数理解析研究所講究録
414 巻 1981 年 224-226

On a certain difference-differential equation related to nonlinear lattices

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In this note I should like to point out that the solutions of equations of the form

$$\frac{\psi(u_1+v_1, u_2+v_2)\psi(u_1-v_1, u_2-v_2) - \psi(u_1, u_2)^2}{\psi(u_1, u_2)^2}$$

$$= \left(F_1^2(v_1)\frac{\partial^2}{\partial u_1^2} + 2X(v_1,v_2)F_1(v_1)F_2(v_2)\frac{\partial^2}{\partial u_1 \partial u_2} + F_2^2(v_2)\frac{\partial^2}{\partial u_2^2}\right) \log \psi(u_1, u_2) \quad (1)$$

are related to multi-soliton solutions of a discrete-time lattice equation.

Example 1. The function

$$\psi(u) = 1 + e^u$$

satisfies

$$\frac{\psi(u+v)\psi(u-v) - \psi(u)^2}{\psi(u)^2} = F(v)\frac{d^2}{du^2}\log \psi(u)$$

where

$$F^2(v) = 4\sinh^2\frac{v}{2}$$

Example 2. The function

$$\psi(u_1, u_2) = 1 + e^{u_1} + e^{u_2} + Be^{u_1+u_2}$$

satisfies (1) with

$$F_1 = 2\sinh\frac{v_1}{2}, \quad F_2 = 2\sinh\frac{v_2}{2} \quad (v_1, v_2 \gtrless 0)$$

and

$$X = \frac{1}{2(B-1)\sinh\frac{v_1}{2}\sinh\frac{v_2}{2}}\left[B\sinh^2\frac{v_1+v_2}{2} + \sinh^2\frac{v_1-v_2}{2} - (B+1)\left\{\sinh^2\frac{v_1}{2} + \sinh^2\frac{v_2}{2}\right\}\right]$$

Application. Let n be lattice position and t the discrete-time, and

$$u_1 = 2\eta_1 = 2(P_1 n + \beta_1 t + \eta_1^0)$$
$$u_2 = 2\eta_2 = 2(P_2 n + \beta_2 t + \eta_2^0)$$

and let

$$\psi_n(t) = 1 + e^{2\eta_1} + e^{2\eta_2} + B\, e^{2\eta_1 + 2\eta_2} .$$

We assume a discrete-time lattice equation

$$\delta^{-2}[\psi_n(t+\delta)\psi_n(t-\delta) - \psi_n(t)^2] = \psi_{n+1}(t)\psi_{n-1}(t) - \psi_n(t)^2 .$$

Then we see that β_i (i=1,2) and B are determined as

$$\delta^{-1}\sinh \beta_i \delta = \sinh P_i \qquad (i=1,2)$$

$$B = \frac{\sinh^2(P_1 - P_2) - \delta^{-2}\sinh(\beta_1 - \beta_2)\delta}{\delta^{-2}\sinh(\beta_1 + \beta_2)\delta - \sinh^2(P_1 + P_2)} ,$$

and ψ_n(t) gives a two-soliton solution. ψ_n(t) reduces to a solution for the ordinary exponential lattice in the limit as t $\to$ 0.

Extension 1. As an extension of (1) we have

$$\frac{\psi(u_1+v_1, u_2+v_2, \ldots)\psi(u_1-v_1, u_2-v_2, \ldots)}{\psi(u_1, u_2, \ldots)^2}$$
$$= 1 + \sum_{i,j=1}^{N} E_{ij}(v_1, v_2, \ldots)\frac{\partial^2}{\partial u_i \partial u_j}\log \psi(u_1, u_2, \ldots) .$$

Solutions to this equation will lead us to a N-soliton solution.

Extension 2. As is well known, the elliptic ϑ-function $\vartheta_3(u)$ satisfies an equation of the form

$$\frac{\vartheta_3(u+v)\vartheta_3(u-v)}{\vartheta_3(u)^2} = C(v) + A(v)\frac{d^2}{du^2}\log\vartheta_3(u)$$

and ϑ_3 is related to the simplest periodic solution of the exponential lattice. For the future problem to obtain general periodic solutions of the discrete-time lattice equation, we will have to deal with a certain equation of the form

$$\frac{\psi(u_1+v_1,\ u_2+v_2,\ \dots)\,\psi(u_1-v_1,\ u_2-v_2,\ \dots)}{\psi(u_1,\ u_2,\ \quad)^2}$$

$$= C(v_1, v_2, \dots) + \sum_{i,j} E_{ij}(v_1, v_2, \quad)\frac{\partial^2}{\partial u_i \partial u_j}\log\psi(u_1, u_2, \dots).$$

Reference

R. Hirota, J. Phys. Soc. Japan 43 (1977) 2074.