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On a certain difference-differential equation related to nonlinear lattices

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In this note I should like to point out that the solutions of equations of the form

\[
\frac{\psi(u_1 + v_1, u_2 + v_2)\psi(u_1 - v_1, u_2 - v_2) - \psi(u_1, u_2)^2}{\psi(u_1, u_2)^2} = \left( F_1(v_1) \frac{\partial^2}{\partial u_1^2} + F_2(v_2) \frac{\partial^2}{\partial u_2^2} \right) \log \psi(u_1, u_2)
\]

are related to multi-soliton solutions of a discrete-time lattice equation.

**Example 1.** The function

\[
\psi(u) = 1 + e^u
\]

satisfies

\[
\frac{\psi(u + v)\psi(u - v) - \psi(u)^2}{\psi(u)^2} = \frac{d^2}{du^2} \log \psi(u)
\]

where

\[
F(v) = 4 \sinh^2 \frac{v}{2}
\]

**Example 2.** The function

\[
\psi(u_1, u_2) = 1 + e^{u_1} + e^{u_2} + B e^{u_1 + u_2}
\]

satisfies (1) with

\[
F_1 = 2 \sinh \frac{v_1}{2}, \quad F_2 = 2 \sinh \frac{v_2}{2} \quad (v_1, v_2 \geq 0)
\]

and

\[
X = \frac{1}{2(B-1) \sinh \frac{v_1}{2} \sinh \frac{v_2}{2}} \left[ B \sinh^2 \frac{v_1 + v_2}{2} + \sinh^2 \frac{v_1 - v_2}{2} \right] - (B+1) \left\{ \sinh \frac{v_1}{2} + \sinh \frac{v_2}{2} \right\}
\]

(1)
Application. Let $n$ be lattice position and $t$ the discrete-time, and

$$U_1 = 2\eta_1 = 2(\rho, n + \beta, t + \eta_1)$$
$$U_2 = 2\eta_2 = 2(\rho, n + \beta, t + \eta_2)$$

and let

$$\psi_n(t) = 1 + \epsilon^2 \eta_1 + \epsilon^2 \eta_2 + B \epsilon^2 \eta_1^2 \eta_2.$$  

We assume a discrete-time lattice equation

$$\delta^{-2}[\psi_n(t+\delta)\psi_n(t-\delta) - \psi_n(t)^2] = \psi_{nt}(t)\psi_{nt}(t) - \psi_n(t)^2.$$  

Then we see that $\beta_i$ (i=1,2) and $B$ are determined as

$$\delta^{-1} \sinh \beta_i \delta = \sinh \rho_i \quad (i = 1, 2)$$

$$B = \frac{\sinh^2(\rho_1 - \rho_2) - \delta^{-2} \sinh(\beta_1 - \beta_2) \delta}{\delta^{-2} \sinh(\beta_1 + \beta_2) \delta - \sinh^2(\rho_1 + \rho_2)}$$

and $\psi_n(t)$ gives a two-soliton solution. $\psi_n(t)$ reduces to a solution for the ordinary exponential lattice in the limit as $t \to 0$.

Extension 1. As an extension of (1) we have

$$\psi(U_1 + \nu_1, U_2 + \nu_2, \ldots) \psi(U_1 - \nu_1, U_2 - \nu_2, \ldots)$$

$$= 1 + \sum_{i, j = 1}^{N} E_{ij}(v_1, v_2, \ldots) \frac{\partial^2}{\partial U_i \partial U_j} \log \psi(U_1, U_2, \ldots).$$

Solutions to this equation will lead us to a $N$-soliton solution.

Extension 2. As is well known, the elliptic $\vartheta$-function $\vartheta_3(u)$ satisfies an equation of the form

$$(2)$$
\[
\frac{\xi_{3}(u+v)}{\xi_{3}(u)} = C(v) + \frac{4(v)}{v} \frac{d}{dv} \log \xi_{3}(v)
\]

and \(\xi_{3}\) is related to the simplest periodic solution of the exponential lattice. For the future problem to obtain general periodic solutions of the discrete-time lattice equation, we will have to deal with a certain equation of the form

\[
\left( \psi(u_{1} + v_{1}, u_{2} + v_{2}, ...) \right) \left( \psi(u_{1} - v_{1}, u_{2} - v_{2}, ...) \right)
\]

\[
\left( \psi(u_{1}, u_{2}, ...) \right)^{2}
\]

\[
= C(v_{1}, v_{2}, ...) + \sum_{i,j} E_{ij}(v_{1}, v_{2}, ...) \frac{\delta^{2}}{\delta u_{i} \delta u_{j}} \log \psi(u_{1}, u_{2}, ...)
\]

Reference