = 1\}$ is compact, $(c_1(X) \cdot Y)/(M_j/m_j \cdot Y)$ converge uniformly to $(c_1(X) \cdot Y)/(M \cdot Y)$ when $j \to \infty$ as functions on V_1. Hence $(c_1(X) \cdot Y)/(M_j/m_j \cdot Y)$ $(j \geq 0,$ $Y \in V - \{0\})$ are uniformly bounded. We have $(c_1(X) \cdot Z_j)/(M_j/m_j \cdot Z_j)$ $\to +\infty$ as $j \to \infty$ because $(M \cdot Z) = 0$ and $(c_1(X) \cdot Z) > 0$ $(Z \notin V)$. Hence for a sufficiently large j, we have

$$\frac{(c_1(X) \cdot Z_j)}{(M_j \cdot Z_j)} > \frac{(c_1(X) \cdot Y)}{(M_j \cdot Y)} \quad \text{for all } Y \in V - \{0\}.$$

If we apply Lemma 5 to these Z_j and M_j (note that $(c_j(X) \cdot Z_j)$ > 0), we get a rational curve ℓ such that

$$\frac{n+1}{(M_j \cdot \ell)} \geq \frac{(c_1(X) \cdot \ell)}{(M_j \cdot \ell)} \geq \frac{(c_1(X) \cdot Z_j)}{(M_j \cdot Z_j)} .$$

This inequality (together with the above) means that $\ell \notin V$ and $(c_1(X) \cdot \ell) \leq n+1$. Since $V \supseteq \overline{NE}_\varepsilon(X)$, we have $\ell \notin \overline{NE}_\varepsilon(X)$ and

$\ell \in \Phi$. This implies $[\ell] \in V$, which is a contradiction. Thus Theorem 1.3, (b) is proved.

§3. Concluding remarks.

A half line $R = \mathbb{R}_+[Z]$ in $N(X)$ is called an extremal ray if (1) $(Z \cdot c_1(X)) > 0$, and (2) Z_1 and Z_2 in $\overline{NE}(X)$ belong to R if $Z_1 + Z_2 \in R$. A rational curve ℓ in X is an extremal rational curve if $(\ell \cdot c_1(X)) \leq n+1$ and $\mathbb{R}_+[\ell]$ is an extremal ray.

It is not hard to restate Theorem 1.3 without using H and ε (cf. [4]). Here we simply state an immediate corollary.

Corollary 1.7. X has an extremal rational curve if and only if K_X is not numerically effective.

Only if part is obvious. If K_X is not numerically effective, then $\overline{NE}(X) \neq \overline{NE}_\varepsilon(X, H)$ for sufficiently small positive ε. Then at least one of ℓ_i's in Theorem 1.3 is an extremal rational curve.

Chapter 2. Threefolds with extremal rays.

Let k be an algebraically closed field of an arbitrary characteristic $p \geq 0$, and X a non-singular projective variety over k whose canonical bundle K_X is not numerically effective. By Corollary 1.7, X has an extremal ray R, which we fix in this chapter.

§1. Case of surfaces.

Let us begin with the case of surfaces X with an extremal ray R as an introduction to the case of 3-folds.

Theorem 2.1. If X is a surface with $\rho(X) > 1$, then we have one of the following 2 cases:

(1) X contains an exceptional curve E of the first kind such that $R = \mathbb{R}_+[E]$, or

(2) X is a $\mathbb{P}^1$-bundle over a curve C with structure morphism $\pi : X \longrightarrow C$ such that $R = \mathbb{R}_+[X_t]$, where $t \in C$.

This result treats the case where "adjunction terminates" except for the case $\rho(X) = 1$, which implies that $X \cong \mathbb{P}^2$. The following lemma is essential for Theorem 2.1.

<u>Lemma</u> 2.2. Under the assumption of Theorem 2.1, let ℓ be an extremal rational curve generating R. Then $(\ell^2) \leq 0$, where (ℓ^2) is the self-intersection number of ℓ on X.

<u>Proof</u>. Assuming $(\ell^2) > 0$, we will get a contradiction. Let H be an ample divisor on X. Let D be an arbitrary divisor on X such that $[D]/\|D\|$ is close enough to $\ell/\|\ell\|$ so that $(D^2) > 0$ and $(D.H) > 0$. Then $H^2(\mathcal{O}(nD)) = H^0(\mathcal{O}(K_X - nD)) = 0$ for $n >> 0$ because $((K_X - nD).H) < 0$ for $n >> 0$. Thus $h^0(\mathcal{O}(nD)) \geq \chi(\mathcal{O}(nD)) = n^2(D^2)/2 - n(D.K)/2 + \chi(\mathcal{O}_X)$ and $h^0(\mathcal{O}(nD)) > 0$ for $n >> 0$ because $(D^2) > 0$. Whence $[D] \in NE(X)$ which contradicts the assumption that ℓ is on the boundary of $\overline{NE}(X)$. Thus $(\ell^2) \leq 0$. q.e.d.

<u>Lemma</u> 2.3. Under the assumption and notation of Lemma 2.2, we have (1) ℓ is an exceptional curve of the first kind if $(\ell^2) < 0$, and (2) if $(\ell^2) = 0$, then an arbitrary irreducible curve D such that $[D] \in R$ has the property that $D \cong \mathbb{P}^1$,

$(D^2) = 0$, and $(D. K_X) = -2$.

Proof. From
$0 \leq p_a(\ell) = 1 + (\ell^2)/2 + (\ell.K_X)/2 \leq 1 + 0/2 + (-1)/2 = 1/2$, we see that $p_a(\ell) = 0$ and $(\ell^2) = (\ell.K) = -1$ if $(\ell^2) < 0$. Thus case (1) is proved. In case (2), we have $(D^2) = 0$ because $[D] \in \mathbb{R}_+[\ell]$ and $(\ell^2) = 0$. Hence just like case (1), we see that $D \cong \mathbb{P}^1$ and $(D. K_X) = -2$. q.e.d.

Proof of Theorem 2.1. By Lemma 2.3, it is enough to show that $|n\ell|$ is composite with a $\mathbb{P}^1$-bundle $X \longrightarrow C$ for $n >> 0$ if $(\ell^2) = 0$. By Riemann-Roch theorem, we see that $h^0(\mathcal{O}(n\ell)) \geq \chi(\mathcal{O}(n\ell)) = \chi(\mathcal{O}_X) + n$ for $n >> 0$. Thus, for some $n >> 0$, $|n\ell|$ is free from base points and fixed components (note: $(\ell^2) = 0$) and is composite with a pencil $\pi : X \longrightarrow C$ such that $\pi_*\mathcal{O}_X = \mathcal{O}_C$. Since $R = \mathbb{R}_+[\ell]$ is an extremal ray (an "edge" of $\overline{NE}(X)$), every irreducible component D of an arbitrary fiber of π belongs to R, whence $D \cong \mathbb{P}^1$ and $(D. K_X) = -2$ by Lemma 2.3. From $(D. K_X) = -2$, we see that every fiber of π is irreducible reduced and hence π is a $\mathbb{P}^1$-bundle. q.e.d.

Now Theorem 2.1 is generalized to the case of 3-folds X with $\rho(X) > 1$ as follows. Case (1) of an exceptional curve of the first kind is generalized to an exceptional divisor of several types which we classify explicitly. The complexity of the case reflects that of birational geometry of 3-folds. Case (2) of $\mathbb{P}^1$-bundle is generalized to two cases, a fiber space structure $X \rightarrow Y$ where (a) every fiber is a conic, or (b) every fiber

X_y is an irreducible reduced surface and ω_{X_y} is negative.

§2. Case of 3-folds.

In this section, we assume that the base field is of characteristic 0 and X is a 3-fold over k, with an extremal ray R.

Theorem 2.4. There exists a morphism $\phi : X \to Y$ to a projective variety Y such that (1) $\phi_*\mathcal{O}_X = \mathcal{O}_Y$ and (2) for any irreducible curve C in X, $[C] \in R$ if and only if $\dim \phi(C) = 0$. Furthermore, such a ϕ is unique up to isomorphisms.

The structure of this ϕ is given by the following theorems.

Theorem 2.5. The extremal ray R is not numerically effective if and only if $\dim Y = 3$. If these conditions are satisfied, then there exists an irreducible divisor D of X such that X is the blowing-up of Y by the ideal defining $\phi(D)$ (given the reduced structure), and we have either

(1) $\phi(D)$ is a non-singular curve and Y is non-singular ; $\phi|_D : D \to \phi(D)$ is a $\mathbb{P}^1$-bundle and $(D.\phi^{-1}(\eta)) = -1$ for any $\eta \in \phi(D)$,

(2) $Q = \phi(D)$ is a point and Y is non-singular ; $D \simeq \mathbb{P}^2$ and $\mathcal{O}_D(D) \simeq \mathcal{O}_{\mathbb{P}}(-1)$,

(3) $Q = \phi(D)$ is an ordinary double point of Y ; $D \simeq \mathbb{P}^1 \times \mathbb{P}^1$ and $\mathcal{O}_D(D) \simeq p_1{}^*\mathcal{O}_{\mathbb{P}}(-1) \otimes p_2{}^*\mathcal{O}_{\mathbb{P}}(-1)$, where p_i is the i-th projection,

(4) $Q = \phi(D)$ is a double point of Y ; D = an irreducible

reduced singular quadric surface S in $\mathbb{P}^3$, $\mathcal{O}_D(D) \simeq \mathcal{O}_S \otimes \mathcal{O}_{\mathbb{P}}(-1)$, or

(5) $Q = \phi(D)$ is a quadruple point of Y ; $D \simeq \mathbb{P}^2$, $\mathcal{O}_D(D) \simeq \mathcal{O}_{\mathbb{P}}(-2)$.

Let $\mathcal{O}_{Y,Q}$ be the local ring of Y at Q for cases (3), (4), and (5) in Theorem 2.5. Then we have

Theorem 2.6. (1) The divisor class group of $\mathcal{O}_{Y,Q}$ is 0 in cases (3) and (4), and $\mathbb{Z}/2\mathbb{Z}$ in case (5), and

(2) the completion $\mathcal{O}_{Y,Q}\hat{}$ of $\mathcal{O}_{Y,Q}$ is given by

$$\mathcal{O}_{Y,Q}\hat{} \simeq \begin{cases} k[[x,y,z,u]]/(x^2+y^2+z^2+u^2) & \text{case (3)}, \\ k[[x,y,z,u]]/(x^2+y^2+z^2+u^3) & \text{case (4)}, \\ k[[x,y,z]]^{(2)} & \text{case (5)}, \end{cases}$$

where $k[[x,y,z]]^{(2)}$ is the invariant subring of $k[[x,y,z]]$ under the action of the involution $(x,y,z) \to (-x,-y,-z)$.

The remaining cases are treated by

Theorem 2.7. If R is numerically effective, then Y is non-singular, $\rho(X) = \rho(Y) + 1$, and we have either

(1) $\dim Y = 2$, and for an arbitrary geometric point η of Y, the scheme-theoretic fiber X_η is isomorphic to a conic of $\mathbb{P}^2_{k(\eta)}$, where $k(\eta)$ is the field of η(i.e. X_η is isomorphic to either a smooth conic, a reducible conic, or a double line),

(2) $\dim Y = 1$, and for an arbitrary geometric point η of Y, X_η is an irreducible reduced surface such that $\omega_{X_\eta}^{-1}$ is ample, or

(3) $\dim Y = 0$, and X is a Fano 3-fold, (these 3-folds are

classified by Iskovski [6]).

References

[1] A. Grothendieck, Fondements de la geometrie algebrique, Secretariat Math., 11 Rue Pierre Curie, Paris 5^e (1962).

[2] R. Hartshorne, Ample subvarieties of algebraic varieties, Lecture Notes in Math., 156, Springer-Verlag (1970).

[3] S. Mori, Projective manifolds with ample tangent bundles, to appear in Annals of Math.

[4] S. Mori, Threefolds whose canonical bundles are not numerically effective, to appear in Proc. Nat. Acad. Sci. U.S.A. (1980).

[5] S. Mori and H. Sumihiro, On Hartshorne's conjecture, J. Math. Kyoto U., 18-3 (1978), 523-533.

[6] V.A. Iskovskih, Fano 3-folds II, Math. USSR Izv. 11 (1977).