On special values of zeta functions associated with a self-dual cone

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以下に述べるのは松島幸三氏還暦記念論文集（Birkhäuser）のための寄稿の一部である。京都の研究集会ではこの御近年について不読しがちなので、その要約を提出する予定であったが、都合上寄稿（の寄稿）しか出させて頂くことにした。本文で説明し難く、この寄稿法は本質的に数解析一般[11]のアイディアにあるものである。$r=2$（circular cone）の場合にはより精密な計算をすることになるが、栗東氏を独立に結果を導き出されたが、これについては将来別の機会に触れたいと思う。
\section{Introduction}

To explain the main idea of this paper, and also to fix some notations, we start with reviewing the classical case of Riemann zeta function. As usual, we set

\[
\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}, \quad (\text{Re } s > 1), \\
\Gamma(s) = \int_{0}^{\infty} x^{s-1}e^{-x} \, dx, \quad (\text{Re } s > 0),
\]

Then, for \( \text{Re } s > 1 \), one obtains

\[
\Gamma(s) \zeta(s) = \sum_{n=1}^{\infty} \frac{n^{-s}}{s} \int_{0}^{\infty} x^{s-1} e^{-nx} \, dx
\]

\[
= \sum_{n=1}^{\infty} \int_{0}^{\infty} x^{s-1} e^{-nx} \, dx' \\
= \int_{0}^{\infty} \frac{x^{s-1}}{e^x - 1} \, dx
\]

We put

\[
b(x, y) = \frac{e^{xy}}{e^x - 1} = \sum_{v=0}^{\infty} \frac{B_v(y)}{v!} x^{v-1} \quad (|x| < 2 \pi),
\]

where

\[
B_v(y) = \sum_{\mu=0}^{v} \binom{v}{\mu} b_{\mu} y^{v-\mu}
\]

is the Bernoulli polynomial, in which the \( b_{\mu} \) are the Bernoulli numbers:

\[
b_0 = 1, \quad b_1 = -\frac{1}{2},
\]

\[
b_v = \begin{cases} (-1)^{v-1} B_{\frac{v}{2}} & (v \text{ even}, \geq 2), \\ 0 & (v \text{ odd}, \geq 3). \end{cases}
\]

Then the above integral can be transformed into a contour integral of the form

\[(1.1) \quad \Gamma(s) \zeta(s) = (e^{2\pi i s} - 1)^{-1} \int_{I(\varepsilon, \infty)} x^{s-1} b(x, 0) \, dx,
\]

where \( I(\varepsilon, \infty) \) denotes the contour consisting of the half-line \([\varepsilon, \infty)\) taken twice in opposite directions and of a (small) circle of radius \( \varepsilon \).
about the origin taken in the counterclockwise direction. The contour integral is absolutely convergent for all \( s \in \mathbb{C} \), so that the function \( \Gamma(s) \zeta(s) \) can be analytically continued to a meromorphic function on \( \mathbb{C} \). Moreover, in virtue of the functional equation of the gamma function:

\[
\Gamma(s)\Gamma(1-s) = \frac{\pi}{\sin \pi s} = 2\pi i \frac{e^{\pi i s}}{e^{2\pi i s} - 1},
\]

one obtains

\[
\zeta(s) = e^{-\pi i s} \Gamma(1 - s) \frac{1}{2\pi i} \int_{I(t, \infty)} x^{s-1} b(x, 0) \, dx.
\]

This shows that \( \zeta(s) \) is holomorphic for \( \text{Re } s < 1 \). In particular, for \( s = 1 - m, m \in \mathbb{Z}^+ \) (positive integers), the contour integral reduces to the residue of \( x^{-m} b(x, 0) \) at \( x = 0 \), i.e., \( b_m/m! \). Hence one obtains

\[
\zeta(1 - m) = (-1)^{m-1}(m-1)! \frac{b_m}{m!} = (-1)^{m-1} \frac{b_m}{m}.
\]

Thus \( \zeta(1 - m) \) (\( m \in \mathbb{Z}^+ \)) is rational. In particular,

\[
\zeta(0) = -\frac{1}{2}, \quad \zeta(-1) = \frac{1}{12},
\]

\[
\zeta(-2\mu) = 0, \quad \zeta(1 - 2\mu) = (-1)^{\mu} \frac{B_{2\mu}}{2\mu} \quad (\mu \geq 1).
\]

This result has been generalized by Hecke, Klingen and Siegel [13] to the case of Dedekind zeta functions of totally real number fields. More recently, Shintani [11] gave a proof based on a direct generalization of the classical method explained above. Zeta functions attached to self-dual homogeneous cones have been studied by Siegel [3] in a special case of quadratic cones, and by Sato-Shintani [8] in a more general context of "prehomogeneous spaces". (Cf. also Shintani [7], [10].) On the other hand, the gamma functions attached to self-dual homogeneous cones were studied by Koecher [5], Gindikin [3] and others (cf. e.g., Resnikoff [6]). In this paper, we try to extend Shintani's method (i.e., the classical method) to examine the rationality of the special values of zeta functions attached to self-dual homogeneous cones.
\section{The gamma function of a self-dual homogeneous cone}

Let $U$ be a real vector space of dimension $n$, endowed with a positive definite inner product $\langle \cdot, \cdot \rangle$. By a "cone" in $U$ we always mean a non-degenerate open convex cone in $U$ with vertex at the origin, i.e., a non-empty open set $\mathcal{L}$ in $U$ such that

$$x, y \in \mathcal{L}, \lambda, \mu \in \mathbb{R}^+ \implies \lambda x + \mu y \in \mathcal{L}$$

and such that $\mathcal{L}$ does not contain any straight line. A cone $\mathcal{L}$ in $U$ is called homogeneous if the group of linear automorphisms

$$G(\mathcal{L}) = \{ g \in GL(U) \mid g(\mathcal{L}) = \mathcal{L} \}$$

is transitive on $\mathcal{L}$; and $\mathcal{L}$ is called self-dual if the "dual" of

$$\mathcal{L}^* = \{ x \in U \mid \langle x, y \rangle > 0 \text{ for all } y \in \mathcal{L} - \{0\} \}$$

coincides with $\mathcal{L}$.

Let $\mathcal{L}$ be a self-dual homogeneous cone in $U$ and $G = G(\mathcal{L})^\circ$. Then it is well-known (e.g., Satake [7]) that the Zariski closure of $G$ (in $GL(U)$) is a reductive algebraic group, containing $G(\mathcal{L})$ as a subgroup of finite index, and $g \mapsto t_g^{-1}$ is a Cartan involution of $G$; the corresponding maximal compact subgroup $K = G \cap O(U)$ coincides with the isotropy subgroup of $G$ at a "base point" $e \in \mathcal{L}$ (which is not unique, but will be fixed once and for all). Let

$$\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$$

be the corresponding Cartan decomposition of $\mathfrak{g} = \text{Lie } G$. Then $\mathfrak{k} = \text{Lie } K$ and one has for $T \in \mathfrak{g}$

$$T \in \mathfrak{k} \iff t_T = -T \iff Te = 0.$$

It follows that, for each $u \in U$, there exists a uniquely determined element $T_u \in \mathfrak{g}$ such that $T_u e = u$. It is well-known that the vector space $U$ endowed with a product
\[ u \cdot u' = T_u u' \quad (u, u' \in U) \]

becomes a formally real Jordan algebra (cf. Braun-Koecher [2], or Satake [7]).

We define the (regular) trace on \( U \) by

\[(2.2) \quad \tau(u) = \text{tr}(T_u). \]

For the given \((\mathcal{L}, e)\), one may assume (by Schur's lemma) that the inner product \( \langle \cdot, \cdot \rangle \) is so normalized that one has

\[(2.3) \quad \langle u, u' \rangle = \tau(u \cdot u') \quad (u, u' \in U). \]

Next, let \( u \in \mathcal{L} \). Then, since \( G \) is transitive on \( \mathcal{L} \), there exists \( g_1 \in G \) such that \( u = g_1 e \). We define the (regular) norm \( N(u) \) by

\[ N(u) = \det(g_1), \]

which is clearly independent of the choice of \( g_1 \). There exists a unique element \( u_1 \in U \) such that \( u = \exp u_1 \) (which is defined to be \( \exp T_{u_1} e \)); then by definition one has

\[(2.4) \quad N(u) = \det(\exp T_{u_1}) = e^{\tau(u_1)}. \]

In terms of the "quadratic multiplication" \( P(u) = 2 T_u^2 - T_u^+ \), one can also write \( N(u) = \det(P(u))^{1/2} \). By the definition, it is clear that

\[(2.5) \quad N(e) = 1, \quad N(gu) = \det(g) N(u) \quad (g \in G(\mathcal{L}), \ u \in \mathcal{L}), \]

which characterizes the norm uniquely. Denoting the Euclidean measure on \( U \) by \( du \), we see that \( d\mathcal{L}(u) = N(u)^{-1} du \) is an invariant measure on \( \mathcal{L} \).

Example. Let \( U = \text{Sym}_r(\mathbb{R}) \) (the space of real symmetric matrices of degree \( r \)) and \( \mathcal{L} = \mathcal{P}_r(\mathbb{R}) \) (the cone of positive definite elements in \( U \)).

Then one has

\[ T_u(u') = \frac{1}{2} (uu' + u'u) \]

and so

\[ \tau(u) = \frac{r+1}{2} \text{tr}(u), \quad N(u) = \det(u)^{\frac{r+1}{2}}. \]
2.2. We define the gamma function of the cone $\mathcal{L}$ by

\begin{equation}
\Gamma_{\mathcal{L}}(s) = \int_{\mathcal{L}} N(u)^{s-1} e^{-\tau(u)} \, du
\end{equation}

which converges absolutely for $\Re s$ sufficiently large (actually for $\Re s > 1 - \frac{1}{n}$ as we will see later).

**Lemma 2.1.** Suppose that the inner product $\langle \cdot, \cdot \rangle$ is normalized by (2.3). Then one has for any $v \in \mathcal{L}$

\begin{equation}
\int_{\mathcal{L}} N(u)^{s-1} e^{-\langle u, v \rangle} \, du = \Gamma_{\mathcal{L}}(s) N(v)^{-s}.
\end{equation}

**Proof.** Let $v = g_i e$ with $g_i \in G$ and put $u' = t g_i u$. Then one has

\[ \langle u, v \rangle = \langle u, g_i e \rangle = \langle u', e \rangle = \tau(u'). \]

Hence by (2.5) the left-hand side of (2.7) is equal to

\[ \int_{\mathcal{L}} N(v)^{s} e^{-\langle u, v \rangle} \, d\mathcal{L}(u) \]

\[ = \int_{\mathcal{L}} (\det(g_i)^{-1}N(u'))^s e^{-\tau(u')} \, d\mathcal{L}(u') \]

\[ = N(v)^{-s} \Gamma_{\mathcal{L}}(s), \text{ q.e.d.} \]

It is known that the function $\Gamma_{\mathcal{L}}(s)$ can be expressed as a product of ordinary gamma functions (cf. e.g., Resnikoff loc. cit.). For the sake of completeness, we sketch a proof. First, it is clear that, if

\[ \mathcal{L} = \mathcal{L}_1 \times \cdots \times \mathcal{L}_m \]

is the decomposition of $\mathcal{L}$ into the direct product of irreducible (self-dual homogeneous) cones, then one has

\[ \Gamma_{\mathcal{L}}(s) = \Gamma_{\mathcal{L}_1}(s) \cdots \Gamma_{\mathcal{L}_m}(s). \]

Hence, for our purpose, we may assume that $\mathcal{L}$ is irreducible.

We need the root structure of $\mathcal{L}$, which can be determined as follows. Let

\begin{equation}
e = \sum_{i=1}^r e_i, \quad e_i e_j = \delta_{ij} e_i
\end{equation}
be a decomposition of $e$ (in the Jordan algebra $U$) into the sum of mutually orthogonal primitive idempotents. ("Primitive" means that each $e_i$ cannot be decomposed into the sum of mutually orthogonal idempotents any more.) Then we obtain the following decomposition of $U$ into the direct sum of subspaces ("Peirce decomposition").

\[(2.9) \quad U = \bigoplus_{i \neq j} U_{ij},\]

where

\[U_{ii} = \left\{ u \in U \mid e_i u = u \right\},\]
\[U_{ij} = \left\{ u \in U \mid e_i u = e_j u = \frac{1}{2} u \right\} \quad (i \neq j).\]

Then one has $e_k u = 0$ for $u \in U_{ij}$, $k \neq i, j$. Moreover

\[(2.10) \quad \dim U_{ii} = 1, \quad \dim U_{ij} = d \quad (i \neq j),\]

where $d$ is a positive integer depending on the irreducible cone $\mathcal{L}$. (For instance, one has $d = 1$ for $\mathcal{L} = \mathcal{P}_x(\mathbb{R})$.) From (2.9), (2.10) one has the relation

\[(2.11) \quad n = r + \frac{1}{2} r(r - 1)d, \quad \text{i.e.,} \quad d = \frac{2(n - r)}{r(r - 1)}.\]

It follows that

\[(2.12) \quad \tau(e_i) = \text{tr}(T_{e_i}) = 1 + \frac{1}{2} (r - 1)d = \frac{n}{r},\]

\[\begin{equation}
\text{Put}
\end{equation}\]

\[(2.13) \quad \mathcal{A}_r = \left\{ T_{e_i} \mid 1 \leq i \leq r \right\}.\]

Then $\mathcal{A}_r$ is an abelian subalgebra of $\mathcal{P}_r$ of dimension $r$ contained in $\mathcal{P}_x$.

We denote by $(\lambda_i)$ the basis of $\mathcal{A}_r^*$ (the dual space of $\mathcal{A}_r$) dual to $(T_{e_i})$, i.e., one has the relation

\[T = \sum_{i=1}^{r} \lambda_i(T) T_{e_i} \quad (T \in \mathcal{A}_r).\]

We put $\alpha_{ij} = \frac{1}{2} (\lambda_i - \lambda_j) (i \neq j)$. 

PROPOSITION 1. The root system of $\mathfrak{g}$ relative to $\mathfrak{a}$ is given by $\Phi = \left\{ \alpha_{i,j} \mid i \neq j \right\}$. The root space $\mathfrak{g}(\alpha_{i,j})$ corresponding to $\alpha_{i,j}$ is given by

$$\mathfrak{g}(\alpha_{i,j}) = \left\{ T_u + \left[ T_{e_i} - e_j, T_u \right] \mid u \in U_{ij} \right\}.$$  

(2.14)

This can be verified by a straightforward computation; see e.g., Ash et al. [1] Ch. II, §3. Proposition 1 implies that the R-rank of $\mathfrak{g}$ is equal to $r$ and the root system $\Phi$ is of type $(A_{r-1})$.

2.3. Next we determine the Haar measure of $G$. Put

$$\nu = \sum_{i<j} \mathfrak{g}(\alpha_{i,j})$$

and let $A, N$ be the analytic subgroups of $G$ corresponding to $\mathfrak{a}, \nu$, respectively. Then one has an Iwasawa decomposition $G = NA \cdot K(\mathbb{R} \times A \times K)$, which gives rise to the following formula for (the volume element of) a (biinvariant) Haar measure on $G$:

$$dg = c_1 e^{-2r} (\log a) d\eta \, da \, dk$$  

(2.15)

for $g = n a k$ with $n \in N, a \in A, k \in K$, where $dn, da, dk$ denote Haar measures on $N, A, K$, respectively, $c_1$ is a positive constant depending on the normalization of the Haar measures, and $\mathfrak{f}$ is a linear form on $\mathfrak{a}$ defined by

$$\mathfrak{f}(T) = \frac{1}{2} \text{tr}(\text{ad} T \mid \nu) \quad (T \in \mathfrak{a});$$

by Proposition 1 one has

$$\mathfrak{f} = \frac{d}{2} \sum_{i<j} \alpha_{i,j} = \frac{d}{2} \sum_{i=1}^{r} (r - 2i + 1) \lambda_i.$$  

(2.16)

The Haar measure of $K$ is always normalized by $\int_{K} dk = 1$. We make an identification $A = (\mathbb{R}^+)^r$ by the correspondence $a \leftrightarrow (t_i)$ defined by the relation $a = \exp(\sum \lambda_i T_{e_i}), t_i = e^{\lambda_i}$; then one has $da = \prod_{i=1}^{r} (dt_i / t_i)$. Moreover one has

$$\det(a) = e^{\mathfrak{f}(\sum \lambda_i e_i)} = e^{\frac{d}{2} \sum \lambda_i} = (\prod_{i=1}^{r} t_i)^{\frac{d}{2}},$$

$$a \cdot e = \sum_{i=1}^{r} e^{\lambda_i} e_i = \sum_{i=1}^{r} t_i e_i.$$  

(2.17)
\[ e^2 \rho(\log a) = \prod_{i=1}^{r} t_i^{\frac{1}{4}(r-2i+1)}. \]

Since \( \mathcal{L} = G/K \), we can normalize the Haar measure of \( G \) by the relation
\[ dg = d_{\mathcal{L}}(u) \cdot dk \text{ where } u = ge. \]
Then by (2.15), (2.16), (2.17) one has
\[
\int_{\mathcal{G}}(s) = \int_{\mathcal{N}} N(ge)^s e^{-\tau(ge)} dg
\]
\[
= c_1 \int_{\mathcal{N}} \det(a)^s e^{-2\rho(\log a) da} \int_{\mathcal{N}} e^{-\tau(n \cdot ae)} dn
\]
\[
= c_1 \int_{\mathcal{N}} \cdots \int_{\mathcal{N}} \left( t_i \prod_{i=1}^{r} t_i^{\frac{1}{4}(r-2i+1)-1} dt_i \right)
\]
\[ \times \int_{\mathcal{N}} e^{-\tau(n \cdot \sum t_i e_i)} dn. \]

To compute the integral over \( \mathcal{N} \), we introduce some notations. For \( u = \sum_{i<j} u_{ij} e_i e_j \in U \) with \( u_{ij} \in U_{ij} \), we put
\[
T_u(+) = \frac{1}{2} (T_u + \sum_{i<j} [T_u e_i, T_u e_j]),
\]
\[
\epsilon(+) = \sum_{i<j} \sum_{\nu=1}^{\infty} \frac{1}{\nu!} \sum_{i<k \leq t_i} u_{ik} u_{jk} \cdots u_{k \leq t_i}.
\]
(Then one has The \( U_{ij} \)-component of \( \epsilon(+) \) is denoted by \( \epsilon^{(+)}_{ij}(u) \).)

**Lemma 2.** The notation being as above, one has
\[
(\exp T_u(+) ) (\sum_i t_i e_i) = \sum_i (t_i + \frac{1}{r} \sum_{k \geq i} t_k \epsilon^{(+)}_{ik}(u)^2) e_i
\]
\[ + \frac{1}{2} \sum_{i<j} (t_j \epsilon^{(+)}_{ij}(u) + \sum_{k \geq j} t_k \epsilon^{(+)}_{ik}(u) \epsilon^{(+)}_{jk}(u)). \]

This may be regarded as a generalization of the so-called "Jacobi transformation". The proof is again straightforward. It follows that, if \( n = \exp T_u(+) \uparrow (u \in \sum_{i<j} U_{ij}) \), one has
\[
\tau(n \cdot \sum_i t_i e_i) = \frac{n}{r} \sum_i t_i + \frac{1}{8} \sum_{i<k} \tau(\epsilon^{(+)}_{ik}(u)^2) t_k.
\]
We denote the Euclidean measure on \( U_{ij} \) (\( i < j \)) (relative to the inner product \( \langle \rangle \)) by \( du_{ij} \) and define the Haar measure on \( \mathcal{N} \) by
\[ dn = \prod_{i<j} du_{ij} \quad \text{for } n = \exp T_u(+) \uparrow. \]
Since the map \( \epsilon(+) \) is a bijection of \( \sum_{i<j} U_{ij} \) onto itself with Jacobian
equal to one, one has

\[ du = \prod_{i < j} du_{ij} = \prod_{i < j} du'_{ij}, \]

where \( u' = E^u(u) \). Hence by (2.21) one has

\[
\int_N e^{-\tau(n \geq t, e_i)} du = e^{-\frac{n}{r} \sum t_i} \prod_{i < j} \int_{U_{ij}} e^{-\frac{t_j}{\pi} \tau(u_{ij}^2)} du'_{ij} \\
= e^{-\frac{n}{r} \sum t_i} \prod_{i < j} \left( \frac{8\pi}{t_j} \right)^{\frac{1}{2}} \\
= \left( \frac{8\pi}{t_j} \right)^{\frac{1}{2}} \prod_{i < j} \left( t_j - \frac{1}{2}(j-1) \right) e^{-\frac{n}{r} t_j}.
\]

Inserting this in (2.18), one obtains

\[
\Gamma_e(s) = c_1 \left( \frac{8\pi}{t} \right)^{\frac{n-r}{2}} \prod_{j=1}^r \left( \int_{t_j}^{\infty} s - \frac{d}{2}(r-j-1) e^{-\frac{n}{r} t_j} dt_j \right) \\
= c_1 \left( \frac{8\pi}{t} \right)^{\frac{n-r}{2}} \prod_{j=1}^r \left( \frac{n}{r} - \frac{r}{2}(r-j) \right) \Gamma \left( \frac{n}{r} s - \frac{d}{2}(r-j) \right) \\
= c_1 \left( \frac{8\pi}{t} \right)^{\frac{n-r}{2}} \prod_{j=1}^r \Gamma \left( \frac{n}{r} s - \frac{d}{2}(j-1) \right).
\]

The constant \( c_1 \) can be determined by the following observation. We set

\[ U_0 = \sum_{i=1}^r U_{ij} = \left\{ e_1, \ldots, e_r \right\}_R \]

and denote by \( du_0 \) the Euclidean measure on \( U_0 \) (relative to \( \langle \cdot, \cdot \rangle \)). Then, since \( \langle e_i, e_j \rangle = \frac{n}{r} \delta_{ij} \), the bijection \( A \rightarrow U_0 \) defined by \( a = \exp T_{u_0} \)

or equivalently by \( ae = \exp u \), gives the relation

\[ du_0 = \left( \frac{n}{r} \right)^{\frac{r}{2}} da. \]

Hence, when

\[ u = (pa)e = \mathcal{B} \left( \sum_{i=1}^r t_i e_i \right), \]

\[ \mathcal{N} = \exp T_{x'}^+, x \in \sum_{i < j} U_{ij}, \quad x' = E^{(+)}(x), \]

one has by Lemma 2

\[ \frac{\mathcal{T}(u)}{\mathcal{T}(t, x)} = \frac{\mathcal{T}(u_0, u_{ij})}{\mathcal{T}(t_i, x'_{ij})} = \left( \frac{n}{r} \right)^{\frac{r}{2}} \prod_{j=1}^r \left( \frac{r}{2}(j-1) \right) d \]
\[ = 2^{r-n} \left( \frac{n}{r} \right)^{\frac{r}{2}} \prod_{j=1}^{r} t_j^{(j-1)d}. \]

It follows that
\[ d_n(u) = 2^{r-n} \left( \frac{n}{r} \right)^{\frac{r}{2}} \prod_{j=1}^{r} (t_j^{(j-1)d} - \frac{n}{r} t_j) \ dx, \]

which, in view of (2.11) and (2.16), implies (2.15) and the relation
\[ (2.22) \quad c_1 = 2^{r-n} \left( \frac{n}{r} \right)^{\frac{r}{2}}. \]

Thus we obtain the formula
\[ (2.23) \quad \prod_{\alpha} (s) = (2\pi)^{\frac{n-r}{2}} \left( \frac{n}{r} \right)^{\frac{n-r}{2}} \prod_{j=1}^{r} (s - \frac{d}{2}(j-1)) \]

Our computation also shows that the integral for \( \prod_{\alpha} (s) \) converges absolutely for \( \text{Res} > 1 - \frac{r}{n} \).

From the relation (1.2) one obtains
\[ \prod_{\alpha} (s) \prod_{\alpha} (1-s) = (2\pi)^{n-r} \prod_{j=1}^{r} \left( s - \frac{d}{2}(j-1) \right)^{\frac{n}{r}(1-s) - \frac{d}{2}(r-j))} \]
\[ = (2\pi)^{n-r} (2\pi i)^{r} \prod_{j=1}^{r} \frac{e^{\pi i(s - \frac{d}{2}(j-1))}}{e^{\pi i(s - \frac{d}{2}(j-1))} - 1} \]

Since one has by (2.11)
\[ n - r = d \frac{r(r-1)}{2} = \begin{cases} \ 0 \mod 2 \ & \text{for } d \text{ even} \\ \left[ \frac{r}{2} \right] \mod 2 \ & \text{for } d \text{ odd,} \end{cases} \]

one has
\[ \prod_{j=1}^{r} e^{-\pi i \frac{d}{2}(j-1)} = (-1)^d \frac{r(r-1)}{2} = \begin{cases} i^{n-r} \ & \text{for } d \text{ even} \\ i^{n-r}\left[ \frac{r}{2} \right] \ & \text{for } d \text{ odd.} \end{cases} \]

Hence one obtains the following functional equation:
\[ (2.24) \quad \prod_{\alpha} (s) \prod_{\alpha} (1-s) = (2\pi i)^n e^{n \pi i s} \begin{cases} (e^{2\pi i \frac{n}{r} s} - 1)^{-r} \ & \text{for } d \text{ even} \\ (e^{2\pi i \frac{n}{r} s} - 1)^{-\frac{r+1}{2}} \ & \text{for } d \text{ odd.} \end{cases} \]
§ 3. Zeta functions of a self-dual homogeneous cone.

3.1. We fix a $\mathbb{Q}$-structure on $U$ and assume that (the Zariski closure of) $G$ is defined over $\mathbb{Q}$ and $e \in U_{\mathbb{Q}}$; then (the Zariski closure of) $K$ is also defined over $\mathbb{Q}$. We also fix a lattice $L$ in $U$ compatible with that $\mathbb{Q}$-structure, i.e., such that $U_{\mathbb{Q}} = L \otimes \mathbb{Q}$, and an arithmetic subgroup $\Gamma$ fixing $L$, i.e., a subgroup of $G_L = \{ g \in G \mid gL = L \}$ of finite index; for simplicity we assume that $\Gamma$ has no fixed point in $\mathcal{L}$. We then define the zeta function associated with $\mathcal{L}, \Gamma, L$ as follows:

$$\zeta_{\mathcal{L}}(s; \Gamma, L) = \sum_{u \in \Gamma \backslash \mathcal{L} \cap L} N(u)^{-s},$$

the summation being taken over a complete set of representatives of $\mathcal{L} \cap L$ modulo $\Gamma$. It can be shown easily that this series is absolutely convergent for $\Re s > 1$.

By the reduction theory, $\Gamma$ has a fundamental domain in $\mathcal{L}$ which is a rational polyhedral cone. More precisely, there exists a finite set of simplicial cones

$$C^{(i)} = \left\{ v_1^{(i)}, \ldots, v_{k_i}^{(i)} \right\}_{R+},$$

$$= \left\{ \sum_{j \in I} \lambda_j v_j^{(i)} \mid \lambda_j \in R_+ \right\} \quad (1 \leq i \leq m),$$

where $v_1^{(i)}, \ldots, v_{k_i}^{(i)}$ are linearly independent elements in $\overline{\mathcal{L}} \cap L$, such that

$$\mathcal{L} = \bigsqcup_{1 \leq i \leq m} \gamma C^{(i)}.$$

It follows that

$$\zeta(s; \Gamma, L) = \sum_{i=1}^{m} \sum_{u \in C^{(i)} \cap L} N(u)^{-s}.$$

For a set of linearly independent vectors $v_1, \ldots, v_l \in L$, we put

$$R((v_j), L) = \left\{ \sum_{j=1}^{l} \lambda_j v_j \mid 0 < \lambda_j \leq 1 \right\} \cap L,$$

which is finite. Then $u \in C^{(i)} \cap L$ can be written uniquely in the form

$$u = v_0 + \sum_{j=1}^{l} m_j v_j^{(i)}, \quad v_0 \in R((v_j^{(i)}), L), \quad m \in \mathbb{Z}, \quad m \geq 0.$$
For a set of linearly independent vectors \( v_1, \ldots, v_k \in \mathcal{M} \cap V_\mathcal{Q} \) and \( v_0 = \sum_j \alpha_j v_j \) \((\alpha_j \in \mathbb{Q}_\mathcal{Q})\), we define a "partial zeta function" by

\[
\zeta_{\mathcal{M}}(s; (v_j), v_0) = \sum_{\mathcal{M} \subseteq \mathcal{M}} N(v_0) + \sum_{j=1}^k m_j v_j^{-s},
\]

which will also be written as \( \zeta_{\mathcal{M}}(s; (v_j), (\alpha_j)) \). Then the zeta function (3.1) can be written as a finite sum of partial zeta functions as follows:

\[
\zeta_{\mathcal{M}}(s; L, L) = \sum_{i=1}^\infty \sum_{v_0 \in \mathcal{R}(l_{v_j} \omega_j, L)} \zeta_{\mathcal{M}}(s; (v_j^{(i)}), v_0).
\]

Hence the study of special values of \( \zeta_{\mathcal{M}}(s; L) \) is reduced to that of the partial zeta functions of the form (3.2).

3.2. Let \( (v_j) \) and \( v_0 \) be as above. Then by (2.7) one obtains

\[
\Gamma_{\mathcal{M}}(s) \zeta_{\mathcal{M}}(s; (v_j), v_0) = \sum_{\mathcal{M} \subseteq \mathcal{M}} \Gamma_{\mathcal{M}}(s) N(v_0) + \sum_{j=1}^k m_j v_j^{-s} \]

\[
= \sum_{\mathcal{M} \subseteq \mathcal{M}} \int_{\mathcal{M}} N(u)^{s-1} e^{-\frac{\mathcal{M}}{k} (\alpha_j + m_j) <v_j, u>} \, du
\]

\[
= \int_{\mathcal{M}} N(u)^{s-1} \prod_{j=1}^k b(<v_j, u>, l - \alpha_j) \, du
\]

\[
= \int_G \det(g)^{s-1} \prod_{j=1}^k b(<v_j, g e>, l - \alpha_j) \, dg.
\]

In the notation of §2, but this time using the decomposition \( G = K A K \), one has

\[
\det(g) = c \, A(a) \, dk \cdot da \cdot dk'.
\]

for \( g = k a k \), \( k, k' \in K, a \in A \). Here \( c \) is a positive constant and

\[
A(a) = \prod_{\alpha \in \Phi_+} (e^{\alpha(\log a)} - e^{-\alpha(\log a)})^d
\]

\[
= (\prod_{i=1}^r t_i^{-1})^d (r-1) \, |A(t_1, \ldots, t_r)|^d,
\]

where \( A(t_1, \ldots, t_r) = \prod_{i<j} (t_i - t_j) \) (cf. Helgason [5], Ch. X, §1). Hence in view of (2.11) and (2.17) one has

\[
\Gamma_{\mathcal{M}}(s) \zeta_{\mathcal{M}}(s; (v_j), (\alpha_j)) = c \int \cdots \int (\prod_{i=1}^r t_i^{\frac{\mathcal{N}(s-1)}{s-1}} |A(t)|^d \gamma(t) \prod_{t=1}^r dt_i,
\]

where
\[ F(t_1, \ldots, t_r) = \int \prod_{j=1}^{l} b(\nu_j, k \sum t_i e_i, 1 - \alpha_j) \, dk. \]

It is clear that \( F(t_1, \ldots, t_r) \) is holomorphic for \( \text{Re} \ t_i > 0 \ (1 \leq i \leq r) \).

Since \( K \) contains an element which induces any given permutation of \( e_1, \ldots, e_r \), the function \( F \) is symmetric. Hence, denoting by \( B_i \) an open simplicial cone in \( \mathbb{R}^r \) defined by \( t_1 > \ldots > t_r > 0 \), one has

\[ (3.5') \quad \varrho(s, \xi) = c \cdot r! \int_{B_i} (\prod t_i)^{r-1} s^{r-1} \Delta(t)^d F(t) \prod dt_i. \]

3.3. Still following Shintani [11], we make a change of variables \( (t_i) \rightarrow (t_i, \tau_1, \ldots, \tau_r) \) with \( \tau_i = t_i/t_{i+1} \ (2 \leq i \leq r) \). Then \( B_i \) can be expressed as

\[ B_i = \left\{ (t_i) \mid t_i = t_1 \prod_{j=2}^{i} \tau_j, \ 0 < t_i < \infty, \ 0 < \tau_i < 1 \right\}. \]

Putting \( \tau_1 = t_1 \), one has

\[ \frac{\varrho(t_1, \ldots, t_r)}{\varrho(t_1, \tau_2, \ldots, \tau_r)} = \prod_{i=1}^{r} \tau_i^{r-i}, \]

\[ \prod t_i = \prod \tau_i^{r-i+1}, \]

\[ \Delta(t) = \prod_{i=1}^{r} \tau_i^{r-i+1} \prod_{2 \leq i < j \leq r} (1 - \tau_i \cdots \tau_j). \]

It follows that the exponent of \( \tau_i \) in the integrand in (3.5') is equal to

\[ (r-i+1)\frac{n}{r}(s-1) + \frac{d}{r}(r-i+1)(r-1) + r - 1 \]

\[ = (r-i+1)\left\{ \frac{n}{r}s - \frac{d}{r}(i-1) \right\} - 1. \]

Hence one has

\[ (3.6) \quad \varrho(s, \xi) = c \cdot r! \int_{0}^{\infty} t^{n-1} dt \]

\[ \int_{0}^{1} \prod_{i=1}^{r} \tau_i^{r-i+1} \left\{ \frac{n}{r}s - \frac{d}{r}(i-1) \right\} \Delta(t)^d F(t, \tau) \prod dt_i, \]

where

\[ (3.7) \quad \tilde{F}(t_1, \tau) = \prod_{2 \leq i < j \leq r} (1 - \tau_i \cdots \tau_j)^d F(t_1, t_1 \tau_2, \ldots, t_1 \tau_i \cdots \tau_r). \]
3.4. We now assume that all \( v_j \)'s are in \( \mathcal{Q} \) (not on the boundary of \( \mathcal{Q} \)). (In the situation explained in 3.1, this means that the \( \mathfrak{g} \)-rank of \( G \) is equal to 1.) Then for any \( v \in \mathcal{Q} \setminus \{ 0 \} \), one has \( < v_j, v > > 0 \); in particular,

\[
< v_j, ke_i > > 0 \quad \text{for all} \quad k \in K, 1 \leq i \leq r.
\]

Put

\[
\zeta_j = < v_j, k \sum t_i e_i >
\]

\[
= t_1 < v_j, ke_1 > (1 + \sum_{i=2}^{r} t_i \ldots t_i \frac{< v_j, ke_i >}{< v_j, ke_i >}).
\]

For the fixed \( e_i, v_j \), choose \( \beta, \beta_i > 0 \) in such a way that

\[
\begin{cases}
\sum_{i=2}^{r} \beta_i^{i-1} \frac{< v_j, ke_i >}{< v_j, ke_i >} < 1 & \text{for all} \quad k \in K, 1 \leq j \leq \ell, \\
\beta_i < \frac{\tau_i}{< v_j, ke_i >} & \text{for all} \quad 1 \leq j < \ell.
\end{cases}
\]

The

\[
0 < |t_i| < \beta, \quad |\tau_i| < \beta \quad (2 \leq i \leq r),
\]

one has \( 0 < |\zeta_j| < 2 \pi \) and so \( b(\zeta_j, 1 - \zeta_j) \) is holomorphic. Hence the function \( F(t) = F(t_1, t_\tau, \ldots, t_1 \tau \ldots \tau_r) \) has a Laurent expansion in \( t_1, \tau, \ldots, \tau_r \) in the domain defined by (3.11). The coefficients in this expansion are a \( \mathfrak{g} \)-linear combination of the integrals of the form

\[
I((v_j)) = \int_{K \times \mathcal{Q}} \prod_{i \leq j \leq \ell} < v_j, ke_i >^{\psi} dk
\]

where \( \psi \geq 0 \) for \( 2 \leq i \leq r \) and \( \psi \in \mathbb{Z} \) for all \( i, j \).
3.5. Let \( I(\xi, 1) \) denote the contour consisting of the line segment \([\xi, 1]\) taken twice in opposite directions and of a (small) circle of radius \( \xi \) about the origin taken in the counterclockwise direction. When the \( \tau_i \) (2 \( \leq i \leq r \)) are on \( I(\xi, 1) \), one has by (2.12)

\[
|<v_j, k(e_1 + \sum_{i=2}^{r} \tau_i \cdots \tau_i e_i)| | \leq |v_j| \cdot \sum_{i=1}^{r} |e_i| = \sqrt{n/r} \cdot |v_j|
\]

and

\[
\Re <v_j, k(e_1 + \sum_{i=2}^{r} \tau_i \cdots \tau_i e_i) = <v_j, ke_1> + \sum_{i=2}^{r} \Re(\tau_1 \cdots \tau_i) <v_j, ke_i> \geq <v_j, ke_1> - \xi |v_j| \cdot \sum_{i=2}^{r} |e_i| = <v_j, ke_1> - \xi (r-1) \sqrt{n/r} |v_j|.
\]

We choose \( \xi \) so that one has

\[(3.13) \quad \xi \sqrt{n} |v_j| < \text{Min} \left\{ 2\pi, <v_j, ke_1> (k \in K) \right\} \quad \text{for all } 1 \leq j \leq l,
\]

The above inequalities show that \( <v_j, k(e_1 + \sum_{i=2}^{r} \tau_i \cdots \tau_i e_i) > \) belongs to the domain

\[
\left\{ z \in \mathbb{C} \mid |z| < \frac{2\pi}{\xi}, \Re z > \xi \sqrt{n/r} |v_j| \right\}.
\]

It follows that, if \( t_1 \) is on the contour \( I(\xi, \infty) \), one has

\[
|\xi_j| < 2\pi \quad \text{or} \quad \Re \xi_j > 0,
\]

so that the function \( b(\xi_j', 1 - \alpha_j') \) is holomorphic.

From this observation, it is clear that the integral on the r.h.s. of (3.6) is equal to the contour integral

\[
(e^{2\pi i n s} - 1)^{-1} \int_{t_1 \in I(\xi, \infty)} \cdots \int_{t_{r-1} \in I(\xi, 1)} \left(e^{2\pi i \tau_i - i+1 ns} - 1\right)^{-1} \left(e^{2\pi i - i+1 ns} - 1\right)^{-1} t_i, \in I(\xi, 1)
\]

which is independent of the choice of \( \xi \) satisfying (3.13). As is easily seen, the contour integral converges for all \( s \in \mathbb{C} \). Hence, the integral \( \int_{\mathbb{C}} (\xi^v - (r-i+1)n)^{-1} \) viewed as a function in \( s \), can be continued to a meromorphic function on the whole plane; the possible poles are of the form

\[
\frac{1}{(r-i+1)n} (v \in \mathbb{Z}).
\]
§ 4. The special values of the zeta functions.

4.1. As a preliminary, we check the rationality of the constant $c$ in (3.4). For that purpose, we compute $\Gamma_\mathfrak{K}(s)$ by using the decomposition $G = \mathfrak{K} \mathfrak{K} \mathfrak{K}$.

\[
\Gamma_\mathfrak{K}(s) = \int \mathfrak{N}(u)^s e^{-\tau(u)} d\mathfrak{K}(u) = \int_G \mathfrak{N}(ge)^s e^{-\tau(ge)} dg = c \int_A \det(a)^s e^{-\tau(ae)} \Delta(a) da = c \int_0^\infty \left( \prod_{i=1}^r t_i \right)^{s-1} |\Delta(t)|^d e^{-\frac{1}{r} \sum_{i=1}^r t_i} \prod_{i=1}^r dt_i.
\]

We make another change of variables:

\[
t = \sum_{i=1}^r t_i^i, \quad t_i^i = t_i / t.
\]

Then

\[
\mathfrak{R}(t_1, \ldots, t_r) = (\mathfrak{R}(t, t_1^i, \ldots, t_r^i)^r = (-1)^{r-l} t^{r-1},
\]

and the exponent of $t$ in the integrand in the last member of (4.1) is equal to

\[
n(s - 1) + \frac{d}{2} r(r - 1) + r - 1 = ns - 1.
\]

Hence one has

\[
\Gamma_\mathfrak{K}(s) = c \mathfrak{R}(s) \beta(s),
\]

where

\[
\begin{align*}
\mathfrak{R}(s) &= \int_0^\infty t^{ns-1} e^{-\frac{1}{r} t} dt = (\frac{r}{n})^{ns} \Gamma(ns), \\
\beta(s) &= \int_{t_i^i > 0} \left\{ t_1^i \ldots t_r^i, (1 - \sum_{i=1}^r t_i^i) \right\}^{s-1} \times \\
&\quad \times \left| \Delta(t_1^i, \ldots, t_r^i, 1 - \sum_{i=1}^r t_i^i) \right| d \prod_{i=1}^r dt_i.
\end{align*}
\]

For $s = 1$, one has

\[
\Gamma_\mathfrak{K}(1) = c \mathfrak{R}(1) \beta(1) = c (\frac{r}{n})^n (n-1)! \beta(1),
\]
\[ \beta(1) = \int \Delta(t_{1}^{r}, \ldots, t_{r_{-1}}^{r}, 1-\sum_{i} t_{i}^{r}) \prod_{i} \mathrm{d}t_{i}^{r} \in \mathbb{Q}. \]

By (2.23) one has

\[
\begin{align*}
\mathcal{G}(1) &= (2\pi)^{\frac{n-r}{2}} \left( \frac{\pi}{n} \right)^{\frac{r}{2}} \prod_{j=1}^{r} \Gamma \left( 1 + \frac{d}{2} (j-1) \right) \\
&\sim \left\{ \begin{array}{ll}
\pi^{\frac{n-r}{2}} & (\text{d even}) \\
\pi^{\frac{n-\lfloor \frac{r+1}{2} \rfloor}{2}} & (\text{d odd})
\end{array} \right.
\end{align*}
\]

where \( \sim \) means that \( a/b \in \mathbb{Q} \). Thus one has

\[
\mathcal{G}(1) \sim \frac{(2\pi)^{\frac{n-r}{2}} \left( \frac{\pi}{n} \right)^{\frac{r}{2}} \prod_{j=1}^{r} \Gamma \left( 1 + \frac{d}{2} (j-1) \right)}{(n-1)! \zeta(1)} \sim \mathcal{G}(1).
\]

Since \( \mathcal{G}(1) \sim \mathcal{G}(1 + \frac{1}{n}, \nu) \) for \( \nu \in \mathbb{Z} \), one obtains

\[
\mathcal{G}(s) \sim \frac{c \mathcal{G}(1)}{\mathcal{G}(1)^{2}} \sim \left\{ \begin{array}{ll}
\pi^{\frac{n-r}{2}} & (\text{d even}) \\
\pi^{\frac{n-\lfloor \frac{r+1}{2} \rfloor}{2}} & (\text{d odd})
\end{array} \right.
\]

4.2. We first consider the case where \( d \) is even. Then by (2.24) one has

\[
\mathcal{G}(s) \mathcal{G}(1-s) = (2\pi i)^{n} e^{\pi i \text{ns}} (e^{2\pi i \nu s} - 1)^{-r}.
\]

Hence

\[
\zeta_{\mathcal{G}}(s; (v_{j}), (\nu_{j})) = \frac{c \mathcal{G}(1-s)}{(2\pi i)^{n} e^{\pi i \text{ns}}} \times R(s),
\]

where

\[
R(s) = \left( \frac{e^{2\pi i \nu s} - 1}{2\pi i} \right)^{r} \int_{\mathbb{B}_{t}} (\prod_{j=1}^{r} \Delta(t_{j}^{r}) \Delta(t_{j}^{r}) \prod_{i=1}^{r} \mathrm{d}t_{i}^{r})
\]

\[
= \left( \prod_{j=1}^{r} \frac{e^{2\pi i r \nu s} - 1}{e^{2\pi i r \nu s} - 1} \right) \times \left( \frac{1}{(2\pi i)^{r}} \right)^{r} \int_{\mathbb{B}_{t}} \left( \prod_{i=1}^{r} \mathrm{d}t_{i}^{r} \right)
\]

\[
= \left( \prod_{j=1}^{r} \frac{1}{r} \int_{\mathbb{B}_{t}} \frac{1}{\prod_{i=1}^{r} t_{i}^{r \nu i} s - \frac{1}{2} (1-1)^{r-1} r! \mathcal{F}(t_{i}, T_{i})} \right) \times \left( \prod_{j=1}^{r} \frac{1}{r} \int_{\mathbb{B}_{t}} \frac{1}{\prod_{i=1}^{r} t_{i}^{r \nu i} s - \frac{1}{2} (1-1)^{r-1} r! \mathcal{F}(t_{i}, T_{i})} \right)
\]

We are interested in the values of \( \zeta_{\mathcal{G}} \) at \( s = -\frac{r}{n} \nu \) (\( \nu = 0, 1, \ldots \)). The
first factor in the right hand side of (4.7) is holomorphic for \( \text{Re } s < \frac{1}{n} \) and by (4.6) the value at \( s = -\frac{r}{n} \) is rational:

\[
(4.10) \quad \frac{c \int_0^\infty (1 + \frac{r}{n} \nu) \, e^{-\nu \pi i}}{(2\pi i)^{n-r} e^{-\nu \pi i}} = (-1)^{\frac{n-r}{r}} c \int_0^\infty (1 + \frac{r}{n} \nu) \, e^{-\nu \pi i} \in \mathbb{Q}.
\]

On the other hand, it is clear that

\[
\frac{e^{2\pi i \frac{r}{n} s} - 1}{e^{2\pi i \frac{r}{n} s} - 1} \quad \rightarrow \quad \frac{1}{r-i+1} \quad \text{when } s \rightarrow -\frac{r}{n} \nu.
\]

Hence we see that \( R(-\frac{r}{n} \nu) \) is equal to the coefficient of

\[
t^{r-i+1} \prod_{i=1}^r \gamma_i (r-i+1) \nu + \frac{d}{2} (i-1)
\]

in the Laurent expansion of \( \tilde{F}(t_i, \tau) \),

which is a \( \mathbb{Q} \)-linear combination of \( I((\nu y')) \).

### 4.3

From now on we assume that \( d \) is odd. By the classification theory, it is known that this assumption implies that \( r = 2 (n = d + 2) \) or \( d = 1 \) \((n = \frac{1}{2} r(r+1))\). By (2.24) one has

\[
\Gamma(s) \Gamma(1-s) = (2\pi i)^n e^{\pi is} (e^{2\pi i \frac{r}{n} s} - 1)^{-\frac{1}{2}} (e^{2\pi i \frac{n}{r} s} + 1)^{-\frac{1}{2}}.
\]

Hence

\[
(4.11) \quad \zeta(s; (\nu, \omega), (\alpha, \beta)) = \frac{c \Gamma(1-s)}{(2\pi i)^n e^{\pi is}} \times R^{(s)}(s) R^{(s)}(s),
\]

where

\[
R^{(s)}(s) = (2\pi i)^{\frac{r}{2}} \frac{\Gamma(s - \frac{r}{2}) \Gamma(s + \frac{1}{2})}{\prod_{k=1}^r (e^{2\pi i (r-k+1) \nu} s - \frac{d}{2} (k-1)^2 - 1) - (e^{2\pi i (r-k+1) \nu} s - \frac{d}{2} (k-1)^2 - 1)}.
\]

\[
R^{(s)}(s) = (2\pi i)^{-r} \int_{I(\tau, \omega)} t_i^{ns-1} dt_i, \int_{\omega} \prod_{i} (t_i^{r-i+1} \frac{\nu}{\pi} s - \frac{d}{2} (i-1)^2 \nu_1 - \Phi(t_i, \nu)) d\tau_i.
\]
The first factor in the right hand side of (4.11) is holomorphic for \( \text{Re } s < \frac{1}{n} \) and by (4.6) the value at \( s = -\frac{r}{n
'} (\nu \geq 0) \) is rational:

\[
(4.12) \quad \frac{c \left( \frac{1}{2} + \frac{r}{n'} \right)}{(2\pi i)^{n-\left[\frac{r+1}{2}\right]} e^{-\pi i vr'}} = (-1)^{\frac{1}{2} (n-\left[\frac{r+1}{2}\right])} \frac{c \left( \frac{1}{2} + \frac{r}{n'} \right)}{(2\pi)^{n-\left[\frac{r+1}{2}\right]}} \in \mathbb{Q}.
\]

Note that one has

\[ n \equiv \left[\frac{r+1}{2}\right] \pmod{2}, \]

since

\[ n = d+2 \equiv 1 \equiv \left[\frac{3}{2}\right] \pmod{2} \quad \text{if } r = 2, \text{ and} \]

\[ n = \frac{1}{2} r(r+1) \equiv \left[\frac{r+1}{2}\right] \pmod{2} \quad \text{if } d = 1. \]

4.4. To compute \( R'(s) \), we first note

\[ e^{\pi i d(k-1)(r-k+1)} = \begin{cases} -1 & \text{if } k \equiv r \equiv 0 \pmod{2}, \\ 1 & \text{otherwise}. \end{cases} \]

We put

\[ \left[\frac{r}{2}\right] = r_1, \quad \zeta = e^{2\pi i \frac{r}{r_1}} s. \]

**The case \( r \) is odd.** One has

\[
R'(s) = (2\pi i)^{r_1} r! \left( \frac{(\zeta - 1)^{r_1} (\zeta + 1)^{r_1}}{(\zeta - 1)} \right) = \frac{r!}{\prod_{k=1}^{r_1} (\zeta^{k-1} + \ldots + \zeta + 1)} (2\pi i \frac{\zeta + 1}{\zeta - 1})^{r_1}.
\]

Hence, when \( s \to -\frac{r}{n'} \nu \), one has

\[
(4.13) \quad (s + \frac{r}{n'} \nu)^{r_1} R'(s) \to (2 \frac{r}{n'})^{r_1}.
\]

Thus \( R'(s) \) has a pole of order \( r_1 \) at \( s = -\frac{r}{n'} \nu \).

**The case \( r \) is even.** One has

\[
R'(s) = (2\pi i)^{r_1} r! \left( \frac{(\zeta - 1)^{r_1} (\zeta + 1)^{r_1}}{(\zeta - 1)} \right) \prod_{k=1}^{r_1} \left\{ (-1)^k \zeta^k - 1 \right\}
\]
\[
(-2\pi i)^{r_1} \frac{r!}{\prod_{k \in \text{odd}} (\zeta^{k-1} + \ldots + \zeta + 1) \prod_{k \in \text{even}} (\zeta^{k-1} - \ldots - \zeta + 1)}
\]

Hence \( R^{(s)} \) is holomorphic at \( s = -\frac{r}{n} \nu \) and

\[
(4.14) \quad R^{(s)}(-\frac{r}{n} \nu) = (-2\pi i)^{r_1} \frac{r!}{(2r_1)!} = (-\pi i)^{r_1} \frac{r!}{r_1!}.
\]

4.5. When \( r \) is odd (hence \( d = 1, n = \frac{1}{2} r(r+1) \)), \( R^{(s)}(s) \) for \( s = -\frac{r}{n} \nu \)

\[-\frac{2\nu}{r+1}\]

is given by the coefficient of

\[
t_1^{r_1} \prod_{i=2}^{r} \tau_i^{(r-i+1)(\nu + \frac{i-1}{2})}
\]
in the Laurent expansion of \( \hat{F}(t_1, \tau) \). Hence \( \zeta_{\hat{Q}} (s; (\nu_j), (\alpha_j)) \) has

at most a pole of order \( r_1 = \frac{r-1}{2} \) at \( s = -\frac{2\nu}{r+1} \) and one has

\[
(4.15) \quad \lim_{s \to -\frac{2\nu}{r+1}} (s + \frac{2\nu}{r+1})^{r_1} \zeta_{\hat{Q}} (s; (\nu_j), (\alpha_j)) \sim R^{(s)}(-\frac{2\nu}{r+1}).
\]

To treat the case \( r \) is even, we use the formula

\[
\int \frac{t^m}{t^{r-1}} \, dt = -\frac{\nu}{m} \quad (m \text{ odd}),
\]

which can be verified easily. When \( r \) is even, the value of \( R^{(s)}(s) \) for

\( s = -\frac{r}{n} \nu \) is given by

\[
(4.16) \quad (-\pi i)^{r_1} \sum_{m_1, \ldots, m_r \in \mathbb{Z}} \frac{a_{(m_j)}}{(m_j + (r-2j+1)(\nu + d(j-1)))^{r_1} \prod_{j=1}^{r} \tau^{m_j}_{2j-1}}
\]

where \( a_{(m_j)} \) is the coefficient of

\[
t_1^{r_1} \prod_{j=1}^{r} \tau^{(r-2j+2)(\nu + d(j-1))}_{2j-1} \prod_{j=1}^{r} \tau^{m_j}_{2j}
\]
in \( \hat{F}(t_1, \tau) \). Hence for the value of \( \zeta_{\hat{Q}} \), one has

\[
(4.17) \quad \zeta_{\hat{Q}} (-\frac{r}{n} \nu; (\nu_j), (\alpha_j)) \sim (2\pi i)^{r_1} R^{(s)}(-\frac{r}{n} \nu).
\]
Bibliography


