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Kyoto University
On special values of zeta functions
associated with a self-dual cone

東北大 佐武一郎

以下に述べるは松島与三氏還暦記念論文集[Birdehausser]
のための著稿の一部である。京都の研究集会ではこの後半について不満しがちで、その要約を提出する予定であった。その後、都合上著稿のうえ、出させて頂くことにした。本文で説明した通り、ここに述べた方法は本質的に佐藤新氏[11]
のアイデアによるものである。r = 2 (circular cone) の場合
はより精密な計算をすることになるが、栗東氏で独立に結果
を得られること。これについてはまた別の機会に触れる
と思う。
§1. Introduction

To explain the main idea of this paper, and also to fix some notations, we start with reviewing the classical case of Riemann zeta function. As usual, we set

\[ \zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} \quad (\text{Re} \ s > 1), \]

\[ \Gamma(s) = \int_{0}^{\infty} x^{s-1} e^{-x} \, dx \quad (\text{Re} \ s > 0). \]

Then, for \( \text{Re} \ s > 1 \), one obtains

\[ \Gamma(s) \zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} \int_{0}^{\infty} x^{s-1} e^{-x} \, dx \]

\[ = \sum_{n=1}^{\infty} \int_{0}^{\infty} x^{s-1} e^{-nx} \, dx \quad (x = nx') \]

\[ = \int_{0}^{\infty} \frac{x^{s-1}}{e^{x} - 1} \, dx. \]

We put

\[ b(x, y) = \frac{e^{xy}}{e^x - 1} = \sum_{v=0}^{\infty} \frac{B_{v}(y)}{v!} x^{v-1} \quad (|x| < 2\pi), \]

where

\[ B_{v}(y) = \sum_{\mu=0}^{v} \binom{v}{\mu} b_{\mu} y^{v-\mu} \]

is the Bernoulli polynomial, in which the \( b_{\mu} \) are the Bernoulli numbers:

\[ b_{0} = 1, \quad b_{1} = -\frac{1}{2}, \]

\[ b_{v} = \begin{cases} (-1)^{\frac{v-1}{2}} B_{\frac{v}{2}} & (v \text{ even, } \geq 2), \\ 0 & (v \text{ odd, } \geq 3). \end{cases} \]

Then the above integral can be transformed into a contour integral of the form

\[ (1.1) \quad \Gamma(s) \zeta(s) = (e^{2\pi is} - 1)^{-1} \int_{I(\varepsilon, \infty)} x^{s-1} b(x, 0) \, dx, \]

where \( I(\varepsilon, \infty) \) denotes the contour consisting of the half-line \( [\varepsilon, \infty) \) taken twice in opposite directions and of a (small) circle of radius \( \varepsilon \).
about the origin taken in the counterclockwise direction. The contour integral
is absolutely convergent for all \( s \in \mathbb{C} \), so that the function \( \Gamma(s) \zeta(s) \) can
be analytically continued to a meromorphic function on \( \mathbb{C} \). Moreover, in virtue
of the functional equation of the gamma function:

\[
\Gamma(s)\Gamma(1-s) = \frac{\pi}{\sin \pi s} = 2\pi i \frac{e^{\pi i s}}{e^{2\pi i s} - 1},
\]

one obtains

\[
\zeta(s) = e^{-\pi i s} \frac{\Gamma(1-s)}{2\pi i} \frac{1}{\Gamma(t,\infty)} \int x^{s-1} b(x, 0) \, dx.
\]

This shows that \( \zeta(s) \) is holomorphic for \( \text{Re} \, s < 1 \). In particular, for
\( s = 1 - m, m \in \mathbb{Z}^+ \) (positive integers), the contour integral reduces to the
residue of \( x^{-m} b(x, 0) \) at \( x = 0 \), i.e., \( b_m/m! \). Hence one obtains

\[
\zeta(1 - m) = (-1)^{m-1} (m-1)! \frac{b_m}{m!} = (-1)^{m-1} \frac{b_m}{m}.
\]

Thus \( \zeta(1 - m) \, (m \in \mathbb{Z}^+) \) is rational. In particular,

\[
\zeta(0) = -\frac{1}{2}, \quad \zeta(-1) = -\frac{1}{12},
\]

\[
\zeta(-2\mu) = 0, \quad \zeta(1 - 2\mu) = (-1)^\mu \frac{B_\mu}{2^\mu} \quad (\mu \geq 1).
\]

This result has been generalized by Hecke, Klingens and Siegel [13] to
the case of Dedekind zeta functions of totally real number fields. More
recently, Shintani [14] gave a proof based on a direct generalization of the
classical method explained above. Zeta functions attached to self-dual homog-
oneous cones have been studied by Siegel [13] in a special case of quadratic
cones, and by Sato-Shintani [8] in a more general context of "prehomogeneous
spaces". (Cf. also Shintani [7], [10].) On the other hand, the gamma functions
attached to self-dual homogeneous cones were studied by Koecher [5], Gindikin
[3] and others (cf. e.g., Resnikoff [6]). In this paper, we try to extend
Shintani's method (i.e., the classical method) to examine the rationality of
the special values of zeta functions attached to self-dual homogeneous cones.
§2. The gamma function of a self-dual homogeneous cone

2.1. Let $U$ be a real vector space of dimension $n$, endowed with a positive definite inner product $< >$. By a "cone" in $U$ we always mean a non-degenerate open convex cone in $U$ with vertex at the origin, i.e., a non-empty open set $\mathcal{L}$ in $U$ such that

$$x, y \in \mathcal{L}, \lambda, \mu \in \mathbb{R}^+ \Rightarrow \lambda x + \mu y \in \mathcal{L}$$

and such that $\mathcal{L}$ does not contain any straight line. A cone $\mathcal{L}$ in $U$ is called **homogeneous** if the group of linear automorphisms

$$G(\mathcal{L}) = \{ g \in GL(U) \mid g(\mathcal{L}) = \mathcal{L} \}$$

is transitive on $\mathcal{L}$; and $\mathcal{L}$ is called **self-dual** if the "dual" of

$$\mathcal{L}^* = \{ x \in U \mid < x, y > > 0 \text{ for all } y \in \mathcal{L} - \{ 0 \} \}$$

coincides with $\mathcal{L}$.

Let $\mathcal{L}$ be a self-dual homogeneous cone in $U$ and $G = G(\mathcal{L})$. Then it is well-known (e.g., Satake [7]) that the Zariski closure of $G$ (in $GL(U)$) is a reductive algebraic group, containing $G(\mathcal{L})$ as a subgroup of finite index, and $g \mapsto t^g$ is a Cartan involution of $G$; the corresponding maximal compact subgroup $K = G \cap O(U)$ coincides with the isotropy subgroup of $G$ at a "base point" $e \in \mathcal{L}$ (which is not unique, but will be fixed once and for all). Let

$$\mathfrak{q} = \mathfrak{k} + \mathfrak{z}$$

be the corresponding Cartan decomposition of $\mathfrak{g} = \text{Lie } G$. Then $\mathfrak{k} = \text{Lie } K$ and one has for $T \in \mathfrak{q}$

$$(2.1) \quad T \in \mathfrak{k} \iff t_T = -T \iff T e = 0.$$  

It follows that, for each $u \in U$, there exists a uniquely determined element $T_u \in \mathfrak{z}$ such that $T_u e = u$. It is well-known that the vector space $U$ endowed with a product
\[ u \cdot u' = T_u u' \quad (u, u' \in U) \]

becomes a formally real Jordan algebra (cf. Braun-Koecher [2], or Satake [7]).

We define the (regular) trace on \( U \) by

\[ \tau(u) = \text{tr}(T_u). \]

For the given \((i, e)\), one may assume (by Schur's lemma) that the inner product \( < > \) is so normalized that one has

\[ < u, u' > = \tau(u \cdot u') \quad (u, u' \in U). \]

Next, let \( u \in \mathcal{L} \). Then, since \( G \) is transitive on \( \mathcal{L} \), there exists \( g_1 \in G \) such that \( u = g_1 e \). We define the (regular) norm \( N(u) \) by

\[ N(u) = \det(g_1), \]

which is clearly independent of the choice of \( g_1 \). There exists a unique element \( u_1 \in U \) such that \( u = \exp u_1 \) (which is defined to be \( \exp T_{u_1} e \)); then by definition one has

\[ N(u) = \det(\exp T_{u_1}) = e^{\tau(u_1)}. \]

In terms of the "quadratic multiplication" \( P(u) = 2 T_u^2 - T_u^4 \), one can also write \( N(u) = \det(P(u))^{\frac{1}{2}} \). By the definition, it is clear that

\[ N(e) = 1, \quad N(gu) = \det(g) N(u) \quad (g \in G(\mathcal{L}), u \in \mathcal{L}), \]

which characterizes the norm uniquely. Denoting the Euclidean measure on \( U \) by \( du \), we see that \( d_\mathcal{L}(u) = N(u)^{-1} du \) is an invariant measure on \( \mathcal{L} \).

**Example.** Let \( U = \text{Sym}_r(\mathbb{R}) \) (the space of real symmetric matrices of degree \( r \)) and \( \mathcal{L} = \mathcal{P}_r(\mathbb{R}) \) (the cone of positive definite elements in \( U \)). Then one has

\[ T_u(u') = \frac{1}{2} (uu' + u'u) \]

and so

\[ \tau(u) = \frac{r+1}{2} \text{tr}(u), \quad N(u) = \det(u)^{\frac{r+1}{2}}. \]
2.2. We define the gamma function of the cone $\mathcal{L}$ by

$$ \Gamma_{\mathcal{L}}(s) = \int_{\mathcal{L}} N(u)^{s-1} e^{-\tau(u)} \, du $$

which converges absolutely for $\Re s$ sufficiently large (actually for $\Re s > 1 - \frac{r}{n}$ as we will see later).

**Lemma 2.1.** Suppose that the inner product $< >$ is normalized by (2.3). Then one has for any $v \in \mathcal{L}$

$$ \int_{\mathcal{L}} N(u)^{s-1} e^{-<u, v>} \, du = \Gamma_{\mathcal{L}}(s) N(v)^{-s}. $$

**Proof.** Let $v = g_i e$ with $g_i \in G$ and put $u' = t_{g_i} u$. Then one has

$$ <u, v> = <u, g_i e> = <u', e> = \tau(u'). $$

Hence by (2.5) the left-hand side of (2.7) is equal to

$$ \int_{\mathcal{L}} N(u)^s e^{-<u, v>} \, d\mathcal{L}(u) $$

$$ = \int_{\mathcal{L}} (\det(g_i)^{-1} N(u'))^s e^{-\tau(u')} \, d\mathcal{L}(u') $$

$$ = N(v)^{-s} \Gamma_{\mathcal{L}}(s), \text{ q.e.d.} $$

It is known that the function $\Gamma_{\mathcal{L}}(s)$ can be expressed as a product of ordinary gamma functions (cf. e.g., Resnikoff loc. cit.). For the sake of completeness, we sketch a proof. First, it is clear that, if

$$ \mathcal{L} = \mathcal{L}_1 \times \cdots \times \mathcal{L}_m $$

is the decomposition of $\mathcal{L}$ into the direct product of irreducible (self-dual homogeneous) cones, then one has

$$ \Gamma_{\mathcal{L}}(s) = \Gamma_{\mathcal{L}_1}(s) \cdots \Gamma_{\mathcal{L}_m}(s). $$

Hence, for our purpose, we may assume that $\mathcal{L}$ is irreducible.

We need the root structure of $\mathcal{L}$, which can be determined as follows. Let

$$ e = \sum_{i=1}^{r} e_i, \quad e_i e_j = \delta_{ij} e_i $$

(2.8)
be a decomposition of $e$ (in the Jordan algebra $U$) into the sum of mutually orthogonal primitive idempotents. ("Primitive" means that each $e_i$ can not be decomposed into the sum of mutually orthogonal idempotents any more.) Then we obtain the following decomposition of $U$ into the direct sum of subspaces ("Peirce decomposition").

\[(2.9)\quad U = \bigoplus_{i \neq j} U_{ij},\]

where

\[
U_{ii} = \left\{ u \in U \mid e_i u = u \right\},
\]

\[
U_{ij} = \left\{ u \in U \mid e_i u = e_j u = \frac{1}{2} u \right\} \quad (i \neq j).
\]

Then one has $e_k u = 0$ for $u \in U_{ij}$, $k \neq i, j$. Moreover

\[(2.10)\quad \dim U_{ii} = 1, \quad \dim U_{ij} = d (i \neq j),\]

where $d$ is a positive integer depending on the irreducible cone $\mathcal{Q}$. (For instance, one has $d = 1$ for $\mathcal{Q} = \mathcal{P}_x(R)$.) From (2.9), (2.10) one has the relation

\[(2.11)\quad n = r + \frac{1}{2} r(r - 1)d, \quad \text{i.e.,} \quad d = \frac{2(n - r)}{r(r - 1)}.
\]

It follows that

\[(2.12)\quad \tau(e_i) = \text{tr}(T_{e_i}) = 1 + \frac{1}{2} (r - 1)d = \frac{n}{r},\]

\[\text{Put}\]

\[(2.13)\quad \mathcal{A} = \left\{ T_{e_i} \mid 1 \leq i \leq r \right\}.\]

Then $\mathcal{A}$ is an abelian subalgebra of $\mathcal{P}_x$ of dimension $r$ contained in $\mathcal{P}$. We denote by $(\lambda_i)$ the basis of $\mathcal{A}^*$ (the dual space of $\mathcal{A}$) dual to $(T_{e_i})$, i.e., one has the relation

\[T = \sum_{i=1}^{r} \lambda_i(T) T_{e_i} \quad (T \in \mathcal{A}).\]

We put $\alpha_{ij} = \frac{1}{2}(\lambda_i - \lambda_j)$ (if $i \neq j$).
PROPOSITION 1. The root system of $\Phi$ relative to $\mathfrak{a}_L$ is given by $\Phi = \{ \alpha_{ij} : (i \neq j) \}$. The root space $\gamma_i(\alpha_{ij})$ corresponding to $\alpha_{ij}$ is given by

$$\gamma_i(\alpha_{ij}) = \left\{ t_u + [T_{e_i - e_j}, T_u] \bigg| u \in U_{ij} \right\}.$$

This can be verified by a straightforward computation; see e.g., Ash et al. [1] Ch. II, §3. Proposition 1 implies that the R-rank of $\gamma_i$ is equal to $r$ and the root system $\Phi$ is of type $(A_{r-1})$.

2.3. Next we determine the Haar measure of $G$. Put

$$\nu = \sum_{i < j} \gamma_i(\alpha_{ij})$$

and let $A, N$ be the analytic subgroups of $G$ corresponding to $\mathfrak{a}_L, \nu$, respectively. Then one has an Iwasawa decomposition $G = NA \cdot K (N \times A \times K)$, which gives rise to the following formula for (the volume element of) a (biinvariant) Haar measure on $G$:

$$dg = c_1 e^{-2 \rho (\log a)} d\theta da dk$$

for $g = n a k$ with $n \in N, a \in A, k \in K$, where $dn, da, dk$ denote Haar measures on $N, A, K$, respectively, $c_1$ is a positive constant depending on the normalization of the Haar measures, and $\rho$ is a linear form on $\mathfrak{a}_L$ defined by

$$\rho(T) = \frac{1}{2} \text{tr}(\text{ad} T | \nu) \quad (T \in \mathfrak{a}_L).$$

by Proposition 1 one has

$$\rho = \frac{d}{2} \sum_{i < j} \alpha_{ij} = \frac{d}{2} \sum_{i=1}^{r} (r - 2i + 1) \lambda_i.$$

The Haar measure of $K$ is always normalized by $\int_K dk = 1$. We make an identification $A = (R^+)^r$ by the correspondence $a \leftrightarrow (t_i)$ defined by the relation $a = \exp(\sum \lambda_i T_{e_i}), t_i = e^{\lambda_i}$; then one has $da = \prod (dt_i/t_i)$.

Moreover one has

$$\det(a) = e^{\tau(\sum \lambda_i)} = e^{\frac{d}{2} \sum \lambda_i} = (\prod t_i)^{\frac{d}{2}},$$

$$a \cdot e = \sum e^{\lambda_i} e_i = \sum_{i=1}^{r} t_i e_i.$$
\[ e^2 p(\log a) = \prod_{i=1}^{v} t_i^{\frac{A}{2}(r-2i+1)}. \]

Since \( Q = G/K \), we can normalize the Haar measure of \( G \) by the relation \( dg = d_Q(u) \cdot dk \) where \( u = ge \). Then by (2.15), (2.16), (2.17) one has

\[
\begin{align*}
\overline{\mathcal{G}}(s) &= \int_G \mathcal{N}(ge)^s e^{-\tau(ge)} \, dg \\
&= c_1 \int_A \det(a)^s e^{-2p(\log a)} \, da \int_{\mathcal{N}} e^{-\tau(n \cdot ae)} \, dn \\
&= c_1 \int_0^\infty \cdots \int_0^\infty \left( t_i \prod_{i=1}^{r} \int_{s^{-\frac{d}{2}(r-2i+1)} \cdot i}^t dt_i \right) \\
& \quad \times \int_{\mathcal{N}} e^{-\tau(n(\sum t_i e_i))} \, dn.
\end{align*}
\]

To compute the integral over \( \mathcal{N} \), we introduce some notations. For \( u = \sum_{i<j} u_{ij} e_i e_j \), we put

\[
T_u^{(+)} = \frac{1}{2} (T_u + \sum_{i<j} [T_{e_i e_j}, T_{u_{ij}}]),
\]

\[
\mathcal{E}(u) = \sum_{i<j} \left[ \prod_{v=1}^{r} \sum_{u \leq i} \sum_{v \leq j} \sum_{u \leq k} \sum_{v \leq j} u_{ij}, u_{jk}, k_{k}, \ldots, u_{k,1}, j \right].
\]

Then one has The \( U_{ij} \)-component of \( \mathcal{E}(u) \) is denoted by \( \mathcal{E}_{ij}^{(+)}(u) \).

**Lemma 2.** The notation being as above, one has

\[
(\exp T_u^{(+)})(\sum_{i=j} t_i e_i) = \sum_{i=j} (t_i + \frac{1}{r} \sum_{k \neq j} t_k \mathcal{E}_{ik}^{(+)}(u)^2) e_i \\
+ \frac{1}{2} \sum_{i<j} (t_j \mathcal{E}_{ij}^{(+)}(u) + \sum_{k \neq j} t_k \mathcal{E}_{ik}^{(+)}(u) \mathcal{E}_{jk}^{(+)}(u)).
\]

This may be regarded as a generalization of the so-called "Jacobi transformation". The proof is again straightforward. It follows that, if \( n = \exp T_u^{(+)} (u \in \sum_{i<j} U_{ij}) \), one has

\[
\tau(n(\sum t_i e_i)) = \frac{n}{r} \sum t_i + \frac{1}{8} \sum_{i<k} \tau(\mathcal{E}_{ik}^{(+)}(u)^2) t_k.
\]

We denote the Euclidean measure on \( U_{ij} \) \( (i \neq j) \) (relative to the inner product \( < > \) ) by \( du_{ij} \) and define the Haar measure on \( \mathcal{N} \) by

\[
dn = \prod_{i<j} du_{ij}, \text{ for } n = \exp T_u^{(+)}.
\]

Since the map \( \mathcal{E}(+) \) is a bijection of \( \sum_{i<j} U_{ij} \) onto itself with Jacobian
equal to one, one has

\[ du = \prod_{i<j} du_{ij} = \prod_{i<j} du'_{ij}, \]

where \( u' = \mathcal{E}^{(+)}(u) \). Hence by (2.21) one has

\[
\int_{\mathcal{N}} e^{-\tau(n - t_j e_j)} \prod_{i<j} \left( \int_{U_{ij}} e^{-\frac{t_j}{\tau} \tau(u_{ij}^2)} du'_{ij} \right) dt_j = e^{-\frac{n}{\tau} \tau t_j} \prod_{i<j} \left( \int_{U_{ij}} e^{-\frac{8\pi}{\tau} \frac{d}{2} (j-1)} e^{-\frac{n}{\tau} t_j} \right).
\]

Inserting this in (2.18), one obtains

\[
\Gamma_{\mathcal{A}}(s) = c_1(8\pi)^{\frac{n-r}{2}} \prod_{j=1}^{r} \left( \int_{t_j}^{\infty} t_j^{\frac{n-r}{2} s-\frac{d}{2} (r-j) \cdot 1 - \frac{n}{\tau} t_j} dt_j \right)
\]

\[
= c_1(8\pi)^{\frac{n-r}{2}} \prod_{j=1}^{r} \left( \frac{n-r}{2} s + \frac{d}{2} (r-j) \cdot 1 \right) \Gamma \left( \frac{n}{\tau} s - \frac{d}{2} (r-j) \right)
\]

\[
= c_1(8\pi)^{\frac{n-r}{2}} \cdot c_1(8\pi)^{-\frac{d}{2} (r-j-1)} \prod_{j=1}^{r} \Gamma \left( \frac{n}{\tau} s - \frac{d}{2} (j-1) \right).
\]

The constant \( c_1 \) can be determined by the following observation. We set

\[
U_0 = \bigoplus_{i=1}^{r} U_i, \quad U_i = \left\{ e_i, \ldots, e_r \right\}_R
\]

and denote by \( du_0 \) the Euclidean measure on \( U_0 \) (relative to \( \langle \cdot, \cdot \rangle \)). Then, since \( \langle e_i, e_j \rangle = \frac{n}{\tau} \delta_{ij} \), the bijection \( A \rightarrow U_0 \) defined by \( a = \exp T_{u_0} \), or equivalently by \( a e = \exp u_e \), gives the relation

\[
du_0 = \left( \frac{n}{\tau} \right)^{\frac{1}{2}} da.
\]

Hence, when

\[
u = (ga)e = B(\sum_{i=1}^{r} t_i e_i),
\]

\[
n = \exp T_{x}, \quad x \in \bigoplus_{i<j} U_{ij}, \quad x' = \mathcal{E}^{(+)}(x),
\]

one has by Lemma 2

\[
\varphi(u) = \varphi(u_0, u_{ij}) = \left( \frac{n}{\tau} \right)^{\frac{1}{2}} \prod_{j=1}^{r} \left( \frac{r-d}{2} (j-1) \right) d,
\]

\[
\varphi(t, x) = \varphi(t_i, x_{ij}) = \left( \frac{n}{\tau} \right)^{\frac{1}{2}} \prod_{j=1}^{r} \left( \frac{r-d}{2} (j-1) \right) d
\]
\[
2^{-r-n} \left( \frac{n}{r} \right)^{\frac{r}{2}} \prod_{j=1}^{r} r_j^{\alpha(j-1)} d_j.
\]

It follows that
\[
d_{\alpha}(u) = 2^{-r-n} \left( \frac{n}{r} \right)^{\frac{r}{2}} \prod_{j=1}^{r} (r_j^{\alpha(j-1)} \alpha(j-1)) d_j.
\]

which, in view of (2.11) and (2.16), implies (2.15) and the relation
\[
(2.22) \quad c_\alpha = 2^{-r-n} \left( \frac{n}{r} \right)^{\frac{r}{2}}.
\]

Thus we obtain the formula
\[
(2.23) \quad \int_{\mathbb{R}} (2\pi)^{\frac{n}{2}} \left( \frac{n}{r} \right)^{\frac{r}{2}} \prod_{j=1}^{r} \Gamma \left( \frac{n}{r} s - \frac{d}{2} (j-1) \right).
\]

Our computation also shows that the integral for \( \int_{\mathbb{R}} (s) \) converges absolutely for \( \text{Re} > 1 - \frac{r}{n} \).

From the relation (1.2) one obtains
\[
\int_{\mathbb{R}} (s) \int_{\mathbb{R}} (1-s) = (2\pi)^{n-r} \prod_{j=1}^{r} \Gamma \left( \frac{n}{r} s - \frac{d}{2} (j-1) \right) \Gamma \left( \frac{n}{r} (1-s) - \frac{d}{2} (r-j) \right)
\]

\[
= (2\pi)^{n-r} (2\pi i)^{\frac{r}{2}} \prod_{j=1}^{r} \frac{e^{-\pi i \left( \frac{3}{2} s - \frac{d}{2} (j-1) \right)}}{e^{2\pi i \left( \frac{n}{r} s - \frac{d}{2} (j-1) \right)}} - 1.
\]

Since one has by (2.11)
\[
n - r = d \frac{r(r-1)}{2} = \begin{cases} 0 \pmod{2} & \text{for } d \text{ even} \\ \left\lfloor \frac{r}{2} \right\rfloor \pmod{2} & \text{for } d \text{ odd}, \end{cases}
\]

one has
\[
\prod_{j=1}^{r} e^{-\pi i \frac{d}{2} (j-1)} = (-1)^d \frac{r(r-1)}{2} = \begin{cases} i^{n-r} & \text{for } d \text{ even} \\ i^{n-r} (-1)^{\left\lfloor \frac{r}{2} \right\rfloor} & \text{for } d \text{ odd}. \end{cases}
\]

Hence one obtains the following functional equation:
\[
(2.24) \quad \int_{\mathbb{R}} (s) \int_{\mathbb{R}} (1-s) = (2\pi i)^n \pi i^{n-r} \begin{cases} (e^{2\pi i \frac{n}{r} s} - 1)^{-r} & \text{(d even)} \\ (e^{2\pi i \frac{n}{r} s} - 1)^{-\left\lfloor \frac{r}{2} \right\rfloor} (e^{2\pi i \frac{n}{r} s + 1})^{-r} & \text{(d odd)}. \end{cases}
\]
§3. Zeta functions of a self-dual homogeneous cone.

3.1. We fix a \( \mathbb{Q} \)-structure on \( U \) and assume that (the Zariski closure of) \( G \) is defined over \( \mathbb{Q} \) and \( e \in U_{\mathbb{Q}} \); then (the Zariski closure of) \( K \) is also defined over \( \mathbb{Q} \). We also fix a lattice \( L \) in \( U \) compatible with that \( \mathbb{Q} \)-structure, i.e., such that \( U_{\mathbb{Q}} = L \oplus \mathbb{Z} \), and an arithmetic subgroup \( \Gamma \) fixing \( L \), i.e., a subgroup of \( G_L = \{ g \in G \mid gL = L \} \) of finite index; for simplicity we assume that \( \Gamma \) has no fixed point in \( \mathcal{L} \). We then define the zeta function associated with \( \mathcal{L} \), \( \Gamma \), \( L \) as follows:

\[
\zeta_{\mathcal{L}}(s; \Gamma, L) = \sum_{u \in \Gamma \backslash \mathcal{L} \cap L} N(u)^{-s},
\]

the summation being taken over a complete set of representatives of \( \mathcal{L} \cap L \) modulo \( \Gamma \). It can be shown easily that this series is absolutely convergent for \( \text{Re} \ s > 1 \).

By the reduction theory, \( \Gamma \) has a fundamental domain in \( \mathcal{L} \) which is a rational polyhedral cone. More precisely, there exists a finite set of simplicial cones

\[
\mathcal{C}^{(i)} = \{ v^{(i)}_1, \ldots, v^{(i)}_{k_i} \}_{R_+},
\]

\[
= \left\{ \sum_{j=1}^{k_i} \lambda_j v^{(i)}_j \mid \lambda_j \in \mathbb{R}_+ \right\} \quad (1 \leq i \leq m),
\]

where \( v^{(i)}_1, \ldots, v^{(i)}_{k_i} \) are linearly independent elements in \( \mathcal{L} \cap L \), such that

\[
\mathcal{L} = \bigcup_{1 \leq i \leq m} \mathcal{C}^{(i)}.
\]

It follows that

\[
\zeta(s; \Gamma, L) = \sum_{i=1}^{m} \sum_{u \in \mathcal{C}^{(i)} \cap L} N(u)^{-s}.
\]

For a set of linearly independent vectors \( v_1, \ldots, v_{\ell} \in L \), we put

\[
R((v_j), L) = \left\{ \sum_{j=1}^{\ell} \lambda_j v_j \mid 0 < \lambda_j \leq 1 \right\} \cap L,
\]

which is finite. Then \( u \in \mathcal{C}^{(i)} \cap L \) can be written uniquely in the form

\[
u = v_0 + \sum_{j=1}^{\ell} m_j v_j^{(i)}, \quad v_j \in R((v_j^{(i)}), L), \quad m \in \mathbb{Z}, m \geq 0.
\]
For a set of linearly independent vectors \( v_1, \ldots, v_c \in \mathcal{L} \cap V_q \) and \( v_o = \sum_j \alpha_j v_j \) \((\alpha_j \in \mathbb{Q}_+),\) we define a "partial zeta function" by
\[
(3.2) \quad \zeta_{\mathcal{L}}(s; (v_j), v_o) = \sum_{\nu_j > 0} N(v_o) + \sum_{j=1}^l m_j v_j^{-s},
\]
which will also be written as \( \zeta_{\mathcal{L}}(s; (v_j), (\alpha_j)) \). Then the zeta function \( \zeta_{\mathcal{L}}(s; \Gamma, L) \) can be written as a finite sum of partial zeta functions as follows:
\[
(3.3) \quad \zeta_{\mathcal{L}}(s; \Gamma, L) = \sum_{i=1}^\infty \sum_{v_o \in \mathcal{F}(L_{\nu_i})(\mathfrak{L})} \zeta_{\mathcal{L}}(s; (v_j^{(i)}), v_o).
\]
Hence the study of special values of \( \zeta_{\mathcal{L}}(s; \Gamma, L) \) is reduced to that of the partial zeta functions of the form \( (3.2) \).

3.2. Let \((v_j)\) and \(v_o\) be as above. Then by \((2.7)\) one obtains
\[
(\Gamma_{\mathcal{L}}(s) \Delta_{\mathcal{L}}(s; (v_j), v_o)) = \sum_{\nu_j > 0} N(v_o) \Gamma_{\mathcal{L}}(s; (v_j), v_o) + \sum_{j=1}^l m_j v_j^{-s}
\]
\[
= \sum_{\nu_j > 0} \int_{\mathcal{L}} N(u)^{s-1} e^{-\frac{t}{2} (\alpha_j + m_j)} \langle v_j, u \rangle \, du
\]
\[
= \int_{\mathcal{L}} N(u)^S \prod_{j=1}^l b(\langle v_j, u \rangle, l - \alpha_j) \, du
\]
\[
= \int_G \det(g)^s \prod_{j=1}^l b(\langle v_j, g - e \rangle, l - \alpha_j) \, dg.
\]
In the notation of \( \S 2 \), but this time using the decomposition \( G = KAK \), one has
\[
(3.4) \quad dg = c \Delta(a) \, dk \cdot da \cdot dk'
\]
for \( g = kak' \), \( k, k' \in K, a \in A \). Here \( c \) is a positive constant and
\[
\Delta(a) = \prod_{\alpha \in \Phi^+} \left( e^{\alpha(\log a)} - e^{-\alpha(\log a)} \right) d
\]
\[
= \left( \prod_{i=1}^r t_i \right)^{-\frac{d}{2}} (r-1) \, \Delta(t_1, \ldots, t_r) \, d,
\]
where \( \Delta(t_1, \ldots, t_r) = \prod_{i<j} (t_i - t_j) \) (cf. Helgason [5], Ch. X, \( \S 1 \)). Hence in view of \((2.11)\) and \((2.17)\) one has
\[
(3.5) \quad \Gamma_{\mathcal{L}}(s) \zeta_{\mathcal{L}}(s; (v_j), (\alpha_j)) = c \int_0^\infty \cdots \int_0^\infty \left( \prod_{i=1}^r t_i \right)^{\frac{d}{2}} \Delta(t) \, d\nu(t) \, \prod_{r=1}^\infty dt_i,
\]
where
\[ F(t_1, \ldots, t_r) = \int K \prod_{j=1}^{r} b(<v_j, k > t^j e_i >, 1 - \alpha_j) \, dk. \]

It is clear that \( F(t_1, \ldots, t_r) \) is holomorphic for \( \Re t_i > 0 \) (\( 1 \leq i \leq r \)).

Since \( K \) contains an element which induces any given permutation of \( e_1, \ldots, e_r \), the function \( F \) is symmetric. Hence, denoting by \( B_i \) an open simplicial cone in \( \mathbb{R}^r \) defined by \( t_1 > \ldots > t_r > 0 \), one has

\[ F_{\mathcal{A}}(s) \zeta_{\mathcal{A}} (s; (v_j), (\alpha_j)) = c \, r! \int_{B_1} \left( \prod t_i \right)^{\frac{r}{2} (s-1)} \Delta(t)^d F(t) \prod dt_i. \]

3.3. Still following Shintani [11], we make a change of variables \( (t_i) \rightarrow (t_1, \tau_2, \ldots, \tau_r) \) with \( \tau_i = t_i/t_{i-1} \) (\( 2 \leq i \leq r \)). Then \( B_1 \) can be expressed as

\[ B_1 = \left\{ (t_i) \mid t_i = t_1 \prod_{j=2}^{i} \tau_j, 0 < t_i < \infty, 0 < \tau_i < 1 \right\}. \]

Putting \( \tau_i = t_1 \), one has

\[ \varphi(t_1, \ldots, t_r) = \prod_{i=1}^{r} t_i^{\tau_i-1}, \]

\[ \prod t_i = \prod t_i^{\tau_i-1+1}, \]

\[ \Delta(t) = \prod_{i} \tau_i \frac{1}{2} (r-i+1)(r-1) \prod_{2 \leq i < j \leq r} (1 - \tau_i \ldots \tau_j). \]

It follows that the exponent of \( \tau_i \) in the integrand in (3.5') is equal to

\[ (r-i+1) \frac{r}{2} (s-1) + \frac{d}{2} (r-i+1)(r-1) + r - 1 \]

\[ = (r-i+1) \left\{ \frac{r}{2} s - \frac{d}{2} (i-1) \right\} - 1. \]

Hence one has

\[ \sum_{\mathcal{A}}(s) \zeta_{\mathcal{A}} (s; (v_j), (\alpha_j)) = c \, r! \int_{0}^{\infty} t^{ns-1} dt \]

\[ \prod_{i=2}^{r} \tau_i^{(r-i+1)} \left\{ \frac{r}{2} s - \frac{d}{2} (i-1) \right\} - 1 \frac{\varphi(t, \tau)}{\prod_{i=2}^{r} \tau_i} \]

where

\[ \tilde{F}(t_1, \tau) = \prod_{2 \leq i < j \leq r} (1 - \tau_i \ldots \tau_j) \frac{d}{\prod_{i=2}^{r} \tau_i} \]

\[ \prod_{i} \tau_i^{d} \]
3.4. We now assume that all $v_j$'s are in $\mathcal{D}$ (not on the boundary of $\mathcal{D}$).
(In the situation explained in 3.1, this means that the $\mathfrak{g}$-rank of $G$ is
equal to 1.) Then for any $v \in \mathcal{D} - \{0\}$, one has $\langle v_j, v \rangle > 0$; in particular,
\begin{equation}
\langle v_j, ke_i \rangle > 0 \quad \text{for all } k \in K, 1 \leq i \leq r.
\end{equation}

Put
\begin{equation}
\xi_j = \langle v_j, k \sum t_i e_i \rangle
= t_1 \langle v_j, k(e_i + \sum_{i=2}^{r} t_i \ldots t_i e_i) \rangle
= t_1 \langle v_j, ke_i \rangle (1 + \sum_{i=1}^{r} t_i \ldots t_i \langle v_j, ke_i \rangle).
\end{equation}

For the fixed $e_i, v_j$, choose $\rho, \rho_i > 0$ in such a way that
\begin{equation}
\begin{cases}
\sum_{i=2}^{r} t_i^{r-1} \frac{\langle v_j, ke_i \rangle}{\langle v_j, ke_i \rangle} < 1 & \text{for all } k \in K, 1 \leq j \leq r,
\rho_i < \frac{\tau_i}{\langle v_j, ke_i \rangle} & \text{for all } 1 \leq j \leq r.
\end{cases}
\end{equation}

The for
\begin{equation}
0 < |t_i| < \rho_i, \quad |\tau_i| < \rho \quad (2 \leq i \leq r),
\end{equation}
one has $0 < |\xi_j| < 2\tau_i$ and so $b(\xi_j, 1 - \xi_j)$ is holomorphic. Hence the
function $F(t) = F(t_1, t_1 \tau_i, \ldots, t_1 \tau_i \ldots \tau_i)$ has a Laurent expansion in
t_1, \tau_i, \ldots, \tau_i in the domain defined by (3.11). The coefficients in
this expansion is a $\mathfrak{g}$-linear combination of the integrals of the form
\begin{equation}
I((v_j)) = \int_{\mathbb{C}} \prod_{\xi_j \neq \xi_i} \langle v_j, ke_i \rangle d\xi
\end{equation}
where $\nu_j > 0$ for $2 \leq i \leq r$ and $\nu_j \in \mathbb{Z}$ for all $i, j$. 
3.5. Let \( I(\varepsilon, 1) \) denote the contour consisting of the line segment \([\varepsilon, 1]\) taken twice in opposite directions and of a (small) circle of radius \( \varepsilon \) about the origin taken in the counterclockwise direction. When the \( \tau_i \) \((2 \leq i \leq r)\) are on \( I(\varepsilon, 1) \), one has by (2.12)

\[
|<v_j, k(e_i + \sum_{i=2}^{r} \tau_i \ldots \tau_i e_i)>| \leq |v_j| \sum_{i=1}^{r} |e_i| = \sqrt{\pi r} |v_j|
\]

and

\[
\text{Re} <v_j, k(e_i + \sum_{i=2}^{r} \tau_i \ldots \tau_i e_i)> = <v_j, ke_i> + \sum_{i=2}^{r} \text{Re}(\tau_i \ldots \tau_i k) <v_j, ke_i> \\
\geq <v_j, ke_i> - \varepsilon |v_j| \sum_{i=2}^{r} |e_i| \\
= <v_j, ke_i> - \varepsilon (r-1) \sqrt{\pi r} |v_j|.
\]

We choose \( \varepsilon \) so that one has

\[
(3.13) \quad \varepsilon \sqrt{\frac{\pi}{r}} |v_j| < \min \left\{ 2\pi, <v_j, ke_i> \mid (k \in K) \right\} \quad \text{for all } 1 \leq j \leq l,
\]

The above inequalities show that \(<v_j, k(e_i + \sum_{i=2}^{r} \tau_i \ldots \tau_i e_i)>\) belongs to the domain

\[
\left\{ z \in \mathbb{C} \mid |z| < \frac{2\pi}{\varepsilon}, \text{Re} z > \varepsilon \sqrt{\frac{\pi}{r}} |v_j| \right\}.
\]

It follows that, if \( \tau_i \) is on the contour \( I(\varepsilon, \infty) \), one has

\[
|\xi_j| < 2\pi \quad \text{or} \quad \text{Re} \xi_j > 0,
\]

so that the function \( b(\xi_j, 1 - \alpha_j) \) is holomorphic.

From this observation, it is clear that the integral on the r.h.s. of (3.6) is equal to the contour integral

\[
(e^{2\pi i s} - 1)^{-1} \int_{\tau_i \in I(\xi, \infty)} \int_{\tau_i \in I(\varepsilon, 1)} (e^{2\pi i s} \tau_i^{-1 + i s} - 1)^{-1}
\]

which is independent of the choice of \( \varepsilon \) satisfying (3.13). As is easily seen, the contour integral converges for all \( s \in \mathbb{C} \). Hence the integral \( \int_{\tau_i \in I(\xi, \infty)} \int_{\tau_i \in I(\varepsilon, 1)} \) viewed as a function in \( s \), can be continued to a meromorphic function on the whole plane; the possible poles are of the form \( \frac{1}{(r-i+1)n} \) \((v \in \mathbb{Z})\).
§ 4. The special values of the zeta functions.

4.1. As a preliminary, we check the rationality of the constant \( c \) in (3.4).

For that purpose, we compute \( \Gamma_{\mathcal{D}}(s) \) by using the decomposition \( G = \mathcal{K} \mathcal{K} \).

\[
\Gamma_{\mathcal{D}}(s) = \int_{\mathcal{D}} N(u)^s e^{-\tau(u)} du
= \int_{G} N(ge)^s e^{-\tau(ge)} dg
= c \int_{\mathcal{A}} \det(a)^s e^{-\tau(ae)} \Delta(a) da
= c \int_{0}^{\infty} \cdots \int_{0}^{\infty} (\prod_{i} t_i) \frac{\Phi(s-l)}{\Delta(t)} \prod_{i} e^{-\frac{t_i}{t}} \prod_{i} dt_i.
\]

We make another change of variables:

\[ t = \prod_{i=1}^{r} t_i, \quad t'_i = t_i / t. \]

Then

\[
\frac{\Theta(t_1, \ldots, t_r)}{\Theta(t, t', \ldots, t'_{r-1})} = (-1)^{r-l} t^{r-l},
\]

and the exponent of \( t \) in the integrand in the last member of (4.1) is equal to

\[ n(s-l) + \frac{d}{2} r(r-1) + r - l = ns - l. \]

Hence one has

\[
\Gamma_{\mathcal{D}}(s) = c \cdot \gamma(s) \cdot \beta(s),
\]

where

\[
\begin{cases}
\gamma(s) = \int_{0}^{\infty} t^{ns-1} e^{-\frac{t}{r}} \frac{t}{r} dt = \left(\frac{r}{n}\right)^{ns} \Gamma(ns),
\beta(s) = \int_{t'_i > 0} \left\{ t'_1, \ldots, t'_{r-1}, (1 - \sum t'_i) \right\}^{\frac{s}{2}} \Delta(t'_1, \ldots, t'_{r-1}, 1 - \sum t'_i) \prod_{i} dt'_i \cdot
\end{cases}
\]

For \( s = 1 \), one has

\[
\Gamma_{\mathcal{D}}(1) = c \cdot \gamma(1) \cdot \beta(1) = c \left(\frac{r}{n}\right)^{n} (n-1)! \cdot \beta(1),
\]
\[ \beta(1) = \int_{t_i > 0}^{\infty} |\Delta(t_i, \ldots, t_{i-1}, 1-\sum t_i')|^d \prod dt_i' \in \Omega. \]

By (2.23) one has

\[ (4.4) \quad \gamma_\Omega(1) = (2\pi)^{n-r} \left( \frac{\chi}{\pi} \right)^k \prod_{j=1}^r \Gamma(1 + \frac{d}{2} (j-1)) \]

\[ \sim \begin{cases} \mathcal{P}^{n-r} & (d \text{ even}) \\ \Omega^{n-\left[\frac{r+1}{2}\right]} & (d \text{ odd}) \end{cases} \]

where \( \mathcal{P} \sim \Omega \) means that \( a/b \in \Omega \). Thus one has

\[ (4.5) \quad c = \frac{(2\pi)^{n-r} \left( \frac{\chi}{\pi} \right)^k \prod_{j=1}^r \Gamma(1 + \frac{d}{2} (j-1))}{(n-1)! (\chi(1))} \sim \gamma_\Omega(1). \]

Since \( \gamma_\Omega(1) \sim \gamma_\Omega(1 + \frac{\chi}{n} \nu) \) for \( \nu \in \mathbb{Z} \), one obtains

\[ (4.6) \quad c \gamma_\Omega(1 + \frac{\chi}{n} \nu) \sim \gamma_\Omega(1)^2 \sim \begin{cases} \mathcal{P}^{n-r} & (d \text{ even}) \\ \Omega^{n-\left[\frac{r+1}{2}\right]} & (d \text{ odd}) \end{cases} \]

4.2. We first consider the case where \( d \) is even. Then by (2.24) one has

\[ \gamma_\Omega(s) \gamma_\Omega(1-s) = (2\pi i)^n e^{\pi \mathrm{Ins}(e^{2\pi i \frac{n}{r} s} - 1)^{-r}}. \]

Hence

\[ (4.7) \quad \zeta_\Omega(s; (\nu_j), (\kappa_j)) = \frac{c \gamma_\Omega(1-s)}{(2\pi i)^{n-r} e^{\pi \mathrm{Ins}}} \times R(s), \]

where

\[ R(s) = \left( \frac{e^{2\pi i \frac{n}{r} s} - 1}{2\pi i} \right) \prod_{i=1}^r \Delta(t_i) \gamma(t_i) \prod dt_i \]

\[ = \prod_{j=1}^r \frac{e^{2\pi i \frac{n}{r} s} - 1}{e^{2\pi i \frac{n}{r} s} - 1} x \frac{1}{(2\pi i)^n} \int_{I(\varepsilon, \infty)} t_i^{n-1} dt_i \]

\[ \left( \prod_{i=1}^r \int_{t_i}^{\infty} \prod_{\tau_i} \zeta_i^{(r-i+1)\frac{n}{r} s - \frac{d}{2} (i-1)^{-1} - r} \frac{1}{\prod d \tau_i} \right) \]

We are interested in the values of \( \zeta_\Omega \) at \( s = -\frac{\chi}{n} \nu \) (\( \nu = 0, 1, \ldots \)). The
first factor in the right hand side of (4.7) is holomorphic for \( \text{Re } s < \frac{r}{m} \) and by (4.6) the value at \( s = -\frac{r}{m} \nu \) is rational:

\[
(4.9) \quad \frac{c \int_0^1 (1+ \frac{r}{m} \nu) \frac{e^{2\pi i \frac{r}{m} s} - 1}{e^{2\pi i \frac{r}{m} (r-1) s} - 1}}{(2\pi i)^{n-r}} \frac{e^{-r \nu \pi i}}{e^{-r \nu \pi i}} = (-1)^{\frac{n-r}{2} + r \nu} \frac{c \int_0^1 (1+ \frac{r}{m} \nu)}{(2\pi)^{n-r}} \in \mathbb{Q}.
\]

On the other hand, it is clear that

\[
\frac{e^{2\pi i \frac{r}{m} s} - 1}{2\pi i \frac{r}{m} (r-1) s} \rightarrow \frac{1}{r-i+1} \quad \text{when } s \rightarrow -\frac{r}{m} \nu.
\]

Hence we see that \( R(-\frac{1}{m} \nu) \) is equal to the coefficient of

\[
t_i^{r \nu} \prod_{i=2}^{r} \tau_i^{(r-i+1)\frac{\nu}{2} + \frac{d}{2}(i-1)}
\]

in the Laurent expansion of \( \hat{F}(t_i, \tau) \),

which is a \( \mathbb{Q} \)-linear combination of \( \hat{I}((v_\nu')) \).

4.3. From now on we assume that \( d \) is odd. By the classification theory, it is known that this assumption implies that \( r = 2(n = d + 2) \) or \( d = 1 \) \((n = \frac{1}{2} r(r+1))\). By (2.24) one has

\[
\int_\mathbb{Q} (s) \int_\mathbb{Q} (1-s) = (2\pi i)^n e^{\pi i s} (e^{2\pi i \frac{r}{m} s} - 1)^{-\frac{r}{2}} (e^{2\pi i \frac{r}{m} s} + 1)^{\frac{r}{2}}.
\]

Hence

\[
(4.11) \quad \zeta_\mathbb{Q}(s; (v_\nu'), (\omega_\nu')) = \frac{c \int_\mathbb{Q} (1-s)}{(2\pi i)^n \int_\mathbb{Q} \frac{e^{\pi i s}}{e^{\pi i (r-1) s} - 1}} \times R^{(1)}(s) R^{(2)}(s),
\]

where

\[
R^{(0)}(s) = (2\pi i)^{\frac{r}{2}} \frac{r!}{\prod_{k=1}^{r} (e^{2\pi i (r-k+1)\frac{\nu}{2} s} - \frac{d}{2} (k-1)(1-1))},
\]

\[
R^{(1)}(s) = (2\pi i)^{-r} \int_{I(t_i, \omega)} \int_{I(t_i, \nu)} \prod_{i=1}^{r} (e^{2\pi i (r-i+1)\frac{\nu}{2} s} - \frac{d}{2} (i-1)) \frac{1}{\prod_{i=1}^{r} (r-i+1) \tau_i^{(r-i+1)\frac{\nu}{2} s} - \frac{d}{2} (i-1))} \frac{d\tau_i}{\prod_{i=1}^{r} (r-i+1) \tau_i^{(r-i+1)\frac{\nu}{2} s} - \frac{d}{2} (i-1))}.
\]
The first factor in the right hand side of (4.11) is holomorphic for $\Re s < \frac{r}{n}$ and by (4.6) the value at $s = -\frac{r}{n} \nu$ ($\nu \geq 0$) is rational:

$$
(4.12) \quad \frac{c \int_{\mathbb{C}} (1 + \frac{r}{n} \nu) e^{-\pi i r \nu}}{(2\pi i)^{n-[\frac{r+1}{2}]} e^{-\pi i r \nu}} = (-1)^{\frac{1}{2} (n-[\frac{r+1}{2}])} \frac{c \int_{\mathbb{C}} (1 + \frac{r}{n} \nu) e^{-\pi i r \nu}}{(2\pi i)^{n-[\frac{r+1}{2}]} e^{-\pi i r \nu}} \in \mathbb{Q}.
$$

Note that one has

$$
n \equiv \left[ \frac{r+1}{2} \right] \pmod{2},
$$

since

$$
n = d+2 \equiv 1 \equiv \left[ \frac{3}{2} \right] \pmod{2} \quad \text{if} \quad r = 2, \quad \text{and}
$$

$$
n = \frac{1}{2} r (r+1) \equiv \left[ \frac{r+1}{2} \right] \pmod{2} \quad \text{if} \quad d = 1.
$$

4.4. To compute $R^{(\nu)}(s)$, we first note

$$
e^{\pi i d k - (r-k+1)(r-k-1)} = \begin{cases} 
-1 & \text{if } k \equiv r \equiv 0 \pmod{2}, \\
1 & \text{otherwise}.
\end{cases}
$$

We put

$$
\left[ \frac{r}{2} \right] = r, \quad \zeta = e^{2\pi i \frac{r}{n} s}.
$$

The case $r$ is odd. One has

$$
R^{(\nu)}(s) = (2\pi i)^{\frac{r}{n}} r! \prod_{k=1}^{\frac{r}{2}} \left( \frac{\zeta - 1}{\zeta^k - 1} \right) = \prod_{k=1}^{\frac{r}{2}} \left( \frac{\zeta - 1}{\zeta^k - 1} \right)^{\frac{r}{n}}.
$$

Hence, when $s \to -\frac{r}{n} \nu$, one has

$$
(4.13) \quad \left( s + \frac{r}{n} \nu \right)^{\frac{r}{n}} R^{(\nu)}(s) \to (2\frac{r}{n})^{\frac{r}{n}}.
$$

Thus $R^{(\nu)}(s)$ has a pole of order $r_1$ at $s = -\frac{r}{n} \nu$.

The case $r$ is even. One has

$$
R^{(\nu)}(s) = (2\pi i)^{\frac{r}{n}} r! \prod_{k=1}^{\frac{r}{2}} \left\{ (-1)^k \frac{\zeta^k - 1}{\zeta^k - 1} \right\}.
$$
\[ (-2\pi i)\prod_{k \leq 1 \text{ even}}^{r!} (\zeta^{k-1} + \ldots + \zeta + 1) \prod_{k \leq 1 \text{ odd}}^{r!} (\zeta^{k-1} - \ldots - \zeta + 1) \]

Hence \( R^{(s)} \) is holomorphic at \( s = -\frac{r}{n} \nu \) and

\[ R^{(s)}(-\frac{r}{n} \nu) = (-2\pi i)^r \frac{r!}{(2r)!} = (-\pi i)^r \frac{r!}{r!} \]

4.5. When \( r \) is odd (hence \( d = 1 \), \( n = \frac{1}{2} r(r+1) \)), \( R^{(s)}(s) \) for \( s = -\frac{r}{n} \nu \)

is given by the coefficient of

\[ t^r \prod_{i=2}^{r} \tau_i^{(r-i+1)(\nu + \frac{i-1}{2})} \]

in the Laurent expansion of \( \widetilde{F}(t, \tau) \). Hence \( \zeta_{\mathcal{D}}(s; (\nu), (\alpha_j)) \) has at most a pole of order \( r_i = \frac{r-1}{2} \) at \( s = -\frac{2\nu}{r+1} \) and one has

\[ \lim_{s \to -\frac{2\nu}{r+1}} (s + \frac{2\nu}{r+1})^r \zeta_{\mathcal{D}}(s; (\nu), (\alpha_j)) \sim R^{(s)}(-\frac{2\nu}{r+1}). \]

To treat the case \( r \) is even, we use the formula

\[ \int_{I(\nu,1)} t^{m-1} dt = -\frac{\nu}{m} \quad (m \text{ odd}), \]

which can be verified easily. When \( r \) is even, the value of \( R^{(s)}(s) \) for \( s = -\frac{r}{n} \nu \) is given by

\[ (-\pi i)^r \prod_{m_i, \ldots, m_i \in \mathbb{Z}} \frac{a(s)}{\prod_{j=1}^{r} \tau_i^{(m_j - (r-2j+1)(\nu + d(j-1))}}) \]

where \( a(s) \) is the coefficient of

\[ t^r \prod_{j=1}^{r} \tau_j^{(r-2j+2)(\nu + d(j-1))} \prod_{j=1}^{r} \tau_j^{m_j} \]

in \( \widetilde{F}(t, \tau) \). Hence for the value of \( \zeta_{\mathcal{D}} \), one has

\[ \zeta_{\mathcal{D}}(-\frac{r}{n} \nu ; (\nu), (\alpha_j)) \sim (2\pi i)^r R^{(s)}(-\frac{r}{n} \nu). \]
Bibliography


[6] H. L. Resnikoff,


