数理解析研究所講究録
415巻 1981年 213-225

Conjugation schemes and second homotopy group

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Introduction. In this paper we introduce an algebraic tool, called conjugation scheme. Definitions are given and some results are stated ; proof, precise references and some other results will appear in a forthcoming paper. A preliminary version with detailed proofs of part of this work is published in preprint form by Istituto Matematico Universitá di Pisa, Italy : "Sistemi di Coniugi e secondo gruppo di omotopia", in Italian.

1. Lüroth-Clebsch condition.

A conjugation system is a couple (G,T) where G is a group and $T \subset G$ is invariant under inner automorphisms : $gTg^{-1} \subset T$, $\forall_g \in G$. For $n \in \mathbb{N}$ and $1 \leq i \leq n-1$ we define $\sigma_i : T^n \to T^n$ by:

$$\sigma_i(t_1,..,t_n) = (t'_1,..t'_n) \quad \text{where} \quad t'_j = t_j \quad \text{for} \quad j \neq i,\ i+1 \quad \text{and}$$

$$t'_i = t_i t_{i+1} t_i^{-1},\ t'_{i+1} = t_i .$$

This gives an action on T^n of the braid group $B(n)$ in n strings. Associate with an element $\underline{t} = (t_1,..t_n) \in T$ the following :

i) The element $\nu(\underline{t}) = t_1 \cdots t_n \in G$

ii) The subgroup $G_{\underline{t}}$ in G, generated by $t_1,..,t_n$.

An element $\underline{t} \in T^n$ will be said <u>complete</u> iff $t_1,\ldots,t_n$ are conjugated to each other in the group G_t.

Definition: The couple (G,t) satisfies the Lüroth-Clebsch condition if the following is true:

(L-C) Let $\underline{t}_1, \underline{t}_2 \in T^n$ be complete. Then they are equivalent under braid action iff $G_{\underline{t}_1} = G_{\underline{t}_2}$ and $\nu(\underline{t}_1) = \nu(\underline{t}_2)$.

Let W be a group generated by reflections in $\mathbb{R}^m$ endowed with some bilinear form, and let $T \subset W$ denote the set of reflections. One can prove the following:

<u>Theorem</u>.1 (W,T) satisfies L-C if W is a Coxeter group of type A_d, D_d, or E_i, $i=6,7,8$.

In all other cases we are unable to decide whether the L-C condition holds (especially when W is any other Coxeter group or the local monodromy group of an isolated singularity of hypersurface, the Milnor fibre of which has not a difinite positive intersection form).

One may try to construct "obstructions" to the validity of L-C. One way is to stabilize the problem in the following way: suppose that T is closed under inversion on G (if not, add to T the set $\{t^{-1} | t \in T\}$); for $\underline{s} \in T^n$, $\underline{t} \in T^m$ define $\underline{s}.\underline{t}$ as the element in T^{n+m} obtained by writing the sequente $\underline{t}$ on the right of the sequence $\underline{s}$, so that $S'(T) = \bigcup_{n\geq 0} T^n$ has the structure of a semigroup; the product being compatible with braid actions, one gets a semigroup $S''(T) = S'(T)/\text{braids}$. Finally define $S(T)$ as the semigroup $S''(T)$ divided by the semigroup generated by $\{(t,t^{-1}) | t \in T\}$. The map $\nu: T^n \to G$ defined above, induces an

homomorphism $S(T) \to G$ that will be denoted again by ν.
One can prove the following:

Theorem 2

i) $S(T)$ is a group

ii) The Kernel of $\nu : S(T) \to G$ is an abelian group; it will be denoted by $\mathcal{O}(T)$.

iii) If L-C holds for (G,T), then $\mathcal{O}(T) = 0$

Theorem 3 $\mathcal{O}(T)$ vanishes in the following cases :

i) G a Coxeter group and T the set of conjugated elements to generators.

ii) G the monodromy group of an isolated singularity of hypersurface type and T the associated set of Picard-Lefschetz transformations (obtained by turning around simple points of the discriminant of a versal deformation).

2. Conjugation schemes

One may generalise the notion of conjugation system as follows.
Let G be a group; it acts on itself by inner automorphisms.
A Conjugation scheme over G is a triple $(\psi, \nu, -)$ where $\psi : G \times T \to T$ is an action of G on a set T, $\nu : T \to G$ is a G-equivariant map and $- : t \to \bar{t}$ is an involution on T satisfying for all $t \in T : \nu(\bar{t}) = \nu(t)^{-1}$ and $\nu(t) \cdot (t) = t$. As before, one has braid actions on the T^n, and the semigroups $S'(T)$, $S''(T)$, $S(T)$ may be defined as well as an homomorphism $\nu : S(T) \to G$; again one can prove that $S(T)$ is a group and that $\mathcal{O}(T) = \mathrm{Ker}\,(S(T) \xrightarrow{\nu} G)$ is an abelian group.

Morphisms between conjugation schemes are defined in the obvious way and the construction above gives a functor with values in the

category of exact sequences $o \to \mathcal{O}(T) \to S(T) \to G$ with $\mathcal{O}(T)$ abelian . Call a conjugation scheme <u>reduced</u> if the map $T \to G$ is injective so that reduced conjugation schemes coincide (essentially) with conjugations systems.

Denote a conjugation system simply by T or by (G,T); let $\tilde{T}$ denote the image of T in G; the conjugation system $(G,\tilde{T})$ will be called the <u>reduced system</u> of (G,T). This gives a functor from {conjugation schemes} $\to$ {conjugation systems} with the property that $S(T) \to S(\tilde{T})$ is surjective (hence $\mathcal{O}(T) \to \mathcal{O}(\tilde{T})$ is surjective too). In some occasions, it will be proved that $\mathcal{O}(T)$ is not zero by showing that $\mathcal{O}(\tilde{T})$ is not zero.

3. Second homotopy groups.

Let X be a C^∞ manifold, $Y \subset X$ a closed submanifold of codimension two. Let $x_0 \in X-Y$ and define :

$$R = \{(\sigma,\varepsilon) \mid \sigma:[0,1] \to X \text{ is differentiable, } \sigma(0) = x_0,\ \sigma^{-1}(Y) = 1,\ \sigma \text{ is transversal to } Y \text{ and } \varepsilon \text{ is an orientation of the normal plane to } Y \text{ at } \sigma(1)\} ;$$

consider on R some standard topology (for example the C^1 topology) and denote by T the set of connected components of R. There is an action of $\pi_1(X-Y;x_0)$ on T (induced by composition of arcs: $g \in \pi_1(X-Y;x_0)$ and $(t,\varepsilon) \in T$ then $g(t) = (t*g^{-1},\varepsilon)$), a map $\nu:T \to \pi_1(X-Y,x_0)$ (defined by associating to $(t,\varepsilon) \in T$ the loop that

follows t until near $t(1)$, then turns around Y according to ε and then come back to x_0 along t^{-1}) and an involution $(t,\varepsilon) \to (t,-\varepsilon)$. One has the following:

Theorem 4

i) $(\pi_1(X-Y;x_0), T)$ is conjugation scheme

ii) The associated map $S(T) \xrightarrow{\nu} \pi_1(X-Y;x_0)$ is canonically isomorphic with the natural map $\pi_2(X,X-Y,x_0) \to \pi_1(X-Y,x_0)$. In particular $\mathcal{O}(T)$ is canonically isomorphic with the cokernel of $\pi_2(X-Y;x_0) \to \pi_2(X;x_0)$.

4. The complement of hypersurfaces

Let Δ be an analytic hypersurface in $\mathbb{C} \times \mathbb{C}^n$ s.th. the projection ψ on $\mathbb{C}^n$ induces a proper branched covering map of degree d. Let us re-examine the Van Kampen Zariski procedure to find a presentation of the fundamental group of $\mathbb{C} \times \mathbb{C}^n - \Delta$. Let $\Gamma \subset \mathbb{C}^n$ be the set of critical values of $\psi : \Delta \to \mathbb{C}^n$, $\tilde{\Gamma} = \mathbb{C} \times \Gamma$. For $z \in \mathbb{C}^n$ denote $\mathbb{C} \times \{z\}$ by L_z. In what follows base points are choosen as $z_0 \in \mathbb{C}^n - \Gamma$ and for the other spaces as (m,z_0) where $|m|$ is large. There is a fibration $\mathbb{C} \times \mathbb{C}^n - (\Delta \cup \tilde{\Gamma}) \to \mathbb{C}^n - \Gamma$ induced by ψ and whose fibre is $\mathbb{C} - \{d \text{ points}\}$. One deduces a diagram :

$$(+) \qquad \begin{array}{ccccccccc} 1 \to & \pi_1(L_{z_0}-\Delta) & \xrightarrow{j} & \pi_1(\mathbb{C}^{n+1} - (\Delta \cup \tilde{\Gamma})) & \underset{\gamma}{\overset{\beta}{\rightleftarrows}} & \pi_1(\mathbb{C}^n - \Gamma) & \to 1 \\ & & & \downarrow\alpha & & & \\ & & & \pi_1(\mathbb{C}^{n+1} - \Delta) & & & \end{array}$$

where the horizontal line is exact (homotopy sequence of a fibration) and γ comes from a section of the fibration. The classical method of computing $\pi_1(\mathbb{C}^{n+1} - \Delta)$ comes from the fact that $\alpha \cdot j$ is

surjective and its kernel is generated (as a normal subgroup) by elements of the form $x^{-1}.\ b(x)$ where $x \in \pi_1(L_{z_0} - \Delta)$ and $b \in \pi_1(\mathbb{C}^n - \Gamma)$, the action of b on x being defined through the semiproduct structure of the horizontal line in the diagram (+). In fact this action comes from a natural homomorphism: $\pi_1(\mathbb{C}^n - \Gamma) \to B(d)=$ braid group in d strings, which operates canonically on the free group in d letters $\pi_1(L_{z_0} - \Delta)$.

Now let $X = \mathbb{C} \times \mathbb{C}^n - \tilde{\Gamma} \cap \Delta$, $Y = \Delta - \tilde{\Gamma}$ and let T be the associated conjugation scheme (as in §3). Such a T may be presented, as a conjugation scheme, by a similar Van Kampen-Zariski procedure that we are going to describe.

5. Presentation of conjugation schemes.

Let (G,T) be a conjugation scheme, and $j : X \to T$ a map where X is a set. We shall say that (G,T) is <u>free on X</u> (better : on j) iff for all (G',T') and maps $i : X \to T'$, there exists one and only one homomorphism $(G,T) \to (G',T')$ compatible with i, j.

<u>Theorem 5</u> (G,T) is free on X iff:

i) $X \subset T$ (i.e. j is injective)

ii) $T \subset G$ (i.e. T is reduced)

iii) G is the free group on X

iv) T is the set of elements of G that are conjugated to elements of X or of X^{-1}.

Now suppose (G,T) is the free conjugation scheme over a set X. Let be given a set of "relations" of the following type :

$x_i = P_i\, y_i P_i^{-1}$, where $x_i, y_i \in X \cup X^{-1}$, $P_i \in G$, $i \in I$.

We say that t and $t' \in T$ are equivalent if they can be interchanged by successive steps of the following types:

- interchange $PP_i t_i P_i^{-1} p^{-1}$ with $Pt_i' P^{-1}$, where $P_i t_i P_i = t_i'$ is one of the given relations.
- interchange QtQ^{-1} with PtP^{-1}, where $P \equiv Q$ mod. the normal subgroup generated in G by the given relations.

This will give a conjugation scheme $\tilde{T}$ on the group $\tilde{G} = G/$ given relations, that satifies an obvious universal property, that characterizes $(\tilde{G},\tilde{T})$ as an "universal" object. In this way we can obtain all the conjugation schemes (G,T) where T generates G. In a similar way one can deal also with conjugation schemes where T does not generate G. We want to emphasize here that the "given relations" cannot be changed in the same way as for presentation of groups. For example let $X = \{x,y\}$ and let be given the relation $x = yxy^{-1}$; the corresponding conjugation scheme is not reduced; in fact $\tilde{G} \simeq Z^2$ is generated by x,y and for $\tilde{T}$ there are the followig distinct representatives: x, x^{-1} and $x^n y x^{-n}$, $x^n \cdot y^{-1} \cdot x^{-n}$ for $n \in Z$. Now add the relation $y = xyx^{-1}$; then G is again Z^2 but the conjugation scheme we get is reduced and it is given by $\{x,x^{-1},y,y^{-1}\}$, this example can be described geometrically in the following way:

let $L = \{(0,0,z) \in \mathbb{R}^3 | z \in \mathbb{R}\}$ and $S = \{(x,y,0) \in \mathbb{R}^3 | x^2 + y^2 = 1\}$

The conjugation scheme obtained as in $\S^3$ by considering $X = \mathbb{R}^3$., $Y = L \cup S$ is not reduced and it coincides with that defined above by the relation $x = yxy^{-1}$. Add one point ∞ to $\mathbb{R}^3$ and define $X = S^3$ and $Y = L \cup \{\infty\} \cup S$; the resulting conjugation scheme is now reduced and it coincides with the one

given by the relations $x = yxy^{-1}$, $y = xyx^{-1}$. The situation of the Zariski-van Kampen presentation is analogous: one has braid relations of the type

$$b: (x_1, \ldots, x_n) \to (T_1 x_{\sigma(1)} T_1^{-1}, \ldots, T_n x_{\sigma(n)} T_n^{-1})$$

where σ is a permutation of $\{1, \ldots, n\}$ and $\prod_1^n x_i = \prod_1^n T_i x_{\sigma(i)} T_i^{-1}$; one knows that the last relation $x_n = T_n x_{\sigma(n)} T_n^{-1}$ is always inessential to give a presentation for the fundamental group, since it is a "consequence" of the others. This will not be true for presentations of conjugation schemes, as one can see by modifying the example above: let b be the braid induced by a generic double point in $\mathbb{C}^2$, i.e. $b: (x,y) \to (y^{-1}xy, y^{-1}x^{-1}yxy)$; if one consider only the first relation $x = y^{-1}xy$ we have seen that the associated conjugation scheme is not reduced; on the other hand it is easy to verify that by adding the second relation one gets a reduced scheme.

We do not know any example of not reducedness of the Zariski-Van Kampen presentation. One verifies that it is reduced in the case of the cusp by an argument that seems to apply to the case of the discriminant of the singularity x^n for all n. I would hope that in fact this is always true for any analytic hypersurface Δ in $\mathbb{C} \times \mathbb{C}^n$ as in §4. In fact one could made the following stronger conjucture that we state for $\Delta = \{(x,y) \in \mathbb{C}^2 \mid y^2 = x^3\}$: let $D = \{z \in D \mid z < 1\}$ and let $\sigma_i : D \to \mathbb{C}^2$ be a C^∞ map, transversal to Δ with only positive intersections, $i = 1,2$; suppose that $\sigma_1|_{\partial D}$ is free homotopic to $\sigma_2|_{\partial D}$ in $\mathbb{C}^2 - \Delta$;

then σ_1 can be deformed to σ_2 through some σ_t that remains transversal to Δ. Actually one could extend this conjecture to the case that Δ is either the discriminant or the bifurcation set for simple singularities.

6. π_2 of complement of hypersurfaces.

Consider again the situation of §4. The homotopy sequence of the given fibration contains an isomorphism

$$\pi_2(\mathbb{C}^{n+1} - (\Delta \quad \tilde{\Gamma})) \xrightarrow{\sim} \pi_2(\mathbb{C}^n - \Gamma)$$

(it is injective since $\pi_2(L - \Delta) = 0$ and surjective since the fibration has a section). Since the map $\mathbb{C}^n - \Gamma \to \mathbb{C}^{n+1} - \Delta$ is null homotopic, it follows that $\pi_2(\mathbb{C}^{n+1} - (\Delta \cup \tilde{\Gamma})) \to \pi_2(\mathbb{C}^{n+1} - \Delta)$ is the zero honomorphism. This implies, by theorem 4, that $\pi_2(\mathbb{C}^{n+1} - \Delta)$ coincides with $\mathcal{O}(T)$ where T is the conjugation scheme over $\pi_1(\mathbb{C}^{n+1} - (\Delta \cup \tilde{\Gamma}))$ made with arcs from the base point to $\tilde{\Gamma}$. From this, one deduces the following.

<u>Theorem 6</u> The diagram (+) of §4 determines $\pi_2(\mathbb{C}^{n+1} - \Delta)$.

Here "determines" means one has a "way to compute"; but obviously this computability is only theoretical: this can be understood easily by thinking of the difficulties one gets by trying to compute actually in the case of the presentation of the fundamental group by the Zariski-Van Kampen procedure. What we want to show now is some algebraic procedure to "extimate" somehow the second homotopy group. This can be applied in concrete situations to show the non vanishing of the second homotopy group.

First of all we reduce the conjugation scheme by taking its image in $W = \pi_1(\mathbb{C}^{n+1} - (\Delta \cup \tilde{\Gamma}))$. This group is the semi-direct product of $G = \pi_1(\mathbb{C}^n - \Gamma)$ with the free group $H = \pi_1(L - \Delta)$ through the natural homomorphism $\pi_1(\mathbb{C}^n - \Gamma) \to B(d)$ = braid group in d strings. Consider the conjugation system S in G, made of loops around simple points of Γ; so the elements of S are the ones one considers to find relations for $\pi_1(\mathbb{C}^{n+1} - \Delta)$. Through the section of the fibration, S can be embedded in W; Now T is just the set of conjugated elements in W to elements of S. By identifying W as a set to $H \times G$, we find that $T = \{(x^{-1} \cdot S(x), S) \mid s \in S,\ x \in H\}$; i.e. the elements in T are "fundamental relations" for $\pi_1(\mathbb{C}^{n+1} - \Delta)$(the first entry $x^{-1} \cdot s(x)$) but keeping also where this relation comes from (the second entry s).

Let us consider an explicit example. Consider $\Delta = \{(w,z) \in \mathbb{C}^2 \mid w^2 = z^2 - 1\}$. Here Γ consists of two points $1, -1$ in $\mathbb{C}$ so that G = free group in two letters say a,b. Also H = free group in two letters say x,y and the homomorphism $G \to B(2)$ = braid group in two strings is given by the map that sends both a and b to the generator of $B(2)$. The associated conjugation system can be shown to be reduced. Consider the element q induced in $\mathcal{O}(T)$ by the sequence $((x^{-1}y,a),(1,a^{-1}),(1,b),(x^{-1}y,b^{-1})) \in T^4$. One can prove the following.

<u>Theorem 7</u> $\mathcal{O}(T)$ is a free module of rank 1 on the group ring $Z[\pi_1(\mathbb{C}^{n+1} - \Delta)] = Z[Z]$ and as such is generated by q. The

proof is a topological one: choose a point p_0 in $\mathbb{C}^2$ near the origin. then $\mathbb{C}^2 - \Delta$ can be described by studing the levels of the distant function from p_0 (this is "relative" Morse theor; and one discovers that $\mathbb{C}^2 - \Delta$ is homotopically constructed by adding to a point two cells of dimension 1 and 2; the attaching map of the 2-dimensional cell being homotopically trival, $\mathbb{C}^2 - \Delta$ has the homotopy type of the one point union of S^1 and S^2. A carefull (but elementary) analysis, show that q is identified with the 2-cell. It would be hard, I believe, to find this result by algebraic computations on the conjugation scheme. But if one is interested only in the result that $\pi_2(\mathbb{C}^2 - \Delta)$ is not vanishing, one can use the following procedure that we describe first in a general frame: let be given a semidirect product $1 \to H \xrightarrow{j} W \underset{\beta}{\overset{\alpha}{\rightleftarrows}} G \to 1$ and let $S \subset G$ be a conjugation system. As before one can consider the conjugation system T generated in W by $\beta(S)$; namely $V = \{(x^{-1}\cdot s(x), s) \mid x \in H,\ s \in S\}$. Let χ: $G \to \mathrm{Aut}\,(H)$ be the homomorphism defining the semidirect product and let $H' \subset H$ and $G' \subset G$ be normal subgroups s. th.

i) $g(H') \subset H'$ for all $g \in G$

ii) $x^{-1}\cdot g'(x) \in H'$ for all $x \in H$ and $g' \in G'$.

then χ induces a homomorphism $\overline{\chi}$: $G/G' \to \mathrm{Aut}\,(H/H')$, so that one gets another semiproduct $1 \to H/H' \to \overline{W} \rightleftarrows G/G' \to 1$ which is a "homomorphic image" of the original one. Also one gets a homomorphism $\mathcal{O}(T) \to \mathcal{O}(\overline{T})$, where $\overline{T}$ is the image of T in $\overline{W}$.

Roughly speaking, this procedure can be applied to the semidirect product (+) by considering "permutations" instead of "braids"; more precisely one may choose for H' the commutator subgroup $[H,H]$, so that $H/_{H'} \simeq Z^d$ and for G', the normal subgroup generated by $\{s^2 \mid s \in S\}$. Or even one may reduce the free group H to be $(Z/_{2Z})^d$ and to enlarge G' by introducing some commutativity relation between the generators of G (namely between the $s \in S$). Computations are greatly simplified and the "obstructions" one gets are still strong in many cases. For example one can easily show that for $\Delta = \{(w,z) \in \mathbb{C}^2 \mid w^2 = (z^2-1)^{2n+1}\}$, the group $\pi_2(\mathbb{C}^2 - \Delta)$ is non zero, and in fact, for example for $n = 0$, that the element q described before is non zero in $\mathcal{O}(T)$ - Also it is easy to show that for Δ a set of n lines in general position in $\mathbb{C}^2$, $\pi_2(\mathbb{C}^2 - \Delta)$ is non zero for $n > 3$. The reason of these non vanishing is always the non injectivity of $\pi_1(\mathbb{C}^n - \Gamma) \to B(d)$; now there is some reason to believe that if Γ is the bifurcation set associated with the semiuniversal deformation of an isolated singularity of hypersurface type, then $\pi_1(\mathbb{C}^n - \Gamma) \to B(d)$ is not injective if the singularity is not a simple one. This seems to suggest that the $K(\pi,1)$ conjecture for non simple singularities may be not true, due probably to the non vanishing of the second homotopy group.

7. Some comment.

The situation described above, would be greatly simplified if one would know a Lüroth-Clebsche type condition to be true for the braid group, namely the conjecture noted in §5. In any case, as it has been just remarked, one could get only threoretical results, computations in concrete cases being too difficult; so one can think of this theory as one that can give only "negative results", i.e. it can be applied in some occasion to show the non vanishing of the second homotopy group - Typical elements in $\mathcal{O}(T)$ are as the q describeb in §6 : one starts with $s_1, s_2 \in S$ and $x \in H$, s.th. $s_1(x) = s_2(x)$; consider in T^4 the element $((x^{-1}\cdot s_1(x),s_1),(1,s_1^{-1}),(1,s_2)(x^{-1}\cdot s_2^{-1}(x),s_2))$: then one may try to show that this is net zero by reducing the semiproduct as in §6.

On the other hand, conjugation schemes seems to be a very nátural object ; as we have seen, in many occasion, one may use topology to get algebraic results about them. For example let $\Gamma_0 \cup .. \cup \Gamma_n$ be a link in S^3 ; consider $G = \pi_1(S^3-(\Gamma_0 \ .. \ \Gamma_n))$ and let T be the conjugation system in G, given by simple loops around Γ_0 . If one knows that $\pi_2(S^3-(\Gamma_0 \cup .. \cup \Gamma_n)) = 0$, one may deduce that $\mathcal{O}(T) = 0$. By choosing $n=1$ and Γ_1 to be a fibre of the S^1-normal bundle of Γ_0 in S^3, by asphericity of knots one gets a statement about the group of a knot .