On special values of zeta functions
associated with a self-dual cone

東北大 佐武一郎

以下に述べるは松島与三氏還暦記念論文集 (Birkhäuser) のための寄稿の一部である。京都の研究集会ではこの役半について不討しつつので、その要約を誤出す予定であったが、都合上寄稿の省略を出さざるをえなければならぬ。本文で説明しにくく、ここに述べた手法は本質的に松原沢氏 [11] のアイデアに帰するものである。$r = 2$ (circular cone) の場合にはより精密な計算をすることにでき、果実期で独立に結果を得られるか、これについてはすでに別の機会に触れたいと思う。
§1. Introduction

To explain the main idea of this paper, and also to fix some notations, we start with reviewing the classical case of Riemann zeta function. As usual, we set

\[ \zeta(s) = \sum_{n=1}^{\infty} \frac{n^{-s}}{n^{-s}} \quad (\text{Re } s > 1), \]

\[ \Gamma(s) = \int_{\epsilon}^{\infty} x^{s-1} e^{-x} \, dx \quad (\text{Re } s > 0). \]

Then, for \( \text{Re } s > 1 \), one obtains

\[ \Gamma(s) \zeta(s) = \sum_{n=1}^{\infty} \frac{n^{-s}}{n^{-s}} \int_{\epsilon}^{\infty} x^{s-1} e^{-x} \, dx \]

\[ = \sum_{n=1}^{\infty} \int_{\epsilon}^{\infty} x^{s-1} e^{-nx} \, dx' \quad (x = nx') \]

\[ = \int_{\epsilon}^{\infty} \frac{x^{s-1}}{e^x - 1} \, dx. \]

We put

\[ b(x, y) = \frac{x^y}{e^x - 1} = \sum_{\nu=0}^{\infty} \frac{B_\nu(y)}{\nu!} x^{\nu - 1} \quad (|x| < 2\pi), \]

where

\[ B_\nu(y) = \sum_{\mu=0}^{\nu} \binom{\nu}{\mu} b_\mu y^{\nu - \mu} \]

is the Bernoulli polynomial, in which the \( b_\mu \) are the Bernoulli numbers:

\[ b_0 = 1, \quad b_1 = -\frac{1}{2}, \]

\[ b_\nu = \begin{cases} (-1)^{\nu-1} \frac{B_\nu}{\nu} & (\nu \text{ even, } \geq 2), \\
0 & (\nu \text{ odd, } \geq 3). \end{cases} \]

Then the above integral can be transformed into a contour integral of the form

\[ (1.1) \quad \Gamma(s) \zeta(s) = (e^{2\pi i s} - 1)^{-1} \int_{\Gamma(\epsilon, \infty)} x^{s-1} b(x, 0) \, dx, \]

where \( \Gamma(\epsilon, \infty) \) denotes the contour consisting of the half-line \([\epsilon, \infty)\) taken twice in opposite directions and of a (small) circle of radius \( \epsilon \).
about the origin taken in the counterclockwise direction. The contour integral is absolutely convergent for all \( s \in \mathbb{C} \), so that the function \( \Gamma(s) \zeta(s) \) can be analytically continued to a meromorphic function on \( \mathbb{C} \). Moreover, in virtue of the functional equation of the gamma function:

\[
\Gamma(s)\Gamma(1-s) = \frac{\pi}{\sin \pi s} = 2\pi i \frac{e^{\pi is}}{e^{2\pi is} - 1},
\]

one obtains

\[
\zeta(s) = e^{-\pi is} \frac{\Gamma(1-s)}{2\pi i} \frac{1}{\Gamma(t, \omega)} \int x^{s-1} b(x, 0) \, dx.
\]

This shows that \( \zeta(s) \) is holomorphic for \( \text{Re} \ s < 1 \). In particular, for \( s = 1 - m, m \in \mathbb{Z}^+ \) (positive integers), the contour integral reduces to the residue of \( x^{-m} b(x, 0) \) at \( x = 0 \), i.e., \( b_m / m! \). Hence one obtains

\[
\zeta(1 - m) = (-1)^{m-1} (m-1)! \frac{b_m}{m!} = (-1)^{m-1} \frac{b_m}{m}.
\]

Thus \( \zeta(1 - m) \) \((m \in \mathbb{Z}^+)\) is rational. In particular,

\[
\zeta(0) = -\frac{1}{2}, \quad \zeta(-1) = -\frac{1}{12},
\]

\[
\zeta(-2\mu) = 0, \quad \zeta(1 - 2\mu) = (-1)^{\mu} \frac{B_{2\mu}}{2\mu} \quad (\mu \geq 1).
\]

This result has been generalized by Hecke, Klingen and Siegel [13] to the case of Dedekind zeta functions of totally real number fields. More recently, Shintani [11] gave a proof based on a direct generalization of the classical method explained above. Zeta functions attached to self-dual homogeneous cones have been studied by Siegel [13] in a special case of quadratic cones, and by Sato-Shintani [8] in a more general context of "prehomogeneous spaces". (Cf. also Shintani [7], [10].) On the other hand, the gamma functions attached to self-dual homogeneous cones were studied by Koecher [5], Gindikin [3] and others (cf. e.g., Resnikoff [6]). In this paper, we try to extend Shintani's method (i.e., the classical method) to examine the rationality of the special values of zeta functions attached to self-dual homogeneous cones.
\section{The gamma function of a self-dual homogeneous cone}

\section*{2.1. Let $U$ be a real vector space of dimension $n$, endowed with a positive definite inner product $\langle \cdot, \cdot \rangle$. By a "cone" in $U$ we always mean a non-degenerate open convex cone in $U$ with vertex at the origin, i.e., a non-empty open set $\mathcal{L}$ in $U$ such that $\forall x, y \in \mathcal{L}, \lambda, \mu \in \mathbb{R}^+ \Rightarrow \lambda x + \mu y \in \mathcal{L}$ and such that $\mathcal{L}$ does not contain any straight line. A cone $\mathcal{L}$ in $U$ is called homogeneous if the group of linear automorphisms $G(\mathcal{L}) = \{ g \in \text{GL}(U) \mid g(\mathcal{L}) = \mathcal{L} \}$ is transitive on $\mathcal{L}$; and $\mathcal{L}$ is called self-dual if the "dual" of $\mathcal{L}$ is \[ \mathcal{L}^* = \{ x \in U \mid \langle x, y \rangle > 0 \text{ for all } y \in \overline{\mathcal{L}} - \{0\} \} \] coincides with $\mathcal{L}$.

Let $\mathcal{L}$ be a self-dual homogeneous cone in $U$ and $G = G(\mathcal{L})^\circ$. Then it is well-known (e.g., Satake [7]) that the Zariski closure of $G$ (in $\text{GL}(U)$) is a reductive algebraic group, containing $G(\mathcal{L})$ as a subgroup of finite index, and $g \mapsto {}^t g^{-1}$ is a Cartan involution of $G$; the corresponding maximal compact subgroup $K = G \cap O(U)$ coincides with the isotropy subgroup of $G$ at a "base point" $e \in \mathcal{L}$ (which is not unique, but will be fixed once and for all). Let $\mathcal{K}^\circ = \mathcal{K} + \mathcal{Y}$ be the corresponding Cartan decomposition of $\mathcal{Y} = \text{Lie } G$. Then $\mathcal{K} = \text{Lie } K$ and one has for $T \in \mathcal{Y}$

\begin{equation}
(2.1) \quad T \in \mathcal{K} \iff {}^t T = -T \iff Te = 0.
\end{equation}

It follows that, for each $u \in U$, there exists a uniquely determined element $T_u \in \mathcal{Y}$ such that $T_u e = u$. It is well-known that the vector space $U$ endowed with a product
\( u \cdot u' = T_u u' \quad (u, u' \in U) \)

becomes a formally real Jordan algebra (cf. Braun-Koecher [2], or Satake [7]).

We define the (regular) trace on \( U \) by

\[
\tau(u) = \text{tr}(T_u).
\]

For the given \( (\mathcal{L}, \cdot) \), one may assume (by Schur's lemma) that the inner product \( < \cdot, \cdot > \) is so normalized that one has

\[
<u, u'> = \tau(u \cdot u') \quad (u, u' \in U).
\]

Next, let \( u \in \mathcal{L} \). Then, since \( G \) is transitive on \( \mathcal{L} \), there exists \( g \in G \) such that \( u = g, e \). We define the (regular) norm \( N(u) \) by

\[
N(u) = \det(g_1),
\]

which is clearly independent of the choice of \( g_1 \). There exists a unique element \( u_1 \in U \) such that \( u = \exp u_1 \) (which is defined to be \( \exp T_{u_1} e \)); then by definition one has

\[
N(u) = \det(\exp T_{u_1}) = e^{\tau(u_1)}.
\]

In terms of the "quadratic multiplication" \( P(u) = 2 T_u^2 - T_u^2 \), one can also write \( N(u) = \det(\exp T_{u_1})^{1/2} \). By the definition, it is clear that

\[
N(e) = 1, \quad N(\exp g) = \det(g) N(u) \quad (g \in G(\mathcal{L}), u \in \mathcal{L}),
\]

which characterizes the norm uniquely. Denoting the Euclidean measure on \( U \) by \( du \), we see that \( d_\mathcal{L}(u) = N(u)^{-1} du \) is an invariant measure on \( \mathcal{L} \).

**Example.** Let \( U = \text{Sym}_r(\mathbb{R}) \) (the space of real symmetric matrices of degree \( r \)) and \( \mathcal{L} = \mathcal{P}_r(\mathbb{R}) \) (the cone of positive definite elements in \( U \)).

Then one has

\[
T_u(u') = \frac{1}{2} (uu' + u'u)
\]

and so

\[
\tau(u) = \frac{r+1}{2} \text{tr}(u), \quad N(u) = \det(u)^{\frac{r+1}{2}}.
\]
2.2. We define the gamma function of the cone $\mathcal{L}$ by

$$\Gamma_\mathcal{L}(s) = \int_{\mathcal{L}} N(u)^{s-1} e^{-\tau(u)} \, du$$

which converges absolutely for $\Re s$ sufficiently large (actually for $\Re s > 1 - \frac{1}{n}$ as we will see later).

**Lemma 2.1.** Suppose that the inner product $\langle \cdot, \cdot \rangle$ is normalized by (2.3).

Then one has for any $v \in \mathcal{L}$

$$\int_{\mathcal{L}} N(u)^{s-1} e^{-\langle u, v \rangle} \, du = \Gamma_\mathcal{L}(s) N(v)^{-s}.$$  

**Proof.** Let $v = g_i e$ with $g_i \in G$ and put $u' = t g_i u$. Then one has

$$\langle u, v \rangle = \langle u, g_i e \rangle = \langle u', e \rangle = \tau(u').$$

Hence by (2.5) the left-hand side of (2.7) is equal to

$$\int_{\mathcal{L}} N(u)^{s} e^{-\langle u, v \rangle} \, d\mathcal{L}(u)$$

$$= \int_{\mathcal{L}} (\det(g_i)^{-1} N(u'))^{s} e^{-\tau(u')} \, d\mathcal{L}(u')$$

$$= N(v)^{-s} \Gamma_\mathcal{L}(s), \text{ q.e.d.}$$

It is known that the function $\Gamma_\mathcal{L}(s)$ can be expressed as a product of ordinary gamma functions (cf. e.g., Resnikoff loc. cit.). For the sake of completeness, we sketch a proof. First, it is clear that, if

$$\mathcal{L} = \mathcal{L}_1 \times \cdots \times \mathcal{L}_m$$

is the decomposition of $\mathcal{L}$ into the direct product of irreducible (self-dual homogeneous) cones, then one has

$$\Gamma_\mathcal{L}(s) = \Gamma_{\mathcal{L}_1}(s) \cdots \Gamma_{\mathcal{L}_m}(s).$$

Hence, for our purpose, we may assume that $\mathcal{L}$ is irreducible.

We need the root structure of $\mathcal{L}$, which can be determined as follows. Let

$$e = \sum_{i=1}^{r} e_i, \quad e_i e_j = \delta_{ij} e_i$$

(2.5)
be a decomposition of \( e \) (in the Jordan algebra \( U \)) into the sum of mutually orthogonal primitive idempotents. ("Primitive" means that each \( e_i \) can not be decomposed into the sum of mutually orthogonal idempotents any more.) Then we obtain the following decomposition of \( U \) into the direct sum of subspaces ("Peirce decomposition").

\[
U = \bigoplus_{1 \leq i \leq j \leq l} U_{ij},
\]

where

\[
U_{ii} = \left\{ u \in U \mid e_i u = u \right\},
\]

\[
U_{ij} = \left\{ u \in U \mid e_i u = e_j u = \frac{1}{2} u \right\} \quad (i \neq j).
\]

Then one has \( e_k u = 0 \) for \( u \in U_{ij}, \ k \neq i, j \). Moreover

\[
\dim U_{ii} = 1, \quad \dim U_{ij} = d \quad (i \neq j),
\]

where \( d \) is a positive integer depending on the irreducible cone \( \mathcal{Q} \). (For instance, one has \( d = 1 \) for \( \mathcal{Q} = \mathcal{P}_{r}(R) \).) From (2.9), (2.10) one has the relation

\[
n = r + \frac{1}{2} r(r - 1)d, \quad \text{i.e.,} \quad d = \frac{2(n - r)}{r(r - 1)}. \tag{2.11}
\]

It follows that

\[
\tau(e_i) = \text{tr}(T_{e_i}) = 1 + \frac{1}{2} (r - 1)d = \frac{n}{r}, \tag{2.12}
\]

Put

\[
\mathfrak{a} = \left\{ T_{e_i} \mid 1 \leq i \leq r \right\}.
\]

Then \( \mathfrak{a} \) is an abelian subalgebra of \( \mathfrak{o}_{r} \) of dimension \( r \) contained in \( \mathfrak{p} \).

We denote by \( (\lambda_{i}) \) the basis of \( \mathfrak{a}^{*} \) (the dual space of \( \mathfrak{a} \)) dual to \( (T_{e_i}) \), i.e., one has the relation

\[
T = \sum_{i=1}^{r} \lambda_{i}(T) T_{e_i} \quad (T \in \mathfrak{a}).
\]

We put \( \lambda_{i,j} = \frac{1}{2}(\lambda_{i} - \lambda_{j}) \) \((i \neq j)\).
PROPOSITION 1. The root system of $\mathfrak{g}$ relative to $\mathfrak{a}$ is given by $\Phi = \{ \alpha_{i,j} \mid i \neq j \}$. The root space $\mathfrak{g}(\alpha_{i,j})$ corresponding to $\alpha_{i,j}$ is given by

$$\mathfrak{g}(\alpha_{i,j}) = \left\{ T_u + [T_{e_i}-e_j, T_u] \mid u \in U_{i,j} \right\}.$$  

This can be verified by a straightforward computation; see e.g., Ash et al. [1] Ch. II, §3. Proposition 1 implies that the R-rank of $\mathfrak{g}$ is equal to $r$ and the root system $\Phi$ is of type $(A_{r-1})$.

2.3. Next we determine the Haar measure of $G$. Put

$$\nu = \sum_{i < j} \mathfrak{g}(\alpha_{i,j})$$

and let $A, N$ be the analytic subgroups of $G$ corresponding to $\mathfrak{a}, \nu$, respectively. Then one has an Iwasawa decomposition $G = N A \cdot K$ where $N \times A \times K$, which gives rise to the following formula for (the volume element of) a (biinvariant) Haar measure on $G$:

$$dg = c_1^{-1} e^{2\rho \left( \log a \right)} dn \, da \, dk$$

for $g = n a k$ with $n \in N, a \in A, k \in K$, where $dn, da, dk$ denote Haar measures on $N, A, K$, respectively, $c_1$ is a positive constant depending on the normalization of the Haar measures, and $\rho$ is a linear form on $\mathfrak{a}$ defined by

$$\rho(T) = \frac{1}{2} \text{tr} \left( \text{ad} T \mid \nu \right) \quad (T \in \mathfrak{a}).$$

by Proposition 1 one has

$$\rho = \frac{d}{2} \sum_{i < j} \chi_{ij} \cdot \chi_{ij} = \frac{d}{2} \sum_{i=1}^{r} (r - 2i + 1) \lambda_i.$$

The Haar measure of $K$ is always normalized by $\int_K dk = 1$. We make an identification $A = (\mathbf{R}^*)^r$ by the correspondence $a \longleftrightarrow (t_i)$ defined by the relation $a = \exp \left( \sum \lambda_i T_{e_i} \right)$, $t_i = e^{\lambda_i}$; then one has $da = \prod (dt_i/t_i)$. Moreover one has

$$\det(a) = e^{\rho \left( \sum \lambda_i e_i \right)} = e^{\frac{2}{2} \sum \lambda_i} = \prod_i t_i^{\frac{r}{2}},$$

$$a \cdot e = \sum_{i=1}^{r} e^{\lambda_i} e_i = \sum_{i=1}^{r} t_i e_i.$$
\[ e^2 \rho(\log a) = \prod_{i=1}^{r} t_i \frac{i}{i} (r-2i+1). \]

Since \( \mathcal{Q} = G/K \), we can normalize the Haar measure of \( G \) by the relation \( dg = d_\mathcal{Q}(u) \cdot dk \) where \( u = ge \). Then by (2.15), (2.16), (2.17) one has

\[
\int_{\mathcal{Q}} \xi(N(ge)) e^{-\tau(ge)} \, dg = c_1 \int_{A} \det(a)^{\frac{r}{2}} e^{-2 \rho(\log a)} \, da \int_{N} e^{-\tau(n, ae)} \, dn
\]

\[
= c_1 \int_{0}^{\infty} \cdots \int_{0}^{\infty} \prod_{i=1}^{r} t_i \frac{i}{i} (r-2i+1) \, dt_i \int_{N} e^{-\tau(n(\sum_{i=1}^{r} t_i e_i))} \, dn
\]

To compute the integral over \( N \), we introduce some notations. For \( u = \sum_{i<j} u_{ij} \in U \) with \( u_{ij} \in U_{ij} \), we put

\[
T_u^{(+)} = \frac{1}{2} (T_u + \sum_{i<j} [T_{e_i - e_j}, T_{u_{ij}}]),
\]

\[
\xi^{(+)}(u) = \sum_{i<j} \sum_{v=1}^{\infty} \frac{1}{v!} \sum_{i<k<\cdots<k_{v}} u_{ik} u_{k_{v}} \cdots u_{k_{v-1}} j.
\]

Then one has The \( U_{ij} \)-component of \( \xi^{(+)}(u) \) is denoted by \( \xi_{ij}^{(+)}(u) \).

**Lemma 2.** The notation being as above, one has

\[
(\exp T_u^{(+)})(\sum_{i=1}^{r} t_i e_i) = \sum_{i=1}^{r} (t_i + \frac{1}{r} \sum_{k>i} t_k \xi^{(+)}_{1k}(u)^2) e_i + \frac{1}{r} \sum_{i<j} (t_j \xi_{ij}^{(+)}(u) + \sum_{k>j} t_k \xi_{ik}^{(+)}(u) \xi_{jk}^{(+)}(u))
\]

This may be regarded as a generalization of the so-called "Jacobian transformation". The proof is again straightforward. It follows that, if \( n = \exp T_u^{(+)} \) \( (u \in \sum_{i<j} U_{ij}) \), one has

\[
\tau(n(\sum_{i=1}^{r} t_i e_i)) = \frac{n}{r} \sum_{i=1}^{r} t_i + \frac{1}{8} \sum_{i<k} \tau(\xi_{ik}^{(+)}(u)^2) t_k.
\]

We denote the Euclidean measure on \( U_{ij} \) \( (i<j) \) (relative to the inner product \( <> \)) by \( du_{ij} \) and define the Haar measure on \( N \) by

\[
dn = \prod_{i<j} du_{ij}
\]

for \( n = \exp T_u^{(+)} \).

Since the map \( \xi^{(+)} \) is a bijection of \( \sum_{i<j} U_{ij} \) onto itself with Jacobian
equal to one, one has

\[ du = \prod_{i < j} du_{i,j} = \prod_{i < j} du'_{i,j}, \]

where \( u' = \mathcal{E}'(u) \). Hence by (2.21) one has

\[
\int_N e^{-\tau (m \geq t_i, e_i)} \, dm = e^{-\frac{n}{r} \sum t_i} \prod_{i < j} \int_{U_{i,j}} e^{-\frac{t_j}{r} \tau (u_i \geq t_j)} \, du'_{i,j} \\
= e^{-\frac{n}{r} \sum t_i} \prod_{i < j} \left( \frac{8 \pi^{\frac{d}{2}}}{t_j} \right)^{\frac{d}{2}} \\
= \left( 8 \pi^{\frac{d}{2}} \right)^{\frac{d}{2}} \prod_{j} \left( t_j - \frac{4}{3} (j-1) e^{-\frac{n}{r} t_j} \right). \]

Inserting this in (2.18), one obtains

\[
\Gamma_{\mathcal{A}}(s) = c_1 \left( \frac{8 \pi^{\frac{d}{2}}}{8 \pi^{\frac{d}{2}}} \right)^{\frac{d}{2}} \prod_{j=1}^{\infty} \left( \int_{t_j}^{\infty} e^{\frac{n}{r} (s - \frac{d}{2} (r - j))} dt_j \right) \\
= c_1 \left( \frac{8 \pi^{\frac{d}{2}}}{8 \pi^{\frac{d}{2}}} \right)^{\frac{d}{2}} \prod_{j=1}^{\infty} \left( \frac{8 \pi^{\frac{d}{2}}}{8 \pi^{\frac{d}{2}}} \right)^{\frac{d}{2}} \left( \frac{8 \pi^{\frac{d}{2}}}{8 \pi^{\frac{d}{2}}} \right)^{\frac{d}{2}} \\
= c_1 \left( \frac{8 \pi^{\frac{d}{2}}}{8 \pi^{\frac{d}{2}}} \right)^{\frac{d}{2}} \prod_{j=1}^{\infty} \Gamma \left( \frac{n}{r} s - \frac{d}{2} (r - j) \right). \]

The constant \( c_1 \) can be determined by the following observation. We set

\[ U_0 = \bigcup_{i=1}^{R} U_{i,i} = \{ e_1, \ldots, e_R \}_R \]

and denote by \( du_0 \) the Euclidean measure on \( U_0 \) (relative to \( < > \)). Then, since \( \langle e_i, e_j \rangle = \frac{n}{r} \delta_{i,j} \), the bijection \( A \to \gamma_0 \) defined by \( a = \exp T_{u_o} \)

or equivalently by \( ae = \exp u \), gives the relation

\[ du_0 = \left( \frac{n^{\frac{d}{2}}}{r} \right) da. \]

Hence, when

\[ u = (pa) e = \mathcal{E}(\sum_{i=1}^{\infty} t_i e_i), \]

\[ n = \exp T_{x'}, x \in \sum_{i < j} U_{i,j}, \quad x' = \mathcal{E}'(x), \]

one has by Lemma 2

\[ \frac{\mathcal{E}(u)}{\mathcal{E}(t, x)} = \mathcal{E}(u_o, u_{i,j}) = \left( \frac{n^{\frac{d}{2}}}{r} \right) \prod_{i < j} \left( \frac{8 \pi^{\frac{d}{2}}}{t_j} (j-1) \right) d \]

\[ \frac{\mathcal{E}(u)}{\mathcal{E}(t, x)} = \mathcal{E}(t_i, x_{i,j}') = \left( \frac{n^{\frac{d}{2}}}{r} \right) \prod_{i < j} \left( \frac{8 \pi^{\frac{d}{2}}}{t_j} (j-1) \right) d. \]
\[ = 2^{r-n} \left( \frac{n}{r} \right)^{r} \prod_{j=1}^{r} t_{j}^{j-1} (j-1)^{d}. \]

It follows that
\[
\mathcal{d}_{\mathcal{Q}}(u) = 2^{r-n} \left( \frac{n}{r} \right)^{r} \prod_{j=1}^{r} \left( t_{j}^{j-1} (j-1)^{d} \right) dx,
\]

which, in view of (2.11) and (2.16), implies (2.15) and the relation
\[
(2.22) \quad c_{1} = 2^{r-n} \left( \frac{n}{r} \right)^{r} .
\]

Thus we obtain the formula
\[
(2.23) \quad \Gamma_{\mathcal{Q}}(s) = (2\pi)^{n-r} \left( \frac{n}{r} \right)^{n(1-s)} \prod_{j=1}^{r} \Gamma \left( \frac{n}{r} s - \frac{d}{2} (j-1) \right).
\]

Our computation also shows that the integral for \( \Gamma_{\mathcal{Q}}(s) \) converges absolutely for \( \text{Res} > 1 - \frac{r}{n} \).

From the relation (1.2) one obtains
\[
\Gamma_{\mathcal{Q}}(s) \Gamma_{\mathcal{Q}}(1-s) = (2\pi)^{n-r} \prod_{j=1}^{r} \Gamma \left( \frac{n}{r} s - \frac{d}{2} (j-1) \right) \Gamma \left( \frac{n}{r} (1-s) - \frac{d}{2} (r-j) \right)
\]
\[
= (2\pi)^{n-r} (2\pi i)^{r} \prod_{j=1}^{r} \frac{e^{-\pi i \frac{d}{2} (j-1)}}{e^{2\pi i \left( \frac{n}{r} s - \frac{d}{2} (j-1) \right)}} - 1
\]

Since one has by (2.11)
\[
n - r = d \frac{r(r-1)}{2} = \begin{cases} 0 \pmod{2} & \text{for } d \text{ even} \\ \left\lfloor \frac{r}{2} \right\rfloor \pmod{2} & \text{for } d \text{ odd}, \end{cases}
\]

one has
\[
\prod_{j=1}^{r} e^{-\pi i \frac{d}{r} (j-1)} = (-1)^{d \frac{r(r-1)}{2}} = \begin{cases} i^{n-r} & \text{for } d \text{ even} \\ i^{n-r(-1)^{\left\lfloor \frac{r}{2} \right\rfloor}} & \text{for } d \text{ odd}.
\end{cases}
\]

Hence one obtains the following functional equation:
\[
(2.24) \quad \Gamma_{\mathcal{Q}}(s) \Gamma_{\mathcal{Q}}(1-s) = (2\pi i)^{n} e^{n\pi is} \begin{cases} (e^{2\pi i \frac{n}{r} s} - 1)^{-r} & \text{for } d \text{ even} \\ (e^{2\pi i \frac{n}{r} s} - 1)^{-\left\lfloor \frac{r+1}{2} \right\rfloor} e^{2\pi i \frac{n}{r} s + 1} & \text{for } d \text{ odd}. \end{cases}
\]
§ 3. Zeta functions of a self-dual homogeneous cone.

3.1. We fix a \( \mathbb{Q} \)-structure on \( U \) and assume that (the Zariski closure of) \( G \) is defined over \( \mathbb{Q} \) and \( e \in U_\mathbb{Q} \); then (the Zariski closure of) \( K \) is also defined over \( \mathbb{Q} \). We also fix a lattice \( L \) in \( U \) compatible with that \( \mathbb{Q} \)-structure, i.e., such that \( U_\mathbb{Q} = \mathbb{Z} \oplus \mathbb{Q} \), and an arithmetic subgroup \( \Gamma \) fixing \( L \), i.e., a subgroup of \( G_\mathbb{Q} = \{ g \in G | gL = L \} \) of finite index; for simplicity we assume that \( \Gamma \) has no fixed point in \( \mathcal{L} \). We then define the zeta function associated with \( \mathcal{L}, \Gamma, L \) as follows:

\[
\zeta_{\mathcal{L}}(s; \Gamma, L) = \sum_{u \in \Gamma \backslash \mathcal{L} \cap L} N(u)^{-s},
\]

the summation being taken over a complete set of representatives of \( \mathcal{L} \cap L \) modulo \( \Gamma \). It can be shown easily that this series is absolutely convergent for \( \text{Re } s > 1 \).

By the reduction theory, \( \Gamma \) has a fundamental domain in \( \mathcal{L} \), which is a rational polyhedral cone. More precisely, there exists a finite set of simplicial cones

\[
C^{(i)} = \left\{ v^{(i)}_1, \ldots, v^{(i)}_{k_i} \right\}_{R^+}
\]

\[
= \left\{ \sum_{j=1}^{k_i} \lambda_j v^{(i)}_j \mid \lambda_j \in R^+ \right\} \quad (1 \leq i \leq m),
\]

where \( v^{(i)}_1, \ldots, v^{(i)}_{k_i} \) are linearly independent elements in \( \mathcal{L} \cap L \), such that

\[
\mathcal{L} = \bigcup_{1 \leq i \leq m} \mathcal{C}^{(i)}.
\]

It follows that

\[
\zeta(s; \Gamma, L) = \sum_{i=1}^{m} \sum_{u \in C^{(i)} \cap L} N(u)^{-s}.
\]

For a set of linearly independent vectors \( v_1, \ldots, v_\ell \in L \), we put

\[
R((v_\ell), L) = \left\{ \sum_{j=1}^{\ell} \lambda_j v_j \mid 0 < \lambda_j \leq 1 \right\} \cap L,
\]

which is finite. Then \( u \in \mathcal{C}^{(i)} \cap L \) can be written uniquely in the form

\[
u = v_\circ + \sum_{j=1}^{\ell} m_j v^{(j)}, \quad v \in R((v^{(j)}), L), \quad m \in \mathbb{Z}, \ m \geq 0.
\]
For a set of linearly independent vectors \( v_1, \ldots, v_\ell \in \mathcal{J} \cap V_\mathbb{Q} \) and \( v_0 = \sum_j \alpha_j v_j \) (\( \alpha_j \in \mathbb{Q}_+ \)), we define a "partial zeta function" by

\[
\zeta_{d\ell}(s; (v_j), v_0) = \sum_{\nu_0 > 0} N(v_0) + \sum_{j=1}^L m_j v_j \right)^{-s},
\]
which will also be written as \( \zeta_{d\ell}(s; (v_j), (\alpha_j)) \). Then the zeta function (3.1) can be written as a finite sum of partial zeta functions as follows:

\[
\zeta_{d\ell}(s; \Gamma, L) = \sum_{i=1}^\infty \sum_{v_0 \in K(\nu_0), \nu_0} \zeta_{d\ell}(s; (v_\ell^{(i)}), v_0).
\]

Hence the study of special values of \( \zeta_{d\ell}(s; \Gamma, L) \) is reduced to that of the partial zeta functions of the form (3.2).

3.2. Let \( (v_\ell) \) and \( v_0 \) be as above. Then by (2.7) one obtains

\[
\begin{align*}
\Gamma_{d\ell}(s) \zeta_{d\ell}(s; (v_\ell), v_0) &= \sum_{nu \geq \nu_0} \Gamma_{d\ell}(s) N(v_\ell) + \sum_{j=1}^L m_j v_j \right)^{-s} \\
&= \sum_{nu \geq \nu_0} \int_{\mathcal{J}} N(u)^{s-1} e^{-\sum_{j=1}^L \alpha_j} b(v_\ell, u)^{-s} du \\
&= \int_{\mathcal{J}} N(u)^s \prod_{j=1}^L b(v_\ell, u)^{-1} d\nu \\
&= \int_G \det(g)^s \prod_{j=1}^L b(\nu_\ell, g\nu_\ell^{-1} \alpha_j) dg.
\end{align*}
\]

In the notation of §2, but this time using the decomposition \( G = KAK \), one has

\[
(3.4)
\]

\[
d\nu = c \Delta(a) dk \cdot da \cdot dk'
\]

for \( g = kak' \), \( k, k' \in K, a \in A \). Here \( c \) is a positive constant and

\[
\Delta(a) = \prod_{\alpha \in \Phi_+} (e^{\alpha(\log a)} - e^{-\alpha(\log a)})^d
\]

\[
= (\prod_{i=1}^r t_i)^{-\frac{d}{2}} |A(t_i, \ldots, t_r)|, \]

where \( A(t_i, \ldots, t_r) = \prod_{i \leq j} (t_i - t_j) \) (cf. Helgason [2], Ch. X, §1). Hence in view of (2.11) and (2.17) one has

\[
(3.5) \quad \Gamma_{d\ell}(s) \zeta_{d\ell}(s; (v_\ell), (\alpha_j)) = c \int_0^\infty \cdots \int_0^\infty \left( \prod_{i=1}^r t_i \right)^{\alpha_j - 1} |A(t)| d\tau(t) \prod_{r=1}^r dt_i,
\]

where
\[ F(t_1, \ldots, t_r) = \int_{\mathbb{R}^r} \prod_{j=1}^r b(<v_j, k \sum_{i=1}^r t_i e_i>, 1 - \alpha_j) \; dk. \]

It is clear that \( F(t_1, \ldots, t_r) \) is holomorphic for \( \Re t_i > 0 \) \((1 \leq i \leq r)\).

Since \( K \) contains an element which induces any given permutation of \( e_1, \ldots, e_r \), the function \( F \) is symmetric. Hence, denoting by \( B_i \) an open simplicial cone in \( \mathbb{R}^r \) defined by \( t_1 > \ldots > t_r > 0 \), one has

\[ (3.5') \quad f_{\mathcal{D}_s}(s) \zeta_{\mathcal{D}_s}(s; (v_j), (\alpha_j)) = \frac{c}{r!} \int_{B_i} \left( \prod_{i=1}^r t_i \right)^{\frac{r}{2}(s-1)} \Delta(t)^d F(t) \prod_{i=1}^r dt_i. \]

3.3. Still following Shintani [11], we make a change of variables \((t_i) \rightarrow (t_1, \tau_2, \ldots, \tau_r)\) with \( \tau_i = t_i / t_{i-1} \) \((2 \leq i \leq r)\). Then \( B_i \) can be expressed as

\[ B_i = \left\{ (t_i) \mid t_i = t_1 \prod_{j=2}^{i} \tau_j, 0 < t_1 < \infty, 0 < \tau_i < 1 \right\}. \]

Putting \( \tau_1 = t_1 \), one has

\[ \frac{\partial}{\partial t_i} \left( t_1, \tau_2, \ldots, \tau_r \right) = \prod_{i=1}^r \tau_i^{r-i}, \]

\[ \prod_{i=1}^r t_i = \prod_{i=2}^r \tau_i^{r-i+1}, \]

\[ \Delta(t) = \prod_{i=1}^r \tau_i^{\frac{1}{2}(r-i+1)(r-i-1)} \prod_{2 \leq i < j \leq r} (1 - \tau_i \cdots \tau_j). \]

It follows that the exponent of \( \tau_i \) in the integrand in (3.5') is equal to

\[ (r-i+1) \frac{n}{2} (s-1) + \frac{d}{2} (r-i+1)(r-i) + r - i \]

\[ = (r-i+1) \left\{ \frac{n}{2} s - \frac{d}{2} (i-1) \right\} - 1. \]

Hence one has

\[ (3.6) \quad f_{\mathcal{D}_s}(s) \zeta_{\mathcal{D}_s}(s; (v_j), (\alpha_j)) = c \cdot \frac{r! n^{s-1}}{I_0} \int_0^\infty t^{ns-1} dt \]

\[ \int_0^1 \prod_{i=2}^r \tau_i^{(r-i+1) \left\{ \frac{n}{2} s - \frac{d}{2} (i-1) \right\} - 1} f(t_i, \tau_i) \prod_{i=1}^r d(t_i). \]

where

\[ (3.7) \quad \hat{f}(t_1, \tau) = \prod_{2 \leq i < j \leq r} (1 - \tau_i \cdots \tau_j) \hat{f}(t_1, t_1 \tau_2, \ldots, t_1 \tau_i \cdots \tau_r). \]
3.4. We now assume that all \( v_j \)'s are in \( Q \) (not on the boundary of \( Q \)).

(In the situation explained in 3.1, this means that the \( Q \)-rank of \( G \) is equal to 1.) Then for any \( v \in \tilde{Q} - \{0\} \), one has \( \langle v_j, v \rangle > 0 \); in particular,

\[
\langle v_j, ke_i \rangle > 0 \quad \text{for all} \quad k \in K, \ 1 \leq i \leq r.
\]

Put

\[
\xi_j = \langle v_j, k \sum t_i e_i \rangle = t_1 \langle v_j, k(e_i + \sum_{i=2}^{r} \tau_i \ldots \tau_{i-1} e_i) \rangle = t_1 \langle v_j, ke_i \rangle (1 + \sum_{i=1}^{r} \tau_i \ldots \tau_i \frac{\langle v_j, ke_i \rangle}{\langle v_j, ke_i \rangle}).
\]

For the fixed \( e_i, v_j \), choose \( \rho, \beta, > 0 \) in such a way that

\[
\frac{\sum_{i=2}^{r} t_i^{\rho - 1} \langle v_j, ke_i \rangle}{\langle v_j, ke_i \rangle} < 1 \quad \text{for all} \quad k \in K, \ 1 \leq j \leq \ell,
\]

\[
\beta_i < \frac{\tau_i}{\langle v_j, ke_i \rangle} \quad \text{for all} \quad 1 \leq j \leq \ell.
\]

The for

\[
0 < |t_i| < \rho, \quad |\tau_i| < \beta \quad (2 \leq i \leq r),
\]

one has \( 0 < |\xi_j| < 2 \pi \) and so \( b(\xi_j, 1 - \xi_j) \) is holomorphic. Hence the function \( F(t) = F(t_1, t_1 \tau_1, \ldots, t_1 \tau_{r-1} \tau_r) \) has a Laurent expansion in \( t_1, \tau_1, \ldots, \tau_r \) in the domain defined by (3.11). The coefficients in this expansion is a \( Q \)-linear combination of the integrals of the form

\[
I((v_j)) = \int_{K \times \ell} \prod_{1 \leq i \leq \ell} \langle v_j, ke_i \rangle ^{\xi_j} \, dk
\]

where \( v_j \geq 0 \) for \( 2 \leq i \leq r \) and \( v_j \in \mathbb{Z} \) for all \( i, j \).
3.5. Let \( I(\xi, 1) \) denote the contour consisting of the line segment \([\xi, 1]\) taken twice in opposite directions and of a (small) circle of radius \( \xi \) about the origin taken in the counterclockwise direction. When the \( \tau_i \) (\( 2 \leq i \leq r \)) are on \( I(\xi, 1) \), one has by (2.12)

\[
| \langle v_j, k(e_1 + \sum_{i=2}^r \tau_i \cdots \tau_i e_i \rangle | \leq |v_j| \sum_{i=1}^r |e_i| = Vnr |v_j|
\]

and

\[
\Re \langle v_j, k(e_1 + \sum_{i=2}^r \tau_i \cdots \tau_i e_i \rangle = \langle v_j, k e_1 \rangle + \sum_{i=2}^r \Re(\tau_2 \cdots \tau_i) \langle v_j, k e_i \rangle \\
\geq \langle v_j, k e_1 \rangle - \xi \sum_{i=2}^r |e_i| \\
= \langle v_j, k e_1 \rangle - \xi (r-1) \sqrt{\frac{\pi}{\tau}} |v_j|.
\]

We choose \( \xi \) so that one has

\[
(3.13) \quad \xi \sqrt{\frac{\pi}{\tau}} |v_j| < \min \left\{ 2\pi, \langle v_j, k e_1 \rangle (k \in K) \right\} \quad \text{for all } 1 \leq j \leq l,
\]

The above inequalities show that \( \langle v_j, k(e_1 + \sum_{i=2}^r \tau_i \cdots \tau_i e_i \rangle \) belongs to the domain

\[
\left\{ z \in \mathbb{C} \mid |z| < \frac{2\pi}{\xi}, \Re z > \xi \sqrt{\frac{\pi}{\tau}} |v_j| \right\}.
\]

It follows that, if \( \tau_1 \) is on the contour \( I(\xi, \infty) \), one has

\[
| \xi_j | < 2\pi \quad \text{or} \quad \Re \xi_j > 0,
\]

so that the function \( b(\xi_j, 1 - \alpha_j) \) is holomorphic.

From this observation, it is clear that the integral on the r.h.s. of (3.6) is equal to the contour integral

\[
\left( e^{2\pi is} - 1 \right)^{-1} \int_{\tau_i \in I(\xi, \infty)} \left( e^{2\pi i \frac{\tau_i - 1}{\tau_i} ns} - 1 \right)^{-1} \int_{\tau_i \in I(\xi, 1)} \langle v_j, k(e_1 + \sum_{i=2}^r \tau_i \cdots \tau_i e_i \rangle,
\]

which is independent of the choice of \( \xi \) satisfying (3.13). As is easily seen, the contour integral converges for all \( s \in \mathbb{C} \). Hence the integral

\[
\sum_{v \in \mathbb{C}} \frac{1}{(r-i+1)n} \quad (v \in \mathbb{Z}),
\]

viewed as a function in \( s \), can be continued to a meromorphic function on the whole plane; the possible poles are of the form
§ 4. The special values of the zeta functions.

4.1. As a preliminary, we check the rationality of the constant \( c \) in (3.4). For that purpose, we compute \( \Gamma_Q(s) \) by using the decomposition \( Q = K \Delta K \).

\[
\Gamma_Q(s) = \int_Q N(u)^s e^{-\tau(u)} \, d\lambda(u) \\
= \int_G N(ge)^s e^{-\tau(ge)} \, dg \\
= c \int_A \det(a)^s e^{-\tau(ae)} \Delta(a) \, da \\
= c \int_0^\infty \int_0^\infty (\prod_i t_i) \chi(s-1) |\Delta(t)|^d \, e^{-\frac{\tau}{r} \sum_i t_i} \, dt_i.
\]

We make another change of variables:

\[
t = \sum_{i=1}^r t_i, \quad t_i = t_i / t.
\]

Then

\[
\frac{\varphi(t_1, \ldots, t_r)}{\varphi(t, t_1', \ldots, t_{r-1}')} = (-1)^{r-1} t^{r-1},
\]

and the exponent of \( t \) in the integrand in the last member of (4.1) is equal to

\[
n(s-1) + \frac{d}{2} r(r-1) + r - 1 = ns - 1.
\]

Hence one has

\[
\Gamma_Q(s) = c \cdot \gamma(s) \cdot \beta(s),
\]

where

\[
\left\{
\begin{align*}
\gamma(s) &= \int_0^\infty t^{ns-1} e^{-\frac{\tau}{r} t} \, dt = (\frac{r}{n})^{ns} \Gamma(ns), \\
\beta(s) &= \int_{\sum t_i' > 0} \chi(s-1) \prod_{t_i'} |\Delta(t_1', \ldots, t_{r-1}', 1 - \sum t_i')|^d \prod dt_i'.
\end{align*}
\right.
\]

For \( s = 1 \), one has

\[
\Gamma_Q(1) = c \cdot \gamma(1) \cdot \beta(1) = c \left( \frac{r}{n} \right)^n (n-1)! \cdot \beta(1),
\]
\[
\beta(1) = \int \prod \Delta (t_i', \ldots, t_{i-1}', 1 - \sum t_i') \prod dt_i' \in Q.
\]

By (2.23) one has

\[
\Gamma Q^{1/(d-1)} (1) = (2\pi)^{n-d} \left( \frac{\Gamma}{\pi} \right)^{n/2} \prod_{j=1}^{r} \Gamma \left( 1 + \frac{d}{2}(j-1) \right)
\]

\[
\sim \left\{ \begin{array}{ll}
\pi^{-\frac{n-r}{2}} & (d \text{ even}) \\
\pi^{-\frac{n}{2} \left[ \frac{r+1}{2} \right]} & (d \text{ odd})
\end{array} \right.
\]

where \(a \sim b\) means that \(a/b \in Q\). Thus one has

\[
c = (2\pi)^{n-d} \left( \frac{\Gamma}{\pi} \right)^{n/2} \prod_{j=1}^{r} \Gamma \left( 1 + \frac{d}{2}(j-1) \right)
\]

\[
\sim \Gamma Q^{1/(d-1)} (1).
\]

Since \(\Gamma Q^{1/(d-1)} (1) \sim \Gamma Q^{1/(d-1)} (1 + \frac{r}{n} \nu)\) for \(\nu \in Z\), one obtains

\[
c \Gamma Q^{1/(d-1)} (1 + \frac{r}{n} \nu) \sim \Gamma Q^{1/(d-1)} (1)^2 \sim \left\{ \begin{array}{ll}
\pi^{-\frac{n-r}{2}} & (d \text{ even}) \\
\pi^{-\frac{n}{2} \left[ \frac{r+1}{2} \right]} & (d \text{ odd})
\end{array} \right.
\]

4.2. We first consider the case where \(d\) is even. Then by (2.24) one has

\[
\Gamma Q^{1/(d-1)} (1-s) = (2\pi i)^{-\nu} e^{-\nu \operatorname{\pi \sin} \left( 2\pi i \frac{r}{n} s - 1 \right)}.
\]

Hence

\[
\zeta Q^{1/(d-1)} (s) = \frac{c^{1/(d-1)}(1-s)}{(2\pi i)^{-\nu} e^{\nu \operatorname{\pi \sin}}} \times R(s),
\]

where

\[
R(s) = \left( \frac{\Gamma Q^{1/(d-1)} (1-s)}{2\pi i} \right) \prod_{j=1}^{r} \int_{t_j}^{x} dt_j \Delta(t)^{d} \mathcal{F}(t) \prod dt_i,
\]

\[
= \prod_{j=1}^{r} \frac{e^{2\pi i \frac{r}{n} s - 1}}{e^{2\pi i \frac{r}{n} s - 1}} \times \frac{1}{(2\pi i)^{r}} \int_{t_i}^{x} dt_i \mathcal{F}(t_i, t_i)
\]

\[
(\prod_{i=1}^{r} \prod_{j=1}^{x} \tau_i (r-j+1) \frac{\pi}{2} s - \frac{1}{2} (i-1) \frac{r}{2} - 1 r^i \mathcal{F}(t_i, t_i))
\]

We are interested in the values of \(\zeta Q^{1/(d-1)} (s)\) at \(s = -\frac{r}{n} \nu\) (\(\nu = 0, 1, \ldots\)). The
first factor in the right hand side of (4.7) is holomorphic for \( \Re s < \frac{\frac{1}{2}}{n} \) and by (4.6) the value at \( s = -\frac{1}{n} \nu \) is rational:

\[
(4.9) \quad \frac{c \int_\delta (1 + \frac{1}{n} \nu)}{(2\pi i)^{n-r} e^{-\nu \pi i} x} = (-1)^{\frac{n-r}{2} + r \nu} \frac{c \int_\delta (1 + \frac{1}{n} \nu)}{(2\pi)^{n-r}} \in \mathbb{Q}.
\]

On the other hand, it is clear that

\[
\frac{e^{2\pi i \frac{r}{n} s} - 1}{e^{\nu} - 1} \longrightarrow \frac{1}{r-i+1} \quad \text{when } s \longrightarrow -\frac{1}{n} \nu.
\]

Hence we see that \( R(-\frac{1}{n} \nu) \) is equal to the coefficient of

\[
t_i^{r \nu} \prod_{i=2}^r \gamma_i^{(r-i+1)\frac{n}{2}(i-1)}
\]

in the Laurent expansion of \( \overline{F}(t_i, \tau) \), which is a \( \mathbb{Q} \)-linear combination of \( I((\nu y )) \).

4.3. From now on we assume that \( d \) is odd. By the classification theorem, it is known that this assumption implies that \( r = 2 (n = d + 2) \) or \( d = 1 (n = \frac{1}{2} r(r+1)) \). By (2.24) one has

\[
\int_\delta (s) \int_\delta (1-s) = (2\pi i)^n e^{\nu \pi i s} (e^{2\pi i \frac{r}{n} s} - 1) \frac{e^{2\pi i \frac{r}{n} s} + 1}{e^{2\pi i \frac{r}{n} s} + 1}.
\]

Hence

\[
(4.11) \quad \zeta_\delta (s; (\nu), (\omega)) = \frac{c \int_\delta (1-s)}{(2\pi i)^{n-\frac{1}{2} \nu} e^{\nu \pi i s}} \times R(\nu)(s) R(\omega)(s),
\]

where

\[
R(\nu)(s) = (2\pi i)^{\frac{\nu}{2} r} \frac{e^{2\pi i \frac{r}{n} s} - 1}{e^{2\pi i (r-k+1)\frac{n}{2} s - \frac{d}{2} (k-1) r} - 1}
\]

and

\[
R(\omega)(s) = (2\pi i)^{-r} \int_{I(\nu, \omega)} \int_t \frac{t_{i+1}^{n-1} dt_i}{I(\nu, \omega)} \prod_{i=1}^r \gamma_i^{(r-i+1)\frac{n}{2}(i-1)} - 1 \overline{F}(t_i, \tau) d\tau_i.
\]
The first factor in the right hand side of (4.11) is holomorphic for \( \text{Re } s < \frac{r}{n} \) and by (4.6) the value at \( s = -\frac{r}{n} \nu \) (\( \nu > 0 \)) is rational:

\[
\left( \frac{c}{2\pi i} \left( 1 + \frac{r}{n} \nu \right) \right)^{\frac{n}{2}} \left( 1 + \frac{r+1}{2} \nu \right)^{\nu} \frac{c}{2\pi i} \left( 1 + \frac{r+1}{2} \nu \right)^{\nu} \in \mathbb{Q}.
\]

Note that one has

\[ n \equiv \left[ \frac{r+1}{2} \right] \pmod{2}, \]

since

\[ n = d+2 \equiv 1 \equiv \left[ \frac{3}{2} \right] \pmod{2} \quad \text{if } r = 2, \text{ and } \]

\[ n = \frac{1}{2} r(r+1) \equiv \left[ \frac{r+1}{2} \right] \pmod{2} \quad \text{if } d = 1. \]

4.4. To compute \( R^{(r)}(s) \), we first note

\[ e^{\pi i \nu (k-1)(r-k+1)} = \begin{cases} -1 & \text{if } k \equiv r \equiv 0 \pmod{2}, \\ 1 & \text{otherwise}. \end{cases} \]

We put

\[ \left[ \frac{r}{2} \right] = r_1, \quad \zeta = e^{2\pi i \frac{n}{r}} s. \]

**The case \( r \) is odd.** One has

\[
R^{(r)}(s) = (2\pi i r_1 r!) \frac{(\zeta - 1)^{r_1}(\zeta + 1)^{r_1}}{\prod_{k=1}^{r_1} (\zeta^k - 1)}.
\]

Hence, when \( s \rightarrow -\frac{r}{n} \nu \), one has

\[
(4.13) \quad (s + \frac{r}{n} \nu)^{r_1} R^{(r)}(s) \rightarrow (2 \frac{r}{n} \nu)^{r_1}.
\]

Thus \( R^{(r)}(s) \) has a pole of order \( r_1 \) at \( s = -\frac{r}{n} \nu \).

**The case \( r \) is even.** One has

\[
R^{(r)}(s) = (2\pi i r_1 r!) \frac{(\zeta - 1)^{r_1}(\zeta + 1)^{r_1}}{\prod_{k=1}^{r_1} \left\{ (-1)^k \zeta^k - 1 \right\}}.
\]
\[ (-2\pi i)^{r_1} \prod_{k=1}^{r_1} (\zeta_{k}^{r-1} + \ldots + \zeta_{k}^{r+1}) \prod_{k=l}^{r_1} (\zeta_{k}^{r-1} - \ldots - \zeta_{k}^{r+1}) \]

Hence \( R^{(s)} \) is holomorphic at \( s = -\frac{r}{n} \) and

\[(4.14) \quad R^{(s)}(-\frac{r}{n} \nu) = (-2\pi i)^{r_1} \frac{r!}{(2r_1)!} = (-\pi i)^{r_1} \frac{r!}{r_1!}. \]

4.5. When \( r \) is odd (hence \( d = 1, n = \frac{1}{2} r(r+1) \)), \( R^{(s)}(s) \) for \( s = -\frac{r}{n} \nu \)

is given by the coefficient of

\[ t_1^{r_1} \nu \prod_{i=2}^{r} \tau_i (r-i+1)(\nu + \frac{i}{2}) \]

in the Laurent expansion of \( \tilde{F}(t_1, \tau) \). Hence \( \zeta_{Q}^{(s)}(s; (v_j), (\alpha_j)) \) has
at most a pole of order \( r_1 = \frac{r-1}{2} \) at \( s = -\frac{2\nu}{r+1} \) and one has

\[(4.15) \quad \lim_{s \to -\frac{2\nu}{r+1}} (s + \frac{2\nu}{r+1})^{r_1} \zeta_{Q}^{(s)}(s; (v_j), (\alpha_j)) \sim Q^{(s)}(-\frac{2\nu}{r+1}). \]

To treat the case \( r \) is even, we use the formula

\[ \int_{I(\epsilon, 1)} t^{m-1} dt = -\frac{\nu}{m} \quad (m \text{ odd}), \]

which can be verified easily. When \( r \) is even, the value of \( R^{(s)}(s) \) for
\( s = -\frac{r}{n} \nu \) is given by

\[(4.16) \quad (-\pi i)^{r_1} \sum_{m_1; \ldots, m_l \in \mathbb{Z}} \frac{a_{(m_j)}}{\prod_{j=1}^{l} (m_j^2 - (r-2j+1)(\nu + d(j-1))^2)} \]

where \( a_{(m_j)} \) is the coefficient of

\[ t_1^{r_1} \prod_{j=2}^{l} \tau_j (r-2j+2)(\nu + d(j-1)) \prod_{j=1}^{l} \tau_j^m \]

in \( \tilde{F}(t_1, \tau) \). Hence for the value of \( \zeta_{\tilde{Q}} \), one has

\[(4.17) \quad \zeta_{\tilde{Q}}(-\frac{r}{n} \nu; (v_j), (\alpha_j)) \sim (2\pi i)^{r_1} R^{(s)}(-\frac{r}{n} \nu). \]
Bibliography


H. L. Resnikoff,


