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On special values of zeta functions
associated with a self-dual cone

東北大 佐武一郎

以下に述べるは松島与三氏還暦記念論文集（Birdhäuser）のために準備の一编である。京都の研究集会ではこの後半について不満を残しきりに。その要約を提出する予定であったが、都会上京稲（の稲穂）を出させて頂くことにした。本文では説明し、通り、これに述べる方法は本質的に改新谷氏[11]のアイデアに基づくものである。$r=2$（circular cone）の場合にはより精密な計算をすることのでき、栗原氏の独立に結果を得られること、これについてはは別に別の機会に触れたいと思う。
§ 1. Introduction

To explain the main idea of this paper, and also to fix some notations, we start with reviewing the classical case of Riemann zeta function. As usual, we set

\[ \zeta(s) = \sum_{n=1}^{\infty} \frac{n^{-s}}{} \quad (\mathrm{Re} \, s > 1), \]

\[ \Gamma(s) = \int_{0}^{\infty} x^{s-1} e^{-x} \, dx \quad (\mathrm{Re} \, s > 0). \]

Then, for \( \mathrm{Re} \, s > 1 \), one obtains

\[ \Gamma(s) \zeta(s) = \sum_{n=1}^{\infty} \frac{n^{-s}}{} \int_{0}^{\infty} x^{s-1} e^{-nx} \, dx \]

\[ = \sum_{n=1}^{\infty} \int_{0}^{\infty} x^{s-1} e^{-nx} \, dx \quad (x = nx') \]

\[ = \int_{0}^{\infty} \frac{x^{s-1}}{e^x - 1} \, dx. \]

We put

\[ b(x, y) = \frac{e^{xy}}{e^x - 1} = \sum_{\nu=0}^{\infty} \frac{B_{\nu}(y)}{\nu!} \, x^{\nu-1} \quad (|x| < 2\pi), \]

where

\[ B_{\nu}(y) = \sum_{\mu=0}^{\nu} \binom{\nu}{\mu} b_{\mu} \, y^{\nu-\mu} \]

is the Bernoulli polynomial, in which the \( b_{\mu} \) are the Bernoulli numbers:

\[ b_0 = 1, \quad b_1 = -\frac{1}{2}, \]

\[ b_\nu = \begin{cases} (-1)^{\nu-1} B_{\frac{\nu}{2}} & (\nu \text{ even, } \geq 2), \\ 0 & (\nu \text{ odd, } \geq 3). \end{cases} \]

Then the above integral can be transformed into a contour integral of the form

\[ (1.1) \quad \Gamma(s) \zeta(s) = (e^{2\pi is} - 1)^{-1} \int_{\gamma(\epsilon, \infty)} x^{s-1} b(x, 0) \, dx, \]

where \( \gamma(\epsilon, \infty) \) denotes the contour consisting of the half-line \([ \epsilon, \infty)\) taken twice in opposite directions and of a (small) circle of radius \( \epsilon \).
about the origin taken in the counterclockwise direction. The contour integral is absolutely convergent for all \( s \in \mathbb{C} \), so that the function \( \Gamma(s) \zeta(s) \) can be analytically continued to a meromorphic function on \( \mathbb{C} \). Moreover, in virtue of the functional equation of the gamma function:

\[
\Gamma(s)\Gamma(1-s) = \frac{\pi}{\sin \pi s} = 2\pi i \frac{e^{\pi is}}{e^{2\pi is} - 1},
\]

one obtains

\[
\zeta(s) = e^{-\pi is} \Gamma(1-s) \frac{1}{2\pi i} \int_{t=0} \frac{x^{s-1} b(x, 0)}{I(t, x)} dx.
\]

This shows that \( \zeta(s) \) is holomorphic for \( \text{Re } s < 1 \). In particular, for \( s = 1 - m, m \in \mathbb{Z}^+ \) (positive integers), the contour integral reduces to the residue of \( x^{-m} b(x, 0) \) at \( x = 0 \), i.e., \( b_m / m! \). Hence, one obtains

\[
\zeta(1 - m) = (-1)^{m-1}(m-1)! \frac{b_m}{m!} = (-1)^{m-1} \frac{b_m}{m}.
\]

Thus \( \zeta(1 - m) \) (\( m \in \mathbb{Z}^+ \)) is rational. In particular,

\[
\zeta(0) = -\frac{1}{2}, \quad \zeta(-1) = -\frac{1}{12},
\]

\[
\zeta(-2\mu) = 0, \quad \zeta(1 - 2\mu) = (-1)^\mu \frac{B_\mu}{2^\mu} \quad (\mu \geq 1).
\]

This result has been generalized by Hecke, Klingen and Siegel \([3]\) to the case of Dedekind zeta functions of totally real number fields. More recently, Shintani \([1]\) gave a proof based on a direct generalization of the classical method explained above. Zeta functions attached to self-dual homogeneous cones have been studied by Siegel \([3]\) in a special case of quadratic cones, and by Sato-Shintani \([8]\) in a more general context of "prehomogeneous spaces". (Cf. also Shintani \([7]\), \([10]\).) On the other hand, the gamma functions attached to self-dual homogeneous cones were studied by Koecher \([5]\), Gindikin \([3]\) and others (cf. e.g., Resnikoff \([6]\)). In this paper, we try to extend Shintani's method (i.e., the classical method) to examine the rationality of the special values of zeta functions attached to self-dual homogeneous cones.
§2. The gamma function of a self-dual homogeneous cone

2.1. Let $U$ be a real vector space of dimension $n$, endowed with a positive definite inner product $<\cdot, \cdot>$. By a "cone" in $U$ we always mean a non-degenerate open convex cone in $U$ with vertex at the origin, i.e., a non-empty open set $\mathcal{L}$ in $U$ such that

$$x, y \in \mathcal{L}, \lambda, \mu \in \mathbb{R}^+ \Rightarrow \lambda x + \mu y \in \mathcal{L}$$

and such that $\mathcal{L}$ does not contain any straight line. A cone $\mathcal{L}$ in $U$ is called homogeneous if the group of linear automorphisms

$$G(\mathcal{L}) = \{ g \in GL(U) \mid g(\mathcal{L}) = \mathcal{L} \}$$

is transitive on $\mathcal{L}$; and $\mathcal{L}$ is called self-dual if the "dual" of $\mathcal{L}$

$$\mathcal{L}^* = \{ x \in U \mid <x, y> > 0 \text{ for all } y \in \overline{\mathcal{L}} - \{ 0 \} \}$$

coincides with $\mathcal{L}$.

Let $\mathcal{L}$ be a self-dual homogeneous cone in $U$ and $G = G(\mathcal{L})^\circ$. Then it is well-known (e.g., Satake [7]) that the Zariski closure of $G$ (in $GL(U)$) is a reductive algebraic group, containing $G(\mathcal{L})$ as a subgroup of finite index, and $g \mapsto g^{-1}$ is a Cartan involution of $G$; the corresponding maximal compact subgroup $K = G \cap O(U)$ coincides with the isotropy subgroup of $G$ at a "base point" $e \in \mathcal{L}$ (which is not unique, but will be fixed once and for all). Let

$$\mathfrak{g} = \mathfrak{k} + \mathfrak{g}$$

be the corresponding Cartan decomposition of $\mathfrak{g} = \text{Lie } G$. Then $\mathfrak{k} = \text{Lie } K$ and one has for $T \in \mathfrak{g}$

$$(2.1) \quad T \in \mathfrak{k} \iff t_T = -T \iff Te = 0.$$ 

It follows that, for each $u \in U$, there exists a uniquely determined element $T_u \in \mathfrak{g}$ such that $T_u e = u$. It is well-known that the vector space $U$ endowed with a product
\[ u \ast u' = T_u u' \quad (u, u' \in U) \]

becomes a formally real Jordan algebra (cf. Braun-Koecher [2], or Satake [7]).

We define the (regular) trace on \( U \) by

\[ \tau(u) = \text{tr}(T_u). \]

For the given \( (P, e) \), one may assume (by Schur's lemma) that the inner product \( \langle \cdot, \cdot \rangle \) is so normalized that one has

\[ \langle u, u' \rangle = \tau(u \ast u') \quad (u, u' \in U). \]

Next, let \( u \in L \). Then, since \( G \) is transitive on \( L \), there exists \( g \in G \) such that \( u = g \ast e \). We define the (regular) norm \( N(u) \) by

\[ N(u) = \det(g), \]

which is clearly independent of the choice of \( g \). There exists a unique element \( u_1 \in U \) such that \( u = \exp u_1 \) (which is defined to be \( (\exp T_{u_1})e \)); then by definition one has

\[ N(u) = \det(\exp T_{u_1}) = e^{\tau(u_1)}. \]

In terms of the "quadratic multiplication" \( P(u) = 2 T_u^2 - T_u \), one can also write \( N(u) = \det(P(u))^{\frac{1}{2}} \). By the definition, it is clear that

\[ N(e) = 1, \quad N(gu) = \det(g) N(u) \quad (g \in G(L), \; u \in L), \]

which characterizes the norm uniquely. Denoting the Euclidean measure on \( U \) by \( du \), we see that \( d_{\mu}(u) = N(u)^{-1} du \) is an invariant measure on \( L \).

Example. Let \( U = \text{Sym}_r(\mathbb{R}) \) (the space of real symmetric matrices of degree \( r \)) and \( L = \mathcal{P}_r(\mathbb{R}) \) (the cone of positive definite elements in \( U \)).

Then one has

\[ T_u(u') = \frac{1}{2} (uu' + u'u) \]

and so

\[ \tau(u) = \frac{r+1}{2} \text{tr}(u), \quad N(u) = \det(u)^{\frac{r+1}{2}}. \]
2.2. We define the gamma function of the cone $\mathcal{L}$ by

$$\Gamma_{\mathcal{L}}(s) = \int_{\mathcal{L}} N(u)^{s-1} e^{-\tau(u)} \, du$$

which converges absolutely for $\text{Re } s$ sufficiently large (actually for $\text{Re } s > 1 - \frac{r}{n}$ as we will see later).

**Lemma 2.1.** Suppose that the inner product $\langle , \rangle$ is normalized by (2.3).

Then one has for any $v \in \mathcal{L}$

$$\int_{\mathcal{L}} N(u)^{s-1} e^{-\langle u, v \rangle} \, du = \Gamma_{\mathcal{L}}(s) N(v)^{-s}.$$

**Proof.** Let $v = g_e^t$ with $g_e \in G$ and put $u' = t g_e u$. Then one has

$$\langle u, v \rangle = \langle u, g_e^t \rangle = \langle u', e \rangle = \tau(u').$$

Hence by (2.5) the left-hand side of (2.7) is equal to

$$\int_{\mathcal{L}} N(u)^{s} e^{-\langle u, v \rangle} \, d_{\mathcal{L}}(u)$$

$$= \int_{\mathcal{L}} (\det(g_e)^{-1} N(u'))^s e^{-\tau(u')} \, d_{\mathcal{L}}(u')$$

$$= N(v)^{-s} \Gamma_{\mathcal{L}}(s), \text{ q.e.d.}$$

It is known that the function $\Gamma_{\mathcal{L}}(s)$ can be expressed as a product of ordinary gamma functions (cf. e.g., Resnikoff loc. cit.). For the sake of completeness, we sketch a proof. First, it is clear that, if

$$\mathcal{L} = \mathcal{L}_1 \times \cdots \times \mathcal{L}_m$$

is the decomposition of $\mathcal{L}$ into the direct product of irreducible (self-dual homogeneous) cones, then one has

$$\Gamma_{\mathcal{L}}(s) = \Gamma_{\mathcal{L}_1}(s) \cdots \Gamma_{\mathcal{L}_m}(s).$$

Hence, for our purpose, we may assume that $\mathcal{L}$ is irreducible.

We need the root structure of $\mathcal{L}$, which can be determined as follows.

Let

$$e = \sum_{i=1}^{r} e_i, \quad e_i e_j = \delta_{ij} e_i$$
be a decomposition of \( e \) (in the Jordan algebra \( U \)) into the sum of mutually orthogonal primitive idempotents. ("Primitive" means that each \( e_i \) can not be decomposed into the sum of mutually orthogonal idempotents any more.) Then we obtain the following decomposition of \( U \) into the direct sum of subspaces ("Peirce decomposition").

\[
U = \bigoplus_{1 \leq i < j \leq n} U_{ij},
\]

where

\[
U_{ii} = \left\{ u \in U \mid e_i u = u \right\},
\]

\[
U_{ij} = \left\{ u \in U \mid e_i u = e_j u = \frac{1}{2} u \right\} \quad (i \neq j).
\]

Then one has \( e_k u = 0 \) for \( u \in U_{ij}, \ k \neq i, j \). Moreover

\[
\dim U_{ii} = 1, \quad \dim U_{ij} = d \quad (i \neq j),
\]

where \( d \) is a positive integer depending on the irreducible cone \( \mathcal{L} \). (For instance, one has \( d = 1 \) for \( \mathcal{L} = \mathcal{P}_r(R) \).) From (2.9), (2.10) one has the relation

\[
n = r + \frac{1}{2} r(r - 1)d, \quad \text{i.e.,} \quad d = \frac{2(n - r)}{r(r - 1)}. 
\]

It follows that

\[
\tau(e_i) = \text{tr}(T_{e_i}) = 1 + \frac{1}{2} (r - 1)d = \frac{n}{r},
\]

Put

\[
\mathfrak{U} = \left\{ T_{e_i} \mid 1 \leq i \leq r \right\}.
\]

Then \( \mathfrak{U} \) is an abelian subalgebra of \( \mathcal{P}_r \) of dimension \( r \) contained in \( \mathcal{F} \).

We denote by \( (\lambda_i) \) the basis of \( \mathfrak{U}^* \) (the dual space of \( \mathfrak{U} \)) dual to \( (T_{e_i}) \), i.e., one has the relation

\[
T = \sum_{i = 1}^r \lambda_i(T) T_{e_i} \quad (T \in \mathfrak{U}).
\]

We put \( \lambda_{ij} = \frac{1}{2} (\lambda_i - \lambda_j) \quad (i \neq j) \).
PROPOSITION 1. The root system of $\Phi$ relative to $\mathfrak{a}_E$ is given by $\Phi = \{ \alpha_{i,j} \mid (i \neq j) \}$. The root space $\Phi(\alpha_{i,j})$ corresponding to $\alpha_{i,j}$ is given by

$$\Phi(\alpha_{i,j}) = \left\{ T_u + [T_{e_i}, -e_j], T_u \mid u \in U_{ij} \right\}.$$  

This can be verified by a straightforward computation; see e.g., Ash et al. [1] Ch. II, §3. Proposition 1 implies that the R-rank of $\Phi$ is equal to $r$ and the root system $\Phi$ is of type $(A_{r-1})$.

2.3. Next we determine the Haar measure of $G$. Put

$$\nu = \sum_{i < j} \Phi(\alpha_{i,j})$$

and let $A, N$ be the analytic subgroups of $G$ corresponding to $\mathfrak{a}_E, \nu$, respectively. Then one has an Iwasawa decomposition $G = NA \cdot K(= N \times A \times K)$, which gives rise to the following formula for (the volume element of) a (biinvariant) Haar measure on $G$:

$$dg = c_1 e^{-2P(\log a)} du \, da \, dk$$  

for $g = nk$ with $n \in N, a \in A, k \in K$, where $du, da, dk$ denote Haar measures on $N, A, K$, respectively, $c_1$ is a positive constant depending on the normalization of the Haar measures, and $P$ is a linear form on $\mathfrak{a}_E$ defined by

$$P(T) = \frac{1}{2} \text{tr}(\text{ad} T \mid \nu) \quad (T \in \mathfrak{a}_E);$$

by Proposition 1 one has

$$P = \frac{d}{2} \sum_{i < j} \alpha_{i,j} = \frac{d}{2} \sum_{i=1}^{r} (r - 2i + 1) \lambda_i.$$

The Haar measure of $K$ is always normalized by $\int_K dk = 1$. We make an identification $A = (R^r)^F$ by the correspondence $a \leftrightarrow (t_i)$ defined by the relation $a = \exp(\sum \lambda_i T_{e_i}), t_i = e^{\lambda_i}$; then one has $da = \prod (dt_i/t_i)$.

Moreover one has

$$\det(a) = e^R(\sum \lambda_i e_i) = e^{\sum \lambda_i} = (\prod_{i=1}^r t_i)^{\frac{R}{r}}$$

and

$$a \cdot e = \sum_{i=1}^r e^{\lambda_i} e_i = \sum_{i=1}^r t_i e_i.$$
\[ e^{2 \rho(\log a)} = \prod_{i=1}^{r} t_i^{\frac{r}{2}(r-2i+1)}. \]

Since \( \mathcal{Q} = G/K \), we can normalize the Haar measure of \( G \) by the relation \( dg = d_{\mathcal{Q}}(u) \cdot dk \) where \( u = ge \). Then by (2.15), (2.16), (2.17) one has

\[
\int_{G} n(ge)^{s} e^{-\tau(ge)} \, dg = \int_{A} \det(a)^{s} e^{-2 \rho(\log a)} \, da \int_{N} e^{-\tau(n \cdot ae)} \, dn
\]

\[
= c_{1} \int_{\mathbb{A}} \cdots \int_{\mathbb{T}} \left( t_{i_{1}}^{\frac{\pi}{2}(r-2i+1)} \right) \, dt_{i_{1}} \ldots \int_{\mathbb{T}} \left( t_{i_{r}}^{\frac{\pi}{2}(r-2i+1)} \right) \, dt_{i_{r}} \]

\[
\times \int_{N} e^{-\tau(n(b_{i_{j}}, e_{i_{j}}))} \, dn
\]

To compute the integral over \( N \), we introduce some notations. For \( u = \sum_{i<j} u_{ij} \in U \) with \( u_{ij} \in U_{ij} \), we put

\[
T_{u}^{(+)} = \frac{1}{2} \left( T_{e_{i}} + \sum_{i<j} \sum_{i<k} \frac{1}{2} \left( T_{e_{i}} e_{i} T_{e_{j}} - e_{j} T_{e_{j}} e_{i} \right) \right),
\]

\[
\varepsilon^{(+)}(u) = \sum_{i<j} \sum_{k \neq i} \sum_{i<k} \sum_{j<k} u_{ik} u_{jk} \cdots u_{k-1,j}.
\]

Then one has The \( U_{ij} \)-component of \( \varepsilon^{(+)}(u) \) is denoted by \( \varepsilon_{ij}^{(+)}(u) \).

**Lemma 2.** The notation being as above, one has

\[
(\exp T_{u}^{(+)})(\sum_{i=1}^{r} t_{i} e_{i}) = \sum_{i=1}^{r} \left( t_{i} + \frac{1}{r} \sum_{k > i} t_{k} \varepsilon_{ik}^{(+)}(u)^{2} e_{i} \right)
\]

\[= \frac{1}{2} \sum_{i<j} \sum_{k > j} t_{j} \varepsilon_{ij}^{(+)}(u) + \sum_{k > j} t_{k} \varepsilon_{ik}^{(+)}(u) \varepsilon_{jk}^{(+)}(u). \]

This may be regarded as a generalization of the so-called "Jacobi transformation". The proof is again straightforward. It follows that, if \( n = \exp T_{u}^{(+)}(u \in \sum_{i<j} U_{ij}) \), one has

\[
\tau(n(\sum_{i=1}^{r} t_{i} e_{i})) = \frac{1}{r} \sum_{i=1}^{r} \tau(\varepsilon_{ik}^{(+)}(u)^{2}) t_{i} + \frac{1}{8} \sum_{k < l} \tau(\varepsilon_{ik}^{(+)}(u) \varepsilon_{jl}^{(+)}(u)) t_{k}.
\]

We denote the Euclidean measure on \( U_{ij} \) (i < j) (relative to the inner product \( <> \)) by \( du_{ij} \) and define the Haar measure on \( N \) by

\[
dn = \prod_{i<j} du_{ij} \quad \text{for} \quad n = \exp T_{u}^{(+)}.
\]

Since the map \( \varepsilon^{(+)} \) is a bijection of \( \sum_{i<j} U_{ij} \) onto itself with Jacobian
equal to one, one has
\[ du = \prod_{i < j} du_{ij} = \prod_{i < j} du'_{ij}, \]
where \( u' = E(\omega)(u) \). Hence by (2.21) one has
\[
\int_N e^{-\tau(u_{ij}(u))} \, du = e^{-\frac{n}{r} \sum t_i} \prod_{i < j} \int_{U_{ij}} e^{-\frac{t_j}{r} \tau(u_{ij}^2)} \, du'_{ij}
\]
\[ = e^{-\frac{n}{r} \sum t_i} \prod_{i < j} \left( \frac{8\pi}{t_j} \right)^{\frac{d}{2}}
\]
\[ = \left( \frac{8\pi}{t_j} \right)^{\frac{d}{2} \left( \frac{n}{r} (j-1) \right)} e^{-\frac{n}{r} t_j}. \]

Inserting this in (2.18), one obtains
\[
[\alpha](s) = c_1(8\pi)^{-\frac{n}{2}} \prod_{j=1}^r \left( \int_0^\infty t_j^{\frac{n}{2}-\frac{d}{2}(r-j)-1} e^{-\frac{n}{r} t_j} \, dt_j \right)
\]
\[ = c_1(8\pi)^{-\frac{n}{2}} \prod_{j=1}^r \left( \frac{n}{x_j} - \frac{d}{2} (r-j) \right) \Gamma \left( \frac{n}{x_j} s - \frac{d}{2} (r-j) \right)
\]
\[ = c_1(8\pi)^{-\frac{n}{2}} (\frac{n}{x_j} - ns + \frac{d}{2} (r-j)) \prod_{j=1}^r \left( \frac{n}{x_j} s - \frac{d}{2} (j-1) \right). \]

The constant \( c_1 \) can be determined by the following observation. We set
\[ U_0 = \bigcup_{i=1}^r U_{i1} = \{ e_1, \ldots, e_r \}_R \]
and denote by \( du_0 \) the Euclidean measure on \( U_0 \) (relative to \( < > \)). Then, since \( < e_i, e_j > = \frac{n}{r} \delta_{ij} \), the bijection \( A \rightarrow U_0 \) defined by \( a = \exp T_u \), or equivalently by \( ae = \exp u_0 \), gives the relation
\[ du_0 = \left( \frac{r}{x} \right)^{\frac{d}{2}} da. \]

Hence, when
\[ u = (pa)e = \exp \left( \frac{nr}{x} \sum_{i=1}^r t_i e_i \right), \]
\[ n = \exp T_x^{(+)} \quad x \in \bigcup_{i < j} U_{ij}, \quad x' = E(x), \]
one has by Lemma 2
\[
\frac{\partial (u)}{\partial (t, x)} = \frac{\partial (u_0, u_{ij})}{\partial (t_1^j, x_1^j)} = \left( \frac{n}{r} \right)^{\frac{d}{2}} \prod_{j=1}^r \left( \frac{r}{2} \right)^{(j-1)d}
\]
\[ = 2^{r-n} \left( \frac{n}{r} \right)^{\frac{r}{2}} \prod_{j=1}^{r} t_{j}^{(j-1)d}. \]

It follows that
\[ d_{\lambda}(u) = 2^{r-n} \left( \frac{n}{r} \right)^{\frac{r}{2}} \prod_{j=1}^{r} (t_{j}^{(j-1)d} - \frac{r}{2} dt_{j}) \ dx, \]

which, in view of (2.11) and (2.16), implies (2.15) and the relation
\[ (2.22) \quad c_{1} = 2^{r-n} \left( \frac{n}{r} \right)^{\frac{r}{2}}. \]

Thus we obtain the formula
\[ (2.23) \quad \int_{\mathcal{D}}(s) = (2\pi)^{\frac{n-r}{2}} \left( \frac{n}{r} \right)^{n(\frac{1}{2} - s)} \prod_{j=1}^{r} \Gamma \left( \frac{n}{r} s - \frac{d}{2} (j-1) \right). \]

Our computation also shows that the integral for \( \int_{\mathcal{D}}(s) \) converges absolutely for \( \text{Res} > 1 - \frac{r}{n} \).

From the relation (1.2) one obtains
\[ \int_{\mathcal{D}}(s) \int_{\mathcal{D}}(1-s) = (2\pi)^{n-r} \prod_{j=1}^{r} \Gamma \left( \frac{n}{r} s - \frac{d}{2} (j-1) \right) \Gamma \left( \frac{n}{r} (1-s) - \frac{d}{2} (r-j) \right) \]
\[ = (2\pi)^{n-r} (2\pi i)^{r} \prod_{j=1}^{r} \frac{e^{\pi i \left( \frac{1}{2} s - \frac{d}{2} (j-1) \right)}}{e^{\pi i \left( \frac{n}{r} s - \frac{d}{2} (j-1) \right)} - 1}. \]

Since one has by (2.11)
\[ n - r = d \frac{r(r-1)}{2} = \begin{cases} 0 \pmod{2} & \text{for } d \text{ even} \\ \left[ \frac{r}{2} \right] \pmod{2} & \text{for } d \text{ odd,} \end{cases} \]

one has
\[ \prod_{j=1}^{r} e^{-\pi i \frac{d}{2} (j-1)} = (-1)^{d} \frac{r(r-1)}{2} = \begin{cases} i^{n-r} & \text{for } d \text{ even} \\ i^{n-r} \left[ \frac{r}{2} \right] & \text{for } d \text{ odd.} \end{cases} \]

Hence one obtains the following functional equation:
\[ (2.24) \quad \int_{\mathcal{D}}(s) \int_{\mathcal{D}}(1-s) = (2\pi i)^{n} e^{n \pi i} \begin{cases} (e^{2\pi i \frac{r}{2} s} - 1)^{-r} & \text{for } d \text{ even} \\ (e^{2\pi i \frac{r}{2} s} - \left[ \frac{r}{2} \right]) (e^{2\pi i \frac{r}{2} s} + 1)^{-\left[ \frac{r}{2} \right]} & \text{for } d \text{ odd.} \end{cases} \]
§ 3. Zeta functions of a self-dual homogeneous cone.

3.1. We fix a $\mathbb{Q}$-structure on $U$ and assume that (the Zariski closure of) $G$ is defined over $\mathbb{Q}$ and $e \in U_{\mathbb{Q}}$; then (the Zariski closure of) $K$ is also defined over $\mathbb{Q}$. We also fix a lattice $L$ in $U$ compatible with that $\mathbb{Q}$-structure, i.e., such that $U_{\mathbb{Q}} = L \oplus Z_{\mathbb{Q}}$, and an arithmetic subgroup $\Gamma$ fixing $L$, i.e., a subgroup of $G_L = \{ g \in G \mid gL = L \}$ of finite index; for simplicity we assume that $\Gamma$ has no fixed point in $\mathcal{L}$. We then define the zeta function associated with $\mathcal{L}, \Gamma, L$ as follows:

$$\zeta_{\mathcal{L}}(s; \Gamma, L) = \sum_{u \in \Gamma \backslash \mathcal{L} \cap L} N(u)^{-s},$$

the summation being taken over a complete set of representatives of $\mathcal{L} \cap L$ modulo $\Gamma$. It can be shown easily that this series is absolutely convergent for $\text{Re} \ s > 1$.

By the reduction theory, $\Gamma$ has a fundamental domain in $\mathcal{L}$ which is a rational polyhedral cone. More precisely, there exists a finite set of simplicial cones

$$C^{(i)} = \{ v_1^{(i)}, \ldots, v_{l_i}^{(i)} \}_{R_+}^{R_+} = \left\{ \sum_{j \in I} \lambda_j v_j^{(i)} \mid \lambda_j \in R_+ \right\} \quad (1 \leq i \leq m),$$

where $v_1^{(i)}, \ldots, v_{l_i}^{(i)}$ are linearly independent elements in $\mathcal{L} \cap L$, such that

$$\mathcal{L} = \bigcup_{1 \leq i \leq m} \gamma C^{(i)}.$$

It follows that

$$\zeta(s; \Gamma, L) = \sum_{i=1}^{m} \sum_{v \in \mathcal{L} \cap L} N(u)^{-s}.$$

For a set of linearly independent vectors $v_1, \ldots, v_{l} \in L$, we put

$$R((v_j), L) = \left\{ \sum_{j \in I} \lambda_j v_j \mid 0 < \lambda_j \leq 1 \right\} \cap L,$$

which is finite. Then $u \in C^{(i)} \cap L$ can be written uniquely in the form

$$u = v_0 + \sum_{j \in I} m_j v_j^{(i)}, \quad v \in R((v_j^{(i)}), L), \ m \in \mathbb{Z}, \ m \geq 0.$$
For a set of linearly independent vectors \( v_1, \ldots, v_\ell \in J \cap V_{\mathfrak{q}} \) and \( v_0 = \sum_j \alpha_j v_j \) (\( \alpha_j \in \mathbb{Q}_+ \)), we define a "partial zeta function" by

\[
\zeta_a(s; (v_j), v_0) = \sum_{N(v_0) > 0} \frac{1}{N(v_0)^s} \sum_{j=1}^\ell m_j v_j^{-s},
\]

which will also be written as \( \zeta_a(s; (v_j), (\alpha_j)) \). Then the zeta function (3.1) can be written as a finite sum of partial zeta functions as follows:

\[
\zeta_a(s; \Gamma, L) = \sum_{i=1}^\infty \sum_{v_0 \in K(l^{(i)}_{\mathfrak{q}}, J)} \zeta_a(s; (v^{(i)}_j), v_0).
\]

Hence the study of special values of \( \zeta_a(s; \Gamma, L) \) is reduced to that of the partial zeta functions of the form (3.2).

3.2. Let \((v_j)\) and \(v_0\) be as above. Then by (2.7) one obtains

\[
\Gamma_a(s) \zeta_a(s; (v_j), v_0) = \sum_{\ell_j \geq \ell} \zeta_a(s) N(v_0) \sum_{j=1}^\ell m_j v_j^{-s}
\]

\[
= \sum_{N(v_0) > 0} \int_{\mathfrak{q}} N(u)^{s-1} e^{-\int_{\mathfrak{q}} (\alpha_j + m_j) \langle v_j, u \rangle} du
\]

\[
= \int_{\mathfrak{q}} N(u)^s \prod_{j=1}^\ell b(\langle v_j, u \rangle, l - \alpha_j) d \mathfrak{q}(u)
\]

\[
= \int_G \det(g)^s \prod_{j=1}^\ell b(\langle v_j, g e \rangle, l - \alpha_j) dg.
\]

In the notation of §2, but this time using the decomposition \( G = KAK \), one has

\[
\zeta_a(s; (\alpha_j), (v_j)) = c \sum_{i=1}^\infty \int_{\mathfrak{q}} N(t_i)^{s-1} \Delta(t_i) \frac{d}{d t_i} \prod_{j=1}^\ell |\Delta(t_j)| \frac{t_{i, j}}{t_{i, j}} dt_i,
\]

where \( \Delta(t, \ldots, t_r) = \prod_{i<j} (t_i - t_j) \) (cf. Helgason [2], Ch. X, §1). Hence in view of (2.11) and (2.17) one has

\[
\Gamma_a(s) \zeta_a(s; (v_j), (\alpha_j)) = c \int \cdots \int \left( \prod_{i=1}^\ell t_i \right)^{\frac{n(\ell-1)}{2}} |\Delta(t)| \prod_{r=1}^\ell dt_i,
\]

where

\[
dg = c |\Delta(a)| d(k da da').
\]
\[ F(t_1, \ldots, t_r) = \int_{K} \prod_{j=1}^{r} b(<v_j, k \sum t_i e_i>, l - \alpha_j) \, dk. \]

It is clear that \( F(t_1, \ldots, t_r) \) is holomorphic for \( \text{Re } t_i > 0 \) \((1 \leq i \leq r)\).

Since \( K \) contains an element which induces any given permutation of \( e_1, \ldots, e_r \), the function \( F \) is symmetric. Hence, denoting by \( B_i \) an open simplicial cone in \( \mathbb{R}^r \) defined by \( t_1 > \ldots > t_r > 0 \), one has

\[
\left(3.5'\right) \quad \int_{\partial B_i} \zeta_{\Delta}(s; (v_j), (\alpha_j)) = c \cdot \sqrt{r!} \int_{B_i} \prod_{t_i} \Delta(t)^{\frac{r}{2}(s-1)} \Delta(t)^{(r-1)} \prod_{t_i} \Delta(t_i). 
\]

3.3. Still following Shintani [11], we make a change of variables \((t_1) \rightarrow (t_1, \tau_2, \ldots, \tau_r)\) with \( \tau_i = t_i/t_{i-1} \) \((2 \leq i \leq r)\). Then \( B_i \) can be expressed as

\[
B_i = \left\{ (t_1, \ldots, t_r) \mid t_1 > \prod_{i=2}^{r} \tau_i, \quad 0 < t_i < \infty, \quad 0 < \tau_i < 1 \right\}. 
\]

Putting \( \tau_1 = t_1 \), one has

\[
\frac{\partial(t_1, \ldots, t_r)}{\partial(t_1, \tau_2, \ldots, \tau_r)} = \prod_{i=1}^{r-1} \tau_i^{r-i}, 
\]

\[
\prod t_i = \prod \tau_i^{r-i+1}, 
\]

\[
\Delta(t) = \prod \tau_i^{\frac{1}{2}(r-i+1)(r-i)} \prod_{2 \leq i < j \leq r} (1 - \tau_i \ldots \tau_j). 
\]

It follows that the exponent of \( \tau_i \) in the integrand in \(3.5'\) is equal to

\[
(r-i+1)\frac{r}{2}(s-1) + \frac{1}{2}(r-i+1)(r-i) + r - i \]

\[
= (r-i+1)\left\{ \frac{r}{2} s - \frac{d}{2} (i-1) \right\} - 1. 
\]

Hence one has

\[
\left(3.6\right) \quad \int_{\partial B_i} \zeta_{\Delta}(s; (v_j), (\alpha_j)) = c \cdot \sqrt{r!} \int_{B_i} t^{ns-1} dt 
\]

\[
\int_{0}^{\infty} \prod_{i=2}^{r} \tau_i^{(r-i+1)\left\{ \frac{r}{2} s - \frac{d}{2} (i-1) \right\} - 1} F(t_1, \tau) \prod_{i=2}^{r} \Delta(t_i), 
\]

where

\[
\left(3.7\right) \quad \tilde{F}(t_1, \tau) = \prod_{2 \leq i < j \leq r} (1 - \tau_i \ldots \tau_j) \frac{d}{d\tau} \prod_{i=2}^{r} \tau_i^{(r-i+1)} \Delta(t_1, t_1 \tau_2, \ldots, t_1 \tau_r \ldots \tau_r). 
\]
3.4. We now assume that all \( v_j \)'s are in \( \mathcal{O} \) (not on the boundary of \( \mathcal{O} \)).

(In the situation explained in 3.1, this means that the \( \mathcal{Q} \)-rank of \( G \) is equal to 1.) Then for any \( v \in \mathcal{O} - \{0\} \), one has \( \langle v_j, v \rangle > 0 \); in particular,

\[
\langle v_j, ke_i \rangle > 0 \quad \text{for all} \quad k \in K, \ 1 \leq i \leq r.
\]

Put

\[
\xi_j = \langle v_j, k \sum t_i e_i \rangle
\]

\[= \sum_i t_i \langle v_j, ke_i \rangle (1 + \sum_{i=1}^{r} \tau_i \cdots \tau_{i-i} \langle v_j, ke_i \rangle).\]

For the fixed \( e_i, v_j \), choose \( \rho, \rho_i > 0 \) in such a way that

\[
\begin{cases}
\sum_{i=2}^{r} \frac{\rho_i - 1}{\rho_i} \frac{\langle v_j, ke_i \rangle}{\langle v_j, ke_i \rangle} < 1 \quad \text{for all} \quad k \in K, \ 1 \leq j \leq l, \\
\rho_i < \frac{\langle v_j, ke_i \rangle}{\langle v_j, ke_i \rangle} \quad \text{for all} \quad l \leq j \leq l.
\end{cases}
\]

Then for

\[
0 < |t_i| < \rho_i, \quad |\tau_i| < \rho \quad (2 \leq i \leq r),
\]

one has \( 0 < |\xi_j| < 2 \pi \) and so \( b(\xi_j, 1 - \xi_j) \) is holomorphic. Hence the function \( F(t) = F(t_1, t_\tau, \ldots, t_i \tau_i \ldots \tau_r) \) has a Laurent expansion in \( t_1, \tau_1, \ldots, \tau_r \) in the domain defined by (3.11). The coefficients in this expansion is a \( \mathcal{Q} \)-linear combination of the integrals of the form

\[
I((v_j)) = \int_{K} \prod_{i=1}^{r} \langle v_j, ke_i \rangle \, d\mathfrak{k}
\]

where \( v_j \geq 0 \) for \( 2 \leq i \leq r \) and \( v_j \in \mathbb{Z} \) for all \( i, j \).
3.5. Let $I(\xi, l)$ denote the contour consisting of the line segment $[\xi, l]$ taken twice in opposite directions and of a (small) circle of radius $\xi$ about the origin taken in the counterclockwise direction. When the $\tau_i$ $(2 \leq i \leq r)$ are on $I(\xi, l)$, one has by (2.12)

$$|<v_j, k(e_i + \sum_{i=2}^r \tau_i \cdots \tau_i e_i)| \leq |v_j| \sum_{i=1}^{r-1} |e_i| = \sqrt{\frac{\pi}{r}} |v_j|$$

and

$$\text{Re} <v_j, k(e_i + \sum_{i=2}^r \tau_i \cdots \tau_i e_i) = <v_j, ke_i> + \sum_{i=2}^{r} \text{Re}(\tau_i \cdots \tau_i <v_j, ke_i>

\geq <v_j, ke_i> - \varepsilon |v_j| \sum_{i=2}^{r} |e_i|

= <v_j, ke_i> - \varepsilon (r-1) \sqrt{\frac{\pi}{r}} |v_j|.$$

We choose $\xi$ so that one has

$$(3.13) \quad \xi \sqrt{\frac{\pi}{r}} |v_j| \leq \min \{2\pi, <v_j, ke_i> (k \in K)\} \quad \text{for all} \quad 1 \leq j \leq l,$$

The above inequalities show that $<v_j, k(e_i + \sum_{i=2}^r \tau_i \cdots \tau_i e_i)$ belongs to the domain

$$\left\{ z \in \mathbb{C} \mid |z| < \frac{2\pi}{\xi}, \text{Re} z > \xi \sqrt{\frac{\pi}{r}} |v_j| \right\}.$$

It follows that, if $t_i$ is on the contour $I(\xi, \infty)$, one has

$$|\xi_j| < 2\pi \quad \text{or} \quad \text{Re} \xi_j > 0,$$

so that the function $b(\xi_j, 1 - \alpha_j)$ is holomorphic.

From this observation, it is clear that the integral on the r.h.s. of (3.6) is equal to the contour integral

$$(e^{2\pi rns} - 1)^{-1} \int_{t_i \in I(\xi, \infty)} \frac{1}{(e^{2\pi i \frac{r-i+1}{r} ns} - 1)^{-1}} \int_{\tau_i \in I(\xi,1)}$$

which is independent of the choice of $\xi$ satisfying (3.13). As is easily seen, the contour integral converges for all $s \in \mathbb{C}$. Hence the integral

$$\sum_{\nu} \frac{1}{(r-i+1)n} (\nu \in \mathbb{Z}).$$
§ 4. The special values of the zeta functions.

4.1. As a preliminary, we check the rationality of the constant \( c \) in (3.4). For that purpose, we compute \( \Gamma_{\mathcal{Q}}(s) \) by using the decomposition \( \mathcal{Q} = K \mathcal{K} \).

\[
\Gamma_{\mathcal{Q}}(s) = \int_{\mathcal{Q}} N(u)^s e^{-\tau(u)} d\mathcal{Q}(u) = \int_{G} N(ge)^s e^{-\tau(ge)} dg = c \int_{A} \det(a)^s e^{-\tau(\sigma e)} \Delta(a) da = c \int_{t}^{e} \left( \prod_{i} t_i \right)^{\gamma(s-1)} |\Delta(t)|^d e^{-\frac{\gamma}{\pi} t_i} \prod dt_i.
\]

We make another change of variables:
\[
t = \prod_{i=1}^{r} t_i, \quad t'_i = t_i/t.
\]

Then
\[
\frac{\varphi(t_1, \ldots, t_r)}{\varphi(t, t'_1, \ldots, t'_{r-1})} = (-1)^{r-1}t^{r-1},
\]

and the exponent of \( t \) in the integrand in the last member of (4.1) is equal to
\[
n(s-1) + \frac{d}{2} r(r-1) + r - 1 = ns - 1.
\]

Hence one has
\[
\Gamma_{\mathcal{Q}}(s) = c \cdot \gamma(s) \cdot \beta(s),
\]

where
\[
\begin{align*}
\gamma(s) &= \int_{0}^{\infty} t^{ns-1} e^{-\frac{\gamma}{\pi} t} dt = \left( \frac{\pi}{\gamma} \right)^{ns} \Gamma(ns), \\
\beta(s) &= \int_{\frac{\pi}{\gamma}}^{e} \prod_{t'_i > 0} \left( t'_1, \ldots, t'_{r-1}, 1 - \sum_{i=1}^{r} t'_i \right)^{\gamma(s-1)} \times \\
& \quad \times \prod_{t'_i < 1} |\Delta(t'_1, \ldots, t'_{r-1}, 1 - \sum_{i=1}^{r} t'_i)|^d \prod dt'_i.
\end{align*}
\]

For \( s = 1 \), one has
\[
\Gamma_{\mathcal{Q}}(1) = c \cdot \gamma(1) \cdot \beta(1) = c \left( \frac{\pi}{\gamma} \right)^n (n-1)! \beta(1),
\]
\[ f(1) = \int \prod_{t_i > 0} (\ldots, t_{r-1}, 1 - \sum t_i') \prod dt_i' \in \mathbb{Q}. \]

By (2.23) one has

\[
\int_{\mathbb{Q}} (2\pi)^{\frac{n-r}{2}} \left( \frac{1}{\pi} \right)^{\frac{r}{2}} \prod_{j=1}^{r} \Gamma' \left( 1 + \frac{d}{2} (j-1) \right) \left( \sum_{a \sim b \in \mathbb{Q}} \right) \sum_{a \sim b \in \mathbb{Q}} \prod_{j=1}^{r} \Gamma' \left( \frac{n-\left[ \frac{r+j}{2} \right]}{2} \right),
\]

where \( a \sim b \text{ means that } a/b \in \mathbb{Q}. \) Thus one has

\[
c = \frac{(2\pi)^{\frac{n-r}{2}} \left( \frac{1}{\pi} \right)^{\frac{r}{2}} \prod_{j=1}^{r} \Gamma' \left( 1 + \frac{d}{2} (j-1) \right)}{(n-1)!} \int_{\mathbb{Q}} (1) \sim \int_{\mathbb{Q}} (1).
\]

Since \( \int_{\mathbb{Q}} (1) \sim \int_{\mathbb{Q}} (1 + \frac{1}{n} v) \text{ for } v \in \mathbb{Z}, \) one obtains

\[
c \int_{\mathbb{Q}} (1 + \frac{1}{n} v) \sim \int_{\mathbb{Q}} (1) \sim \int_{\mathbb{Q}} (1) \sim \left\{ \begin{array}{ll} \pi^{n-r} & \text{(d even)} \\ \pi^{n-[\frac{r+1}{2}]} & \text{(d odd)} \end{array} \right\}
\]

4.2. We first consider the case where \( d \) is even. Then by (2.24) one has

\[
\int_{\mathbb{Q}} (1-s) \int_{\mathbb{Q}} (1-s) = (2\pi i)^n \epsilon \sin(2\pi i \frac{n}{s} - 1)^r.
\]

Hence

\[
\zeta_{\mathbb{Q}}(s; (\nu), (\kappa)) = c \frac{\int_{\mathbb{Q}} (1-s) (2\pi i)^{\frac{n}{s} - 1} \epsilon \sin}{(2\pi i)^{\frac{n}{s} - 1} \epsilon \sin} \times R(s),
\]

where

\[
R(s) = \frac{e^{2\pi i \frac{n}{s} - 1}}{2\pi i} \prod_{j=1}^{r} \left( \prod t_i \right)^{\frac{r}{2}} \Delta(t) \prod dt_i,
\]

\[
= \frac{\prod_{j=1}^{r} \frac{e^{2\pi i \frac{n}{s} - 1}}{e^{2\pi i \frac{n}{s} - 1}}}{\prod_{j=1}^{r} \frac{e^{2\pi i \frac{n}{s} - 1}}{e^{2\pi i \frac{n}{s} - 1}} \times \frac{1}{(2\pi i)^r}} \int_{t_i^{ns-1}}^{1} \frac{dt_i}{I(e, \omega)}
\]

We are interested in the values of \( \zeta_{\mathbb{Q}} \) at \( s = -\frac{r}{n} v \) \((v = 0, 1, \ldots). \) The
first factor in the right hand side of (4.7) is holomorphic for \( \Re s < \frac{r}{n} \)
and by (4.6) the value at \( s = -\frac{r}{n} \nu \) is rational:

\[
(4.7) \quad \frac{c}{(2\pi i)^{n-r}e^{-r\nu|\pi i}} \int_0^\infty \frac{(1 + \frac{r}{n} \nu)}{e^{r\nu|\pi i}} \in \mathbb{Q}.
\]

On the other hand, it is clear that

\[
\frac{e^{2\pi i \frac{r}{n} \frac{s}{r} - \frac{1}{r} \frac{ns}{r}}}{e^{s - \frac{1}{r} \frac{ns}{r}}} \rightarrow \frac{1}{s - \frac{1}{r} \frac{ns}{r}} \quad \text{when} \quad s \rightarrow -\frac{r}{n} \nu.
\]

Hence we see that \( R(-\frac{r}{n} \nu) \) is equal to the coefficient of

\[
t_i \nu \prod_{i=2}^{r} \Gamma_i (r-i+1)^{\frac{r}{2}(i-1)}
\]

in the Laurent expansion of \( \tilde{F}(t_i, \tau) \),

which is a \( \mathbb{Q} \)-linear combination of \( I((\nu_j')) \).

4.3. From now on we assume that \( d \) is odd. By the classification theory,

it is known that this assumption implies that \( r = 2 (n = d + 2) \) or \( d = 1 \)
\( (n = \frac{1}{2} r(r+1)) \). By (2.24) one has

\[
\int_0^\infty \frac{\nu}{e^{\nu|\pi i}} (e^{2\pi i \frac{r}{n} \frac{s}{r} - \frac{1}{r} \frac{ns}{r}}) = \frac{c}{(2\pi i)^{n-\frac{\nu}{r} i}} \cdot \frac{e^{\nu|\pi i}}{(e^{2\pi i \frac{r}{n} \frac{s}{r} - \frac{1}{r} \frac{ns}{r}})^{\frac{\nu}{r} i}}.
\]

Hence

\[
(4.11) \quad \zeta_{\mathcal{D}} (s; (\nu_j), (\omega_j)) = \frac{c}{(2\pi i)^{n-\frac{\nu}{r} i}} \cdot \frac{e^{\nu|\pi i}}{(e^{2\pi i \frac{r}{n} \frac{s}{r} - \frac{1}{r} \frac{ns}{r}})^{\frac{\nu}{r} i}} R^{(s)}(s) R^{(s)}(s),
\]

where

\[
R^{(s)}(s) = (2\pi i)^{\frac{r}{2} \frac{s}{r}} \cdot \frac{e^{2\pi i \frac{r}{n} \frac{s}{r} - \frac{1}{r} \frac{ns}{r}}}{\prod_{i=1}^{r} (e^{2\pi i (-k+1)}(\nu s - \frac{\nu}{2} (k-1)) - 1)},
\]

\[
R^{(s)}(s) = (2\pi i)^{-r} \int_{I(s, \omega)} \int_{I(t, \tau)} d\tau_i .
\]
The first factor in the right hand side of (4.11) is holomorphic for \( \Re s < \frac{r}{n} \) and by (4.6) the value at \( s = -\frac{r}{n} \nu \) \( (\nu \geq 0) \) is rational:

\[
(4.12) \quad \frac{c \int \frac{(1+\frac{r}{n} \nu)}{2 \pi i n - \frac{r+1}{2}} e^{-\pi i r \nu}}{\nu^{(n-\frac{r+1}{2})+r \nu}} = (-1)^{\frac{r}{2} (n-\frac{r+1}{2})} \frac{c \int \frac{(1+\frac{r}{n} \nu)}{2 \pi i n - \frac{r+1}{2}}}{\nu^{(n-\frac{r+1}{2})}} \in \mathbb{Q}.
\]

Note that one has

\[ n \equiv \left[ \frac{r+1}{2} \right] \quad (\text{mod } 2), \]

since

\[ n = d+2 \equiv 1 \equiv \left[ \frac{3}{2} \right] \quad (\text{mod } 2) \quad \text{if } r = 2, \text{ and} \]

\[ n = \frac{1}{2} r(r+1) \equiv \left[ \frac{r+1}{2} \right] \quad (\text{mod } 2) \quad \text{if } d = 1. \]

4.4. To compute \( R^{(r)}(s) \), we first note

\[ e^{\pi i d(k-1)(r-k+1)} = \begin{cases} -1 & \text{if } k \equiv r \equiv 0 \pmod{2}, \\ 1 & \text{otherwise}. \end{cases} \]

We put

\[ \frac{r}{2} = r_1, \quad \zeta = e^{2 \pi i \frac{r_1}{r}} s. \]

The case \( r \) is odd. One has

\[ R^{(r)}(s) = (2 \pi i)^{r_1} r! \frac{(\zeta - 1)^{r_1} \zeta}{\prod_{k=1}^{r_1} (\zeta^k - 1)} \]

\[ = \frac{r!}{\prod_{k=1}^{r_1} (\zeta^k + \ldots + \zeta + 1)(\zeta + 1)} (2 \pi i)^{r_1} \frac{\zeta + 1}{\zeta - 1}^{r_1}. \]

Hence, when \( s \rightarrow -\frac{r}{n} \nu \), one has

\[
(4.13) \quad (s + \frac{r}{n} \nu)^{r_1} R^{(r)}(s) \longrightarrow (2 \frac{r}{n} \nu)^{r_1}.
\]

Thus \( R^{(r)}(s) \) has a pole of order \( r_1 \) at \( s = -\frac{r}{n} \nu \).

The case \( r \) is even. One has

\[ R^{(r)}(s) = (2 \pi i)^{r_1} r! \frac{(\zeta - 1)^{r_1} \zeta}{\prod_{k=1}^{r_1} \left\{ (-1)^k \zeta^k - 1 \right\}} \]

\[ \frac{c \int \frac{(1+\frac{r}{n} \nu)}{2 \pi i n - \frac{r+1}{2}}}{\nu^{(n-\frac{r+1}{2})}} \in \mathbb{Q}. \]

\[
\frac{c \int \frac{(1+\frac{r}{n} \nu)}{2 \pi i n - \frac{r+1}{2}} e^{-\pi i r \nu}}{\nu^{(n-\frac{r+1}{2})+r \nu}} = (-1)^{\frac{r}{2} (n-\frac{r+1}{2})} \frac{c \int \frac{(1+\frac{r}{n} \nu)}{2 \pi i n - \frac{r+1}{2}}}{\nu^{(n-\frac{r+1}{2})}} \in \mathbb{Q}.
\]

Note that one has

\[ n \equiv \left[ \frac{r+1}{2} \right] \quad (\text{mod } 2), \]

since

\[ n = d+2 \equiv 1 \equiv \left[ \frac{3}{2} \right] \quad (\text{mod } 2) \quad \text{if } r = 2, \text{ and} \]

\[ n = \frac{1}{2} r(r+1) \equiv \left[ \frac{r+1}{2} \right] \quad (\text{mod } 2) \quad \text{if } d = 1. \]

4.4. To compute \( R^{(r)}(s) \), we first note

\[ e^{\pi i d(k-1)(r-k+1)} = \begin{cases} -1 & \text{if } k \equiv r \equiv 0 \pmod{2}, \\ 1 & \text{otherwise}. \end{cases} \]

We put

\[ \frac{r}{2} = r_1, \quad \zeta = e^{2 \pi i \frac{r_1}{r}} s. \]

The case \( r \) is odd. One has

\[ R^{(r)}(s) = (2 \pi i)^{r_1} r! \frac{(\zeta - 1)^{r_1} \zeta}{\prod_{k=1}^{r_1} (\zeta^k - 1)} \]

\[ = \frac{r!}{\prod_{k=1}^{r_1} (\zeta^k + \ldots + \zeta + 1)(\zeta + 1)} (2 \pi i)^{r_1} \frac{\zeta + 1}{\zeta - 1}^{r_1}. \]

Hence, when \( s \rightarrow -\frac{r}{n} \nu \), one has

\[
(4.13) \quad (s + \frac{r}{n} \nu)^{r_1} R^{(r)}(s) \longrightarrow (2 \frac{r}{n} \nu)^{r_1}.
\]

Thus \( R^{(r)}(s) \) has a pole of order \( r_1 \) at \( s = -\frac{r}{n} \nu \).

The case \( r \) is even. One has

\[ R^{(r)}(s) = (2 \pi i)^{r_1} r! \frac{(\zeta - 1)^{r_1} \zeta}{\prod_{k=1}^{r_1} \left\{ (-1)^k \zeta^k - 1 \right\}} \]
\[ (-2\pi i)^{r_1} \frac{r!}{\prod_{k \text{ odd}} (\zeta^{k-1} - \ldots - \zeta + 1)} \prod_{k \neq r_1} (\zeta^{k-1} + \ldots + \zeta + 1) \]

Hence \( R^{(s)} \) is holomorphic at \( s = -\frac{r}{n} \nu \) and

\[ R^{(s)}(-\frac{r}{n} \nu) = (-2\pi i)^{r_1} \frac{r!}{(2r_1)!} = (-\pi i)^{r_1} \frac{r!}{r_1!} \cdot \]

4.5. When \( r \) is odd (hence \( d = 1, n = \frac{r+1}{2} r(r+1) \)), \( R^{(s)}(s) \) for \( s = -\frac{r}{n} \nu \)

\[ = -\frac{2\nu}{r+1} \]

is given by the coefficient of 

\[ t_1^{r \nu} \prod_{i=2}^{r} \tau_i^{(r_i-1+1)(\nu + \frac{i-1}{2})} \]

in the Laurent expansion of \( \tilde{F}(t_1, \tau) \). Hence \( \zeta_{\ell_1}(s; (\nu_1), (\alpha_1)) \) has

at most a pole of order \( r_1 = \frac{r-1}{2} \) at \( s = -\frac{2\nu}{r+1} \) and one has

\[ \lim_{s \to -\frac{r}{n} \nu} (s + \frac{2\nu}{r+1})^{r_1} \zeta_{\ell_1}(s; (\nu_1), (\alpha_1)) \sim \frac{R^{(s)}}{Q} (-\frac{2\nu}{r+1}) \cdot \]

4.15. To treat the case \( r \) is even, we use the formula

\[ \int I(t_1^{m-1}) dt = -\frac{\nu}{m} \] (m odd),

which can be verified easily. When \( r \) is even, the value of \( R^{(s)}(s) \) for

\[ s = -\frac{r}{n} \nu \]

is given by

\[ (-\pi i) \frac{r!}{\prod_{j=2}^{r} (\zeta^{j-2+1} + \ldots + \zeta + 1)} \sum_{m_1, \ldots, m_r} \frac{a(\mu_j)}{((m_j-1)(r-2j+1) + 1)(2j-1)^{r+1}} \]

where \( a(\mu_j) \) is the coefficient of

\[ t_1^{r \nu} \prod_{j=2}^{r} \tau_j^{(r-j+2)(\nu + d(j-1))} \prod_{j=1}^{r_1} \tau_j^{m_j} \]

in \( \tilde{F}(t_1, \tau) \). Hence for the value of \( \zeta_{\ell_1} \), one has

\[ \zeta_{\ell_1}(-\frac{r}{n} \nu; (\nu_1), (\alpha_1)) \sim (2\pi i)^{r_1} R^{(s)}(-\frac{r}{n} \nu). \]
Bibliography


