Title

On Special Values of Zeta Functions Associated with a Self-Dual Cone (P-Adic L-Functions and Algebraic Number Theory)

Author(s)

SATAKE, ICHIRO

Citation

数理解析研究所講究録 1981, 411: 203-224

Issue Date

1981-01

URL

http://hdl.handle.net/2433/102407

Type

Departmental Bulletin Paper

Textversion

publisher

Kyoto University
On special values of zeta functions
associated with a self-dual cone
§ 1. Introduction

To explain the main idea of this paper, and also to fix some notations, we start with reviewing the classical case of Riemann zeta function. As usual, we set

\[ \zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} \quad (\text{Re } s > 1), \]
\[ \Gamma(s) = \int_{0}^{\infty} x^{s-1} e^{-x} \, dx \quad (\text{Re } s > 0). \]

Then, for \( \text{Re } s > 1 \), one obtains

\[ \Gamma(s) \zeta(s) = \sum_{n=1}^{\infty} n^{-s} \int_{0}^{\infty} x^{s-1} e^{-nx} \, dx \]
\[ = \sum_{n=1}^{\infty} \int_{0}^{\infty} x^{s-1} e^{-nx} \, dx \]
\[ = \int_{0}^{\infty} \frac{x^{s-1}}{e^x - 1} \, dx. \]

We put

\[ b(x, y) = \frac{e^{xy}}{e^x - 1} = \sum_{\nu=0}^{\infty} \frac{B_{\nu}(y)}{\nu!} x^{\nu - 1} \quad (|x| < 2\pi), \]

where

\[ B_{\nu}(y) = \sum_{k=0}^{\nu} \binom{\nu}{k} b_{\nu-k} y^{\nu-k} \]

is the Bernoulli polynomial, in which the \( b_{\nu} \) are the Bernoulli numbers:

\[ b_{0} = 1, \quad b_{1} = -\frac{1}{2}, \]
\[ b_{\nu} = \begin{cases} (-1)^{\frac{\nu-1}{2}} B_{\frac{\nu}{2}} & (\nu \text{ even, } \geq 2), \\ 0 & (\nu \text{ odd, } \geq 3). \end{cases} \]

Then the above integral can be transformed into a contour integral of the form

\[ \Gamma(s) \zeta(s) = (e^{2\pi is} - 1)^{-1} \int_{I(\varepsilon, \infty)} x^{s-1} b(x, 0) \, dx, \]

where \( I(\varepsilon, \infty) \) denotes the contour consisting of the half-line \([ \varepsilon, \infty)\) taken twice in opposite directions and of a (small) circle of radius \( \varepsilon \).
about the origin taken in the counterclockwise direction. The contour integral is absolutely convergent for all \( s \in \mathbb{C} \), so that the function \( \gamma(s) \zeta(s) \) can be analytically continued to a meromorphic function on \( \mathbb{C} \). Moreover, in virtue of the functional equation of the gamma function:

\[
(1.2) \quad \gamma(s) \gamma(1-s) = \frac{\pi}{\sin \pi s} = 2\pi i \frac{e^{\pi is}}{e^{2\pi is} - 1},
\]

one obtains

\[
(1.3) \quad \zeta(s) = e^{-\pi is} \gamma(1-s) \frac{1}{2\pi i} \int \frac{x^{s-1}}{I(t, \omega)} b(x, 0) \, dx.
\]

This shows that \( \zeta(s) \) is holomorphic for \( \text{Re } s < 1 \). In particular, for \( s = 1 - m, m \in \mathbb{Z}^+ \) (positive integers), the contour integral reduces to the residue of \( x^{-m} b(x, 0) \) at \( x = 0 \), i.e., \( \frac{b_m}{m!} \). Hence one obtains

\[
(1.4) \quad \zeta(1 - m) = (-1)^{m-1}(m-1)! \frac{b_m}{m!} = (-1)^{m-1} \frac{b_m}{m}.
\]

Thus \( \zeta(1 - m) \) (\( m \in \mathbb{Z}^+ \)) is rational. In particular,

\[
\zeta(0) = -\frac{1}{2}, \quad \zeta(-1) = -\frac{1}{12},
\]

\[
\zeta(-2\mu) = 0, \quad \zeta(1 - 2\mu) = (-1)^{\mu} \frac{B_{2\mu}}{2\mu} \quad (\mu \geq 1).
\]

This result has been generalized by Hecke, Klingen and Siegel \([3]\) to the case of Dedekind zeta functions of totally real number fields. More recently, Shintani \([1]\) gave a proof based on a direct generalization of the classical method explained above. Zeta functions attached to self-dual homogeneous cones have been studied by Siegel \([3]\) in a special case of quadratic cones, and by Sato-Shintani \([8]\) in a more general context of "prehomogeneous spaces". (Cf. also Shintani \([7]\), \([10]\).) On the other hand, the gamma functions attached to self-dual homogeneous cones were studied by Koecher \([5]\), Gindikin \([3]\) and others (cf. e.g., Resnikoff \([6]\)). In this paper, we try to extend Shintani's method (i.e., the classical method) to examine the rationality of the special values of zeta functions attached to self-dual homogeneous cones.
2. The gamma function of a self-dual homogeneous cone

2.1. Let $U$ be a real vector space of dimension $n$, endowed with a positive definite inner product $\langle \cdot, \cdot \rangle$. By a "cone" in $U$ we always mean a non-degenerate open convex cone in $U$ with vertex at the origin, i.e., a non-empty open set $\mathcal{L}$ in $U$ such that

$$x, y \in \mathcal{L}, \; \lambda, \mu \in \mathbb{R}^+ \implies \lambda x + \mu y \in \mathcal{L}$$

and such that $\mathcal{L}$ does not contain any straight line. A cone $\mathcal{L}$ in $U$ is called **homogeneous** if the group of linear automorphisms

$$G(\mathcal{L}) = \{ g \in GL(U) \mid g(\mathcal{L}) = \mathcal{L} \}$$

is transitive on $\mathcal{L}$; and $\mathcal{L}$ is called **self-dual** if the "dual" of

$$\mathcal{L}^* = \{ x \in U \mid \langle x, y \rangle > 0 \text{ for all } y \in \mathcal{L} - \{0\} \}$$

coinsides with $\mathcal{L}$.

Let $\mathcal{L}$ be a self-dual homogeneous cone in $U$ and $G = G(\mathcal{L})^\circ$. Then it is well-known (e.g., Satake [7]) that the Zariski closure of $G$ (in $GL(U)$) is a reductive algebraic group, containing $G(\mathcal{L})$ as a subgroup of finite index, and $g \mapsto ^t g^{-1}$ is a Cartan involution of $G$; the corresponding maximal compact subgroup $K = G\cap O(U)$ coincides with the isotropy subgroup of $G$ at a "base point" $e \in \mathcal{L}$ (which is not unique, but will be fixed once and for all). Let

$$\mathfrak{g} = \mathfrak{k} + \mathfrak{f}$$

be the corresponding Cartan decomposition of $\mathfrak{g} = \text{Lie } G$. Then $\mathfrak{k} = \text{Lie } K$ and one has for $T \in \mathfrak{f}$

$$T \in \mathfrak{k} \iff ^t T = -T \iff Te = 0.$$ 

It follows that, for each $u \in U$, there exists a uniquely determined element $T_u \in \mathfrak{f}$ such that $T_u e = u$. It is well-known that the vector space $U$ endowed with a product
\[ u \cdot u' = T_u u' \quad (u, u' \in U) \]

becomes a formally real Jordan algebra (cf. Braun-Koecher [2], or Satake [7]).

We define the (regular) trace on \( U \) by

\[ \tau(u) = \text{tr}(T_u). \]

For the given \((L, e)\), one may assume (by Schur's lemma) that the inner product \( \langle \cdot, \cdot \rangle \) is so normalized that one has

\[ \langle u, u' \rangle = \tau(u \cdot u') \quad (u, u' \in U). \]

Next, let \( u \in L \). Then, since \( G \) is transitive on \( L \), there exists \( g_1 \in G \) such that \( u = g_1 e \). We define the (regular) norm \( N(u) \) by

\[ N(u) = \det(g_1), \]

which is clearly independent of the choice of \( g_1 \). There exists a unique element \( u_1 \in U \) such that \( u = \exp u_1 \) (which is defined to be \( \exp T_{u_1} e \)); then by definition one has

\[ N(u) = \det(\exp T_{u_1}) = e^{\tau(u_1)}. \]

In terms of the "quadratic multiplication" \( P(u) = 2 T_u - T_u^* \), one can also write \( N(u) = \det(P(u))^{\frac{1}{2}} \). By the definition, it is clear that

\[ N(e) = 1, \quad N(ge) = \det(g) N(u) \quad (g \in G(L), u \in L), \]

which characterizes the norm uniquely. Denoting the Euclidean measure on \( U \) by \( du \), we see that \( d_{\mathcal{L}}(u) = N(u)^{-1} du \) is an invariant measure on \( L \).

**Example.** Let \( U = \text{Sym}_r(\mathbb{R}) \) (the space of real symmetric matrices of degree \( r \)) and \( \mathcal{L} = \mathcal{P}_r(\mathbb{R}) \) (the cone of positive definite elements in \( U \)).

Then one has

\[ T_u(u') = \frac{1}{2} (uu' + u'u) \]

and so

\[ \tau(u) = \frac{r+1}{2} \text{tr}(u), \quad N(u) = \det(u)^{\frac{r+1}{2}}. \]
2.2. We define the gamma function of the cone $\mathcal{L}$ by

\[(2.6) \quad \Gamma(\mathcal{L}) = \int_{\mathcal{L}} N(u)^{-1} e^{-\tau(u)} \, du \]

which converges absolutely for $\Re s$ sufficiently large (actually for $\Re s > 1 - \frac{r}{n}$ as we will see later).

**Lemma 2.1.** Suppose that the inner product $\langle \rangle$ is normalized by (2.3).

Then one has for any $v \in \mathcal{L}$

\[(2.7) \quad \int_{\mathcal{L}} N(u)^{s-1} e^{-\langle u, v \rangle} \, du = \int_{\mathcal{L}} N(v)^{-s} \Gamma(\mathcal{L}) \, du. \]

**Proof.** Let $v = g_i e$ with $g_i \in G$ and put $u' = t g_i u$. Then one has

\[\langle u, v \rangle = \langle u, g_i e \rangle = \langle u', e \rangle = \tau(u').\]

Hence by (2.5) the left-hand side of (2.7) is equal to

\[\int_{\mathcal{L}} N(u)^{s-1} e^{-\langle u, v \rangle} \, du = \int_{\mathcal{L}} (\det(g_i)^{-1} N(u'))^s e^{-\tau(u')} \, du'. \]

\[= N(v)^{-s} \int_{\mathcal{L}} \Gamma(\mathcal{L}), \text{ q.e.d.} \]

It is known that the function $\Gamma(\mathcal{L})$ can be expressed as a product of ordinary gamma functions (cf. e.g., Resnikoff loc. cit.). For the sake of completeness, we sketch a proof. First, it is clear that, if

\[\mathcal{L} = \mathcal{L}_1 \times \cdots \times \mathcal{L}_m\]

is the decomposition of $\mathcal{L}$ into the direct product of irreducible (self-dual homogeneous) cones, then one has

\[\Gamma(\mathcal{L}) = \Gamma(\mathcal{L}_1) \cdots \Gamma(\mathcal{L}_m).\]

Hence, for our purpose, we may assume that $\mathcal{L}$ is irreducible.

We need the root structure of $\mathcal{L}$, which can be determined as follows. Let

\[(2.8) \quad e = \sum_{i=1}^{r} e_i, \quad e_i e_j = \delta_{ij} e_i.\]
be a decomposition of $e$ (in the Jordan algebra $U$) into the sum of mutually orthogonal primitive idempotents. ("Primitive" means that each $e_i$ cannot be decomposed into the sum of mutually orthogonal idempotents any more.) Then we obtain the following decomposition of $U$ into the direct sum of subspaces ("Peirce decomposition").

\[(2.9) \quad U = \bigoplus_{i \neq j} U_{ij}, \]

where

\[ U_{ii} = \left\{ u \in U \mid e_i u = u \right\}, \]

\[ U_{ij} = \left\{ u \in U \mid e_i u = e_j u = \frac{1}{2} u \right\} \quad (i \neq j). \]

Then one has $e_k u = 0$ for $u \in U_{ij}$, $k \neq i, j$. Moreover

\[(2.10) \quad \dim U_{ii} = 1, \quad \dim U_{ij} = d \quad (i \neq j), \]

where $d$ is a positive integer depending on the irreducible cone $\mathcal{Q}$. (For instance, one has $d = 1$ for $\mathcal{Q} = \mathcal{P}_r(R)$.) From (2.9), (2.10) one has the relation

\[(2.11) \quad n = r + \frac{1}{2} r(r - 1)d, \quad \text{i.e.,} \quad d = \frac{2(n - r)}{r(r - 1)}. \]

It follows that

\[(2.12) \quad \tau(e_i) = \text{tr}(T_{e_i}) = 1 + \frac{1}{2} (r - 1)d = \frac{n}{r}, \]

where $\tau$ is the trace on $\mathcal{M}_n(R)$.

Put

\[(2.13) \quad \alpha_l = \left\{ T_{e_i} \mid 1 \leq i \leq r \right\}. \]

Then $\alpha_l$ is an abelian subalgebra of $\mathcal{M}_r$ of dimension $r$ contained in $\mathcal{P}_r$. We denote by $(\lambda_i)$ the basis of $\alpha^*$ (the dual space of $\alpha$) dual to $(T_{e_i})$, i.e., one has the relation

\[ T = \sum_{i=1}^{r} \lambda_i(T) T_{e_i} \quad (T \in \alpha). \]

We put $\lambda_i = \frac{1}{2} (\lambda_i - \lambda_j) \quad (i \neq j)$. 

PROPOSITION 1. The root system of $\Phi$ relative to $\mathfrak{a}_i$ is given by $\Phi = \{ \alpha_{i,j} \mid i \neq j \}$. The root space $\Phi(\alpha_{i,j})$ corresponding to $\alpha_{i,j}$ is given by

$$(2.14) \quad \Phi(\alpha_{i,j}) = \left\{ T_u + \left[ T_{e_i}, T_{e_j} \right] \mid u \in V_{i,j} \right\}.$$ 

This can be verified by a straightforward computation; see e.g., Ash et al. [1] Ch. II, §3. Proposition 1 implies that the $R$-rank of $\mathfrak{g}$ is equal to $r$ and the root system $\Phi$ is of type $(A_{r-1})$.

2.3. Next we determine the Haar measure of $G$. Put

$$\nu = \sum_{i < j} \Phi(\alpha_{i,j})$$

and let $A, N$ be the analytic subgroups of $G$ corresponding to $\mathfrak{a}_i, \mathfrak{n}$, respectively. Then one has an Iwasawa decomposition $G = NA \cdot K$, which gives rise to the following formula for (the volume element of) a (biinvariant) Haar measure on $G$:

$$(2.15) \quad dg = c_1 e^{-2 \rho (\log a)} d_a da dk$$

for $g = nan$ with $n \in N$, $a \in A$, $k \in K$, where $da, da, dk$ denote Haar measures on $N, A, K$, respectively, $c_1$ is a positive constant depending on the normalization of the Haar measures, and $\rho$ is a linear form on $\mathfrak{n}$ defined by

$$\rho(T) = \frac{1}{2} \text{tr}(\text{ad} T | \mathfrak{n}) \quad (T \in \mathfrak{n}).$$

by Proposition 1 one has

$$(2.16) \quad \rho = \frac{d}{2} \sum_{i < j} \alpha_{i,j} = \frac{d}{2} \sum_{i=1}^r (r - 2i + 1) \lambda_i.$$  

The Haar measure of $K$ is always normalized by $\int_K dk = 1$. We make an identification $A = (\mathbb{R}^*)^r$ by the correspondence $a \leftrightarrow (t_i)$ defined by the relation $a = \exp(\sum \lambda_i T_{e_i})$, $t_i = e^{\lambda_i}$; then one has $da = \prod (dt_i/t_i)$. Moreover one has

$$\det(a) = e^{\rho (\sum \lambda_i e_i)} = e^{\frac{1}{2} \sum \lambda_i} = (\prod t_i)^b,$$

$$(2.17) \quad a \cdot e = \sum_{i=1}^r e^{\lambda_i} e_i = \sum_{i=1}^r t_i e_i.$$
\[ e^2 P(\log a) = \prod_{i=1}^{r} t_i \frac{1}{6i} \delta r - 2i + 1. \]

Since \( \mathcal{Q} = G/K \), we can normalize the Haar measure of \( G \) by the relation \( dg = d_{\mathcal{Q}}(u) \cdot dk \) where \( u = ge \). Then by (2.15), (2.16), (2.17) one has

\[ \frac{1}{c_i} \int_A A \cdot \det(a)^s e^{-2P(\log a)} da \int_{\mathcal{N}} e^{-\tau(n \cdot ae)} dn \]
\[ = c_i \int_0^\infty \cdots \int_0^\infty \left( t_i \prod_{i=1}^s \left( \frac{n^i}{s} \right)^{\delta r - 2i + 1} \right) dt_i \]
\[ \times \int_{\mathcal{N}} e^{-\tau(n \cdot \mathcal{V} t_i e_i)} dn \]

To compute the integral over \( \mathcal{N} \), we introduce some notations. For \( u = \sum_{i<j} u_{ij} \in U \) with \( u_{ij} \in U_{ij} \), we put

\[ T_u(+) = \frac{1}{2} \left( T_u + \sum_{i<j} [T_{e_i e_j}, T_{u_{ij}}] \right), \]
\[ \mathcal{E}(+) = \sum_{i<j} u_{ij} \cdot \sum_{i<k} u_{ik} \cdot \cdots \cdot u_{k\geqj} \cdot j. \]

Then one has the \( U_{ij} \)-component of \( \mathcal{E}(+) \) is denoted by \( \mathcal{E}_{ij}^{(+)}(u) \).

**Lemma 2.** The notation being as above, one has

\[ \exp T_u(+) \left( \sum_{i<j} t_i e_i \right) = \sum_{i<j} (t_i + \frac{1}{s} \sum_{k\geq i} t_k) \mathcal{E}_{ij}^{(+)}(u)^2 e_i \]
\[ + \frac{1}{2} \sum_{i<j} t_j \mathcal{E}_{i\geq j}^{(+)}(u) + \sum_{k\geq j} t_k \mathcal{E}_{ik}^{(+)}(u) \mathcal{E}_{jk}^{(+)}(u) \]

This may be regarded as a generalization of the so-called "Jacobian transformation". The proof is again straightforward. It follows that, if \( n = \exp T_u(+) \) \( (u \in \sum_{i<j} U_{ij}) \), one has

\[ \tau(n \cdot \sum_{i<j} t_i e_i) = \frac{n}{r} \sum_{i<j} t_i + \frac{1}{8} \sum_{k}\tau(\mathcal{E}_{ik}^{(+)}(u)^2) t_k. \]

We denote the Euclidean measure on \( U_{ij} \) \( (i<j) \) (relative to the inner product \( <> \)) by \( du_{ij} \) and define the Haar measure on \( \mathcal{N} \) by

\[ dn = \prod_{i<j} du_{ij} \quad \text{for} \ n = \exp T_u^{(+)}. \]

Since the map \( \mathcal{E}(+) \) is a bijection of \( \sum_{i<j} U_{ij} \) onto itself with Jacobian
equal to one, one has

$$du = \prod_{i < j} du'_{ij} = \prod_{i < j} du''_{ij},$$

where \( u' = \mathcal{E}'(u) \). Hence by (2.21) one has

$$\int_N e^{-\tau(u, t, \epsilon_1)} \, du = e^{-\frac{n}{\tau} \sum_i t_i} \prod_{i < j} \int_{U_{ij}} e^{-\frac{t_j}{\tau} \tau(u_{ij}^2)} du'_{ij}$$

$$= e^{-\frac{n}{\tau} \sum_i t_i} \prod_{i < j} \left( \frac{8\pi}{t_j} \right)^{\frac{d}{2}}$$

$$= (8\pi)^{\frac{d}{2}} \prod_{j=1} \left( t_j - \frac{d}{2}(j-1) e^{-\frac{n}{\tau} t_j} \right).$$

Inserting this in (2.18), one obtains

$$\Gamma_{g}(s) = c_1(8\pi)^{\frac{d}{2}} \prod_{j=1} \left( \int_0^\infty \frac{1}{t_j} s - \frac{d}{2}(r-j) - 1 e^{-\frac{n}{\tau} t_j} dt_j \right)$$

$$= c_1(8\pi)^{\frac{d}{2}} \prod_{j=1} \left( \frac{n}{r} - \frac{n}{s} + \frac{d}{r} (r-j) \right) \prod_{j=1} \left( \frac{n}{r} s - \frac{d}{2} (j-1) \right)$$

The constant \( c_1 \) can be determined by the following observation. We set

$$U_0 = \sum_{i=1}^r U_{1i} = \{ e_1, \ldots, e_r \} \in \mathbb{R}$$

and denote by \( du_0 \) the Euclidean measure on \( U_0 \) (relative to \( < > \)). Then, since \( \langle e_i, e_j \rangle = \frac{n}{r} \delta_{ij} \), the bijection \( A \rightarrow U_0 \) defined by \( a = \exp T_{u_0} \),

or equivalently by \( ae = \exp u' \), gives the relation

$$du_0 = \left( \frac{r}{t_j} \right) da.$$

Hence, when

$$u = (pa)e = \mathbb{R}(\sum_i t_i e_i),$$

$$n = \exp T_{x'}^+, x \in \sum_i U_{1i}, \quad x' = \mathcal{E}(x),$$

one has by Lemma 2

$$\mathcal{E}(u) = \mathcal{E}(u_0, u_{ij}^2) \in \prod_{i < j} \left( \frac{t_j}{2} \right)^{(j-1)d}$$

$$\mathcal{E}(t, x) = \mathcal{E}(t_i, x_{ij}^2) \in \prod_{i < j} \left( \frac{t_j}{2} \right)^{(j-1)d}.$$
\[
2^r \cdot n \cdot \frac{1}{r} \sum_{j=1}^{r} \left( \frac{n}{r} \right)^{\frac{j}{2}} \left( \frac{j-1}{d} \right) dt_j.
\]

It follows that
\[
d_{\varphi}(u) = 2^r \cdot n \cdot \frac{1}{r} \sum_{j=1}^{r} \left( \frac{n}{r} \right)^{\frac{j}{2}} \left( \frac{j-1}{d} \right) dt_j \frac{d}{dx},
\]

which, in view of (2.11) and (2.16), implies (2.15) and the relation
\[
(2.22)
\]
\[
c_1 = 2^r \cdot n \cdot \frac{1}{r} \sum_{j=1}^{r} \left( \frac{n}{r} \right)^{\frac{j}{2}}.
\]

Thus we obtain the formula
\[
(2.23) \quad \left[ \varphi' \left( s \right) \right] = (2\pi)^{n-r} \frac{1}{r} \sum_{j=1}^{r} \left( \frac{n}{r} \right)^{\frac{j}{2}} \left( \frac{j-1}{d} \right) \frac{d}{dx} \left( \frac{n}{r} s - \frac{d}{2} \left( j - 1 \right) \right).
\]

Our computation also shows that the integral for \( \left[ \varphi' \left( s \right) \right] \) converges absolutely for \( \text{Res} > 1 - \frac{r}{n} \).

From the relation (1.2) one obtains
\[
\left[ \varphi \left( s \right) \right] \left[ \varphi \left( 1-s \right) \right] = (2\pi)^{n-r} \frac{1}{r} \sum_{j=1}^{r} \left( \frac{n}{r} \right)^{\frac{j}{2}} \left( \frac{j-1}{d} \right) \frac{d}{dx} \left( \frac{n}{r} s - \frac{d}{2} \left( j - 1 \right) \right)
\]
\[
= (2\pi)^{n-r} (2\pi 1)^{r} \frac{1}{r} \sum_{j=1}^{r} \frac{e^{\pi i \left( \frac{n}{r} s - \frac{d}{2} \left( j - 1 \right) \right)}}{e^{2\pi i \left( \frac{n}{r} s - \frac{d}{2} \left( j - 1 \right) \right)}} - 1.
\]

Since one has by (2.11)
\[
n - r = \frac{d}{\frac{r}{2}} = \begin{cases} 0 \pmod{2} & \text{for } d \text{ even} \\ \left[ \frac{r}{2} \right] \pmod{2} & \text{for } d \text{ odd}, \end{cases}
\]

one has
\[
\prod_{j=1}^{r} e^{-\pi i \left( \frac{d}{r} \right) \left( j - 1 \right)} = (-1)^{d} \frac{r}{r} \left( \frac{d}{r} \right) = \begin{cases} 1^{n-r} & \text{for } d \text{ even} \\ 1^{n-r}(-1)^{\left[ \frac{r}{2} \right]} & \text{for } d \text{ odd}. \end{cases}
\]

Hence one obtains the following functional equation:
\[
(2.24) \quad \left[ \varphi \left( s \right) \right] \left[ \varphi \left( 1-s \right) \right] = (2\pi i)^{n} e^{\pi i \left( \frac{n}{r} s \right)} \left\{ \begin{array}{ll}
(1^{n-r})^{r} & \text{for } d \text{ even} \\
(1^{n-\left[ \frac{r}{2} \right]}) e^{\pi i \left( \frac{n}{r} s \right)} & \text{for } d \text{ odd}. \end{array} \right.
\]
§ 3. Zeta functions of a self-dual homogeneous cone.

3.1. We fix a \( \mathbb{Q} \)-structure on \( U \) and assume that (the Zariski closure of) \( G \) is defined over \( \mathbb{Q} \) and \( e \in U_{\mathbb{Q}} \); then (the Zariski closure of) \( K \) is also defined over \( \mathbb{Q} \). We also fix a lattice \( L \) in \( U \) compatible with that \( \mathbb{Q} \)-structure, i.e., such that \( U_{\mathbb{Q}} = L \oplus \mathbb{Z} \), and an arithmetic subgroup \( \Gamma \) fixing \( L \), i.e., a subgroup of \( G_L = \{ g \in G \mid gL = L \} \) of finite index; for simplicity we assume that \( \Gamma \) has no fixed point in \( \mathcal{L} \). We then define the zeta function associated with \( \mathcal{L}, \Gamma, L \) as follows:

\[
\zeta_{\mathcal{L}}(s; \Gamma, L) = \sum_{u \in \Gamma \backslash \mathcal{L} \cap L} N(u)^{-s},
\]

the summation being taken over a complete set of representatives of \( \mathcal{L} \cap L \) modulo \( \Gamma \). It can be shown easily that this series is absolutely convergent for \( \text{Re } s > 1 \).

By the reduction theory, \( \Gamma \) has a fundamental domain in \( \mathcal{L} \) which is a rational polyhedral cone. More precisely, there exists a finite set of simplicial cones

\[
C^{(i)} = \{ v_1^{(i)}, \ldots, v_{I_i}^{(i)} \}_{R_+}
\]

\[
= \left\{ \sum_{i=1}^{I_i} \lambda_i v_i^{(i)} \mid \lambda_i \in R_+ \right\} \quad (1 \leq i \leq m),
\]

where \( v_1^{(i)}, \ldots, v_{I_i}^{(i)} \) are linearly independent elements in \( \bar{\mathcal{L}} \cap L \), such that

\[
\mathcal{L} = \bigsqcup_{1 \leq i \leq m} \chi C^{(i)}.
\]

It follows that

\[
\zeta(s; \Gamma, L) = \sum_{i=1}^{m} \sum_{u \in C^{(i)} \cap L} N(u)^{-s}.
\]

For a set of linearly independent vectors \( v_1, \ldots, v_l \in L \), we put

\[
R((v_j), L) = \left\{ \sum_{j=1}^{l} \lambda_j v_j \mid 0 < \lambda_j \leq 1 \right\} \cap L,
\]

which is finite. Then \( u \in \mathcal{C}^{(i)} \cap L \) can be written uniquely in the form

\[
u = v_0 + \sum_{j=1}^{l} m_j v_j^{(i)}, \quad v_j \in R(v_j^{(i)}, L), \quad m_j \in \mathbb{Z}, \quad m_j \geq 0.
\]
For a set of linearly independent vectors \( v_1, \ldots, v_\ell \in \mathcal{L} \cap V_\mathcal{Q} \) and \( v_0 = \sum_j \alpha_j v_j (\alpha_j \in \mathcal{Q}_+ \) ), we define a "partial zeta function" by

\[
\zeta_\mathcal{Q}(s; (v_j), v_0) = \sum_{\nu_0 \geq 0} N(v_0) + \sum_{j=1}^\ell m_j v_j)^{-s},
\]

which will also be written as \( \zeta_\mathcal{Q}(s; (v_j), (\alpha_j)) \). Then the zeta function (3.1) can be written as a finite sum of partial zeta functions as follows:

\[
\zeta_\mathcal{Q}(s; \Gamma, L) = \sum_{i=1}^\infty \sum_{v_0 \in K((\mathfrak{v}_j), \mathcal{L})} \zeta_\mathcal{Q}(s; (v_j^{\{i\}}, v_0).
\]

Hence the study of special values of \( \zeta_\mathcal{Q}(s; \Gamma, L) \) is reduced to that of the partial zeta functions of the form (3.2).

3.2. Let \( (v_j) \) and \( v_0 \) be as above. Then by (2.7) one obtains

\[
\Gamma_\mathcal{Q}(s) \zeta_\mathcal{Q}(s; (v_j), v_0) = \sum_{\nu_0 \geq 0} \nu_0^s \Gamma_\mathcal{Q}(s) N(v_0) + \sum_{j=1}^\ell m_j v_j)^{-s} = \sum_{\nu_0 \geq 0} \int_\mathcal{Q} N(u)^{s-1} e^{-\sum_j (\alpha_j + m_j) <v_j, u>} du = \int_\mathcal{Q} N(u)^{s-1} \prod_{j=1}^\ell b(<v_j, u>, l - \alpha_j) d\mathcal{Q}(u) = \int_G \det(g)^{s-1} \prod_{j=1}^\ell b(<v_j, g \cdot , l - \alpha_j) dg.
\]

In the notation of \( \xi \), but this time using the decomposition \( G = KAK \), one has

\[
\Delta(a) = \int (e^{\alpha \log a} - e^{-\alpha \log a})^d
\]

for \( g = kak', k, k' \in K, a \in A \). Here \( c \) is a positive constant and

\[
\Delta(a) = \prod_{\alpha \in \mathcal{F}_+} (e^{\alpha \log a} - e^{-\alpha \log a})^d = \prod_{i=1}^r (t_i)^{-\frac{d}{2}(r-1)} |\Delta(t_1, \ldots, t_r)|^d,
\]

where \( \Delta(t_1, \ldots, t_r) = \prod_{i<j} (t_i - t_j) \) (cf. Helgason [8], Ch. X, \( \xi \)). Hence in view of (2.11) and (2.17) one has

\[
\Gamma_\mathcal{Q}(s) \zeta_\mathcal{Q}(s; (v_j), (\alpha_j)) = c \int_0^\infty \cdots \int_0^\infty \int_0^\infty (\prod_{i=1}^r t_i)^{\frac{n}{2}(s-1)} |\Delta(t)|^d \gamma(t) \prod_{i=1}^r dt_i,
\]

where
\[ F(t_1, \ldots, t_r) = \int K \prod_{j=1}^{r} b(< v_j, k \sum t_i e_i >, 1 - \alpha_j) \, dk. \]

It is clear that \( F(t_1, \ldots, t_r) \) is holomorphic for \( \text{Re} \, t_i > 0 \) (\( 1 \leq i \leq r \)).

Since \( K \) contains an element which induces any given permutation of \( e_1, \ldots, e_r \), the function \( F \) is symmetric. Hence, denoting by \( B_i \) an open simplicial cone in \( \mathbb{R}^r \) defined by \( t_1 > \ldots > t_r > 0 \), one has

\[ (3.5') \quad \Gamma_{\Delta}(s) \zeta_{\Delta}(s; (v_j), (\alpha_j)) = c \, r! \int_{B_1} \left( \prod t_i^{i} \right)^{s-1} \Delta(t)^d \, F(t) \prod dt_i. \]

3.3. Still following Shintani [11], we make a change of variables \( (t_i) \rightarrow (t_i, \tau_1, \ldots, \tau_r) \) with \( \tau_i = t_i/t_{i-1} \) (\( 2 \leq i \leq r \)). Then \( B_1 \) can be expressed as

\[ B_1 = \left\{ (t_i) \mid t_i = t_1^{i} \prod_{j=2}^{i} \tau_j, \quad 0 < t_i < \infty, \quad 0 < \tau_i < 1 \right\}. \]

Putting \( \tau_1 = t_1 \), one has

\[ \frac{\varphi(t_1, \ldots, t_r)}{\varphi(t_1, \tau_2, \ldots, \tau_r)} = \prod_{i=1}^{r} \tau_i^{r-i}, \]

\[ \prod_{i} t_i = \prod_{i} \tau_i^{r-i+1}, \]

\[ \Delta(t) = \prod_{i} \tau_i^{\frac{1}{2}(r-i+1)(r-i)} \prod_{\xi \leq i < j \leq r} (1 - \tau_i \ldots \tau_j). \]

It follows that the exponent of \( \tau_i \) in the integrand in \( (3.5') \) is equal to

\[ (r-i+1)\frac{n}{r}(s-1) + \frac{r}{2} (r-i+1)(r-i) + r - 1 \]

\[ = (r-i+1) \left\{ \frac{n}{r} s - \frac{d}{2} (i-1) \right\} - 1. \]

Hence one has

\[ \gamma_{\Delta}(s) \zeta_{\Delta}(s; (v_j), (\alpha_j)) = c \, r! \int_{0}^{\infty} t^{n s-1} \, dt \]

\[ \left( \prod_{\xi} \tau_i^{(r-i+1)} \left\{ \frac{n}{r} s - \frac{d}{2} (i-1) \right\} - 1 \right) \eta(t_i, \tau_i) \prod_{i=2}^{r} d \, \tau_i, \]

where

\[ (3.6) \quad \tilde{\eta}(t_i, \tau_i) = \prod_{\xi \leq i < j \leq r} (1 - \tau_i \ldots \tau_j) \eta(t_i, t_1 \tau_2, \ldots, t_i \tau_i \ldots \tau_r). \]
3.4. We now assume that all \( v_j \)'s are in \( \mathcal{Q} \) (not on the boundary of \( \mathcal{Q} \)).

(In the situation explained in 3.1, this means that the \( \mathbb{Q} \)-rank of \( G \) is equal to 1.) Then for any \( v \in \mathcal{Q} - \{0\} \), one has \( \langle v_j, v \rangle > 0 \); in particular,

\[
\langle v_j, ke_i \rangle > 0 \quad \text{for all } k \in K, 1 \leq i \leq r.
\]

Put

\[
\xi_j = \langle v_j, k \sum t_i e_i \rangle
= t_1 \langle v_j, k(e_i + \sum_{i=2}^r t_i \ldots t_1 e_i) \rangle
= t_1 \langle v_j, ke_i \rangle (1 + \sum_{i=1}^r t_i \ldots t_i \frac{\langle v_j, ke_i \rangle}{\langle v_j, ke_i \rangle}).
\]

For the fixed \( e_i, v_j \), choose \( \rho, \rho_i, \rho_j > 0 \) in such a way that

\[
\begin{cases}
\frac{\sum_{i=2}^r \rho^{i-1}}{\rho} \frac{\langle v_j, ke_i \rangle}{\langle v_j, ke_i \rangle} < 1 \quad \text{for all } k \in K, 1 \leq j \leq l, \\
\rho_i < \frac{\rho}{\langle v_j, ke_i \rangle} \quad \text{for all } 1 \leq i < l.
\end{cases}
\]

The proof

\[
0 < |t_i| < \rho_i, \quad |\tau_i| < \rho \quad (2 \leq i \leq r),
\]

one has \( 0 < |\xi_j| < 2 \pi \) and so \( b(\xi_j, 1 - \xi_j) \) is holomorphic. Hence the function \( F(t) = F(t_1, t_1 \tau_1, \ldots, t_1 \tau_{i-1} \ldots \tau_r) \) has a Laurent expansion in \( t_1, \tau_1, \ldots, \tau_r \) in the domain defined by (3.11). The coefficients in this expansion is a \( \mathbb{Q} \)-linear combination of the integrals of the form

\[
I((v_j)) = \int_{K} \prod_{i \leq i \leq \ell}^{\ell} \langle v_j, ke_i \rangle^* \text{dk}
\]

where \( v_j \geq 0 \) for \( 2 \leq i \leq r \) and \( v_j \in \mathbb{Z} \) for all \( i, j \).
3.5. Let $I(\varepsilon, l)$ denote the contour consisting of the line segment $[\varepsilon, l]$ taken twice in opposite directions and of a (small) circle of radius $\varepsilon$ about the origin taken in the counterclockwise direction. When the $\tau_i$ $(2 \leq i \leq r)$ are on $I(\varepsilon, l)$, one has by (2.12)

$$|<v_j, k(e_1 + \sum_{i=2}^{\varepsilon} \tau_i \cdots \tau_i e_i)>| \leq |v_j| \left| \sum_{i=1}^{\varepsilon} \left| e_i \right| \right| = \sqrt{n\pi} |v_j|$$

and

$$\text{Re} <v_j, k(e_1 + \sum_{i=2}^{\varepsilon} \tau_i \cdots \tau_i e_i)> = <v_j, ke_1> + \sum_{i=2}^{\varepsilon} \text{Re}(\tau_i \cdots \tau_i) <v_j, ke_i> \geq <v_j, ke_1> - \varepsilon |v_j| \sum_{i=2}^{\varepsilon} |e_i|$$

$$= <v_j, ke_1> - \varepsilon (r-1) \sqrt{\pi \frac{n}{r}} |v_j|.$$ 

We choose $\varepsilon$ so that one has

$$(3.13) \quad \varepsilon \sqrt{n \pi} |v_j| < \min \left\{ 2\pi, <v_j, ke_i> (k \in K) \right\} \quad \text{for all } 1 \leq j \leq l,$$

The above inequalities show that $<v_j, k(e_1 + \sum_{i=2}^{\varepsilon} \tau_i \cdots \tau_i e_i)>$ belongs to the domain

$$\left\{ z \in \mathbb{C} \mid |z| < \frac{2\pi}{\varepsilon}, \text{Re} z > \varepsilon \sqrt{\pi \frac{n}{r}} |v_j| \right\}.$$ 

It follows that, if $\tau_i$ is on the contour $I(\varepsilon, \infty)$, one has

$$|\xi_j| < 2\pi \quad \text{or} \quad \text{Re} \xi_j > 0,$$

so that the function $b(\xi_j, 1 - \alpha_i)$ is holomorphic.

From this observation, it is clear that the integral on the r.h.s. of (3.6) is equal to the contour integral

$$(e^{2\pi i ns} - 1)^{-1} \int_{\tau_i \in I(\varepsilon, \infty)} \left| \prod_{i=2}^{r} (e^{2\pi i \frac{\tau_i - 1}{r} ns} - 1)^{-1} \right|$$

which is independent of the choice of $\varepsilon$ satisfying (3.13). As is easily seen, the contour integral converges for all $s \in \mathbb{C}$. Hence the integral $b^{(r-1)l}(x_0, \ldots, x_n)$, viewed as a function in $s$, can be continued to a meromorphic function on the whole plane; the possible poles are of the form $\frac{e^{i\pi (r-1)l}}{(r-1)l n}$ ($\nu \in \mathbb{Z}$).
§ 4. The special values of the zeta functions.

4.1. As a preliminary, we check the rationality of the constant \( c \) in (3.4).

For that purpose, we compute \( \Gamma_\mathcal{O}(s) \) by using the decomposition \( G = \mathcal{K} \mathcal{A} \).

\[
\Gamma_\mathcal{O}(s) = \int_{\mathcal{O}} N(u)^s e^{-\tau(u)} d\mathcal{O}(u)
= \int_G N(ge)^s e^{-\tau(ge)} dg
= c \int_A \det(a)^s e^{-\tau(ae)} \Delta(a) da
= c \int_0^\infty \int_{\prod_t} (\prod_t t_i)^{s-1} |\Delta(t)|^d e^{-\tau(t)} \prod dt_i.
\]

We make another change of variables:

\[
t = \sum_{i=1}^r t_i, \quad t_i' = t_i/t.
\]

Then

\[
\frac{\varphi(t_1, \ldots, t_r)}{\varphi(t, t_1', \ldots, t_r')} = (-1)^r t_r^{-1},
\]

and the exponent of \( t \) in the integrand in the last member of (4.1) is equal to

\[
n(s - 1) + \frac{d}{2} r(r - 1) + r - 1 = ns - 1.
\]

Hence one has

\[
\Gamma_\mathcal{O}(s) = c \varphi(s) \beta(s),
\]

where

\[
\left\{
\begin{array}{l}
\varphi(s) = \int_0^\infty t^{ns-1} e^{-\frac{t}{r}} t \ dt = (\frac{r}{n})^{ns} \Gamma(ns), \\
\beta(s) = \int_{\sum_{i=1}^r t_i' > 0} \prod_{i=1}^r \left( t_i' \right)^{s-1} |\Delta(t_1', \ldots, t_r', l - \sum_{i=1}^r t_i' )|^d \prod dt_i'.
\end{array}
\right.
\]

For \( s = 1 \), one has

\[
\Gamma_\mathcal{O}(1) = c \varphi(1) \beta(1) = c (\frac{r}{n})^n (n-1)! \beta(1),
\]
\[ F(1) = \int \left| \Delta(t_1', \ldots, t_{r-1}', 1 - \Sigma t_i') \right|^d \prod dt_i' \in \mathbb{Q}. \]

By (2.23) one has

\[ (4.4) \quad \sum_{a \sim b} (2\pi)^{\frac{n-r}{2}} (\frac{n}{r})^\frac{n}{2} \prod_{j=1}^{r} \Gamma(1 + \frac{d}{2}(j-1)) \]

\[ \sim \left\{ \begin{array}{ll}
\frac{n-r}{2} & \text{(d even)} \\
\frac{n-[r+1]}{2} & \text{(d odd)}
\end{array} \right\}
\]

where \( a \sim b \) means that \( a/b \in \mathbb{Q} \). Thus one has

\[ (4.5) \quad c = \frac{(2\pi)^{\frac{n-r}{2}} (\frac{n}{r})^\frac{n}{2} \prod_{j=1}^{r} \Gamma(1 + \frac{d}{2}(j-1))}{(n-1)!} \sim \sum_{a \sim b} \Gamma_\mathcal{Q}(1).
\]

Since \( \sum_{a \sim b} \Gamma_\mathcal{Q}(1) \sim \Gamma_\mathcal{Q}(1 + \frac{r}{n}) \) for \( \forall \in \mathbb{Z} \), one obtains

\[ (4.6) \quad c \sum_{a \sim b} \Gamma_\mathcal{Q}(1 + \frac{r}{n}) \sim \sum_{a \sim b} \Gamma_\mathcal{Q}(1)^2 \sim \left\{ \begin{array}{ll}
\frac{n-r}{2} & \text{(d even)} \\
\frac{n-[r+1]}{2} & \text{(d odd)}
\end{array} \right\} \sum_{a \sim b} \Gamma_\mathcal{Q}(1).
\]

4.2. We first consider the case where \( d \) is even. Then by (2.24) one has

\[ \sum_{a \sim b} \Gamma_\mathcal{Q}(1) \Gamma_\mathcal{Q}(1-s) = (2\pi i)^n e^{-\pi i s} (e^{2\pi i \frac{n}{r}} s - 1)^{-r}.
\]

Hence

\[ (4.7) \quad \zeta_\mathcal{Q}(s; (\nu), (\zeta)) = \frac{c \sum_{a \sim b} \Gamma_\mathcal{Q}(1-s)}{(2\pi i)^{n-r} e^{-\pi i s}} R(s),
\]

where

\[ R(s) = \left( \frac{e^{2\pi i \frac{n}{r}} s - 1}{2\pi i} \right)^{\frac{r}{n}} \int_{B_1} \left( \prod t_i \right)^{\frac{n}{r}} \Delta(t) \frac{d}{d t_i} \prod dt_i
\]

\[ = \frac{r}{n} \left( \frac{e^{2\pi i \frac{n}{r}} s - 1}{e^{2\pi i \frac{n}{r}} s - 1} \right) \int_{B_1} \frac{1}{(2\pi i)^{s-1}} (\prod t_i)^{r-s-1} \frac{1}{t_i} (t_i, \epsilon, \omega) \left( \prod \frac{d t_i}{t_i} \right)
\]

We are interested in the values of \( \zeta_\mathcal{Q} \) at \( s = -\frac{r}{n} \nu \) (\( \nu = 0, 1, \ldots \)). The
first factor in the right hand side of (4.7) is holomorphic for \( \Re s < \frac{1}{n} \)
and by (4.6) the value at \( s = -\frac{1}{n} \) is rational:

\[
(4.9) \quad \frac{c \int_{\mathcal{D}} (1 + \frac{r}{n} \nu)}{(2\pi i)^{n-r} e^{-r\nu \pi i}} = (-1)^{\frac{n-r}{2} + r \nu} \frac{c \int_{\mathcal{D}} (1 + \frac{r}{n} \nu)}{(2\pi)^{n-r}} \in \mathbb{Q}.
\]

On the other hand, it is clear that

\[
\frac{e^{2\pi i \frac{r}{n} s} - 1}{2\pi i \frac{r}{n} s - 1} \rightarrow \frac{1}{r-1+i} \quad \text{when } s \rightarrow -\frac{r}{n} \nu.
\]

Hence we see that \( R(-\frac{1}{n} \nu) \) is equal to the coefficient of

\[
t_{r-1}^\nu \prod_{i=2}^r \tau_i^{(r-i+1)\nu + \frac{d}{2}(i-1)}
\]

in the Laurent expansion of \( \tilde{F}(t, \tau) \),

which is a \( \mathbb{Q} \)-linear combination of \( I((\nu \gamma')) \).

4.3. From now on we assume that \( d \) is odd. By the classification theory,
it is known that this assumption implies that \( r = 2 \) \((n = d + 2)\) or \( d = 1 \) \((n = \frac{1}{2} r(r+1))\). By (2.24) one has

\[
g_{\mathcal{D}}(s) \left[ g_{\mathcal{D}}(1-s) \right] = (2\pi i)^n e^{\pi i s} (e^{2\pi i \frac{r}{n} s} - \frac{[\frac{1}{2}]}{[\frac{1}{2}]}) (e^{2\pi i \frac{r}{n} s} + 1)^{-[\frac{r}{2}]}. \]

Hence

\[
(4.11) \quad \zeta_{\mathcal{D}}(s; \nu, \omega) = \frac{c \int_{\mathcal{D}} (1-s)^n e^{\pi i s}}{(2\pi i)^{n-\left[\frac{1}{2}\right]} e^{\pi i s}} \times R^{(s)}(s) R^{(s)}(s),
\]

where

\[
R^{(0)}(s) = (2\pi i)^{\left[\frac{1}{2}\right]} \frac{(e^{2\pi i \frac{r}{n} s} - 1)^{\left[\frac{1}{2}\right]} (e^{2\pi i \frac{r}{n} s} + 1)^{\left[\frac{r}{2}\right]}}{\prod_{k=1}^r (e^{2\pi i (r-k+1) \frac{n}{2}} s - \frac{d}{2} (k-1)^{2} - 1)},
\]

\[
R^{(1)}(s) = (2\pi i)^{-r} \int_{t_1}^{ns-1} dt_1 \int \prod_{i=1}^r \left[ (r-i+1) \frac{n}{2} s - \frac{d}{2} (i-1)^{2} - 1 \right] e^{\nu \pi i t_1} \int_{(t_1, \tau)} \frac{d \tau}{I(t_1, \tau)}.
\]
The first factor in the right hand side of (4.11) is holomorphic for \( \Re s < \frac{1}{n} \) and by (4.6) the value at \( s = -\frac{1}{n} \) (\( \nu \geq 0 \)) is rational:

\[
(4.12) \quad \frac{c \frac{\Gamma}{2} (1+\frac{1}{n} \nu)}{(2\pi i)^{n-[\frac{r+1}{2}]}} e^{-\pi i r \nu} = (-1)^{\frac{1}{2}} (\frac{n-[\frac{r+1}{2}]}{2}) \cdot \frac{c \frac{\Gamma}{2} (1+\frac{1}{n} \nu)}{(2\pi i)^{n-[\frac{r+1}{2}]}} \in \mathbb{Q}.
\]

Note that one has

\[ n \equiv \left[ \frac{r+1}{2} \right] \pmod{2}, \]

since

\[ n = d+2 \equiv 1 \equiv \left[ \frac{3}{2} \right] \pmod{2} \quad \text{if} \quad r = 2, \quad \text{and} \]

\[ n = \frac{1}{2} r(r+1) \equiv \left[ \frac{r+1}{2} \right] \pmod{2} \quad \text{if} \quad d = 1. \]

4.4. To compute \( R^{(r)}(s) \), we first note

\[ e^{\pi i d(k-1)(r-k+1)} = \begin{cases} 
-1 & \text{if} \quad k \equiv r \equiv 0 \pmod{2}, \\
1 & \text{otherwise}.
\end{cases} \]

We put

\[ \frac{r}{2} = r_1, \quad \zeta = e^{2\pi i \frac{r}{r_1}} s. \]

The case \( r \) is odd. One has

\[
R^{(r)}(s) = (2\pi i)^{r_1} r_1! \prod_{k=1}^{r} \left( \frac{\zeta - 1}{\zeta^k - 1} \right) \prod_{k=1}^{r} \left( \frac{\zeta + 1}{\zeta - 1} \right).
\]

Hence, when \( s \rightarrow -\frac{1}{n} \nu \), one has

\[
(4.13) \quad (s + \frac{1}{n} \nu)^{r_1} R^{(r)}(s) \longrightarrow (2 \frac{r}{r_1})^{r_1}.
\]

Thus \( R^{(r)}(s) \) has a pole of order \( r_1 \) at \( s = -\frac{1}{n} \nu \).

The case \( r \) is even. One has

\[
R^{(r)}(s) = (2\pi i)^{r_1} r_1! \prod_{k=1}^{r} \left( \frac{(-1)^k \zeta^k - 1}{\zeta - 1} \right)
\]
Hence $R^{(s)}$ is holomorphic at $s = -\frac{r}{n} \nu$ and

$$R^{(s)}(-\frac{r}{n} \nu) = (-2\pi i)^{-r} \frac{r!}{(2r_1)^n} (-\pi i)^{r_1} \frac{r_1!}{r_1!}.$$

4.5. When $r$ is odd (hence $d = 1, n = \frac{1}{2} \nu (r+1)$), $R^{(s)}(s)$ for $s = -\frac{r}{n} \nu = -\frac{2\nu}{r+1}$ is given by the coefficient of

$$t_1^{r_1} \frac{r}{r+1} \prod_{i=2}^{r} \tau_i^{(r-i+1)}(\nu + \frac{i-1}{2})$$

in the Laurent expansion of $\tilde{F}(t_1, \tau)$. Hence $\zeta_{\tilde{Q}}(s; (\nu), (\alpha))$ has at most a pole of order $r_1 = \frac{r-1}{2}$ at $s = -\frac{2\nu}{r+1}$ and one has

$$\lim_{s \to -\frac{2\nu}{r+1}} (s + \frac{2\nu}{r+1})^{r_1} \zeta_{\tilde{Q}}(s; (\nu), (\alpha)) \sim \frac{R^{(s)}}{Q} (-\frac{2\nu}{r+1}).$$

To treat the case $r$ is even, we use the formula

$$\int \frac{t^m}{t^{m-1}} dt = -\frac{t^m}{m} \quad (m \text{ odd}),$$

which can be verified easily. When $r$ is even, the value of $R^{(s)}(s)$ for $s = -\frac{r}{n} \nu$ is given by

$$(-\pi i)^{-r} \sum_{m_1; \ldots; m_r, \mu_j \in \mathbb{Z}} \frac{a(\mu_j)}{\prod_{j=1}^{m_r} (m_j - (r-2j+1)(\nu + \frac{d}{2}(2j-1)))^{\nu + \frac{d}{2}(2j-1)}^{r_j - 1}}$$

where $a(\mu_j)$ is the coefficient of

$$t_1^{r_1} \prod_{j=2}^{r} \tau_j^{(r-x+2)}(\nu + d(j-1)) \prod_{j=1}^{r_1} \tau_{2j}^{m_j}$$

in $\tilde{F}(t_1, \tau)$. Hence for the value of $\zeta_{\tilde{Q}}$, one has

$$\zeta_{\tilde{Q}}(-\frac{r}{n} \nu; (\nu), (\alpha)) \sim (2\pi i)^{r_1} R^{(s)}(-\frac{r}{n} \nu).$$
Bibliography


