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New exponents and Betti numbers of complement of hyperplanes

Hiroaki TERAO*

(International Christian University)

§0. Introduction

The aim of this article is to report the results in [8][9][10] and to give the outlines of their proofs. For further details see the original papers.

We define an n-arrangement as a finite family of hyperplanes through the origin O in $\mathbb{C}^{n+1}$. Let X be an n-arrangement. By $|X|$ denote we the union of all hyperplanes belonging to X. Our subject here is the Poincaré polynomial $P_M(t)$ of $M = \mathbb{C}^{n+1} \setminus |X|$. Let $Q \in \mathbb{C}[z_0, \ldots, z_n]$ be a defining equation of $|X|$.

(0.1) Definition. We say that X is free if

$$D(X) := \{\text{germ } \theta \text{ at } O \text{ of holomorphic vector field such that } \theta \cdot Q \in Q \cdot \mathcal{O}\}$$

is a free $\mathcal{O}$-module, where $\mathcal{O} = \mathcal{O}_{\mathbb{C}^{n+1}, O}$.

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A germ θ of holomorphic vector field at 0 is said to be homogeneous of degree d, denoted by $\deg\theta = d$, if θ has a local expression

$$\theta = \sum_{i=0}^{n} f_i \frac{\partial}{\partial z_i}$$

at the origin such that all f_i's are homogeneous polynomials and all non-zero f_i's have the same degree d. A little observation leads us to the existence of a system of homogeneous free basis $\{\theta_0,\dots,\theta_n\}$ for D(X) if X is a free n-arrangement. It is easy to see that the set $\{\deg\theta_0,\dots,\deg\theta_n\}$ of non-negative integers depends only on X.

(0.2) Definition. We call $(\deg\theta_0,\dots,\deg\theta_n)$ the exponents of a free n-arrangement X.

Let $(d_0,\dots,d_n)$ be the exponents of a free n-arrangement X. Then our main result here is:

Main Theorem. $P_M(t) = \prod_{i=0}^{n}(1+d_i t)$.

Let $G \subset GL(n+1;\mathbb{C})$ be a finite unitary reflection groups acting on $\mathbb{C}^{n+1}$. Then the set of the reflecting hyperplanes of the unitary reflections in G makes an n-arrangement X. Such an arrangement is called a unitary reflection arrangement. Then we can prove that X is free. Moreover its exponents coincide with the exponents of G which were recently introduced by Orlik-Solomon ([3]). In this special case our Main Theorem is nothing other than the main result in [3]. For details see [10].

Especially when G is real, our Main Theorem was first proved by Brieskorn ([1] Theorem 6(ii)).

Remark. The class of the free arrangements is far wider than that of the unitary reflection arrangements. In fact many examples suggest that the freeness of arrangement is a combinatorial property ([6]).

In Sect. 1, we study an n-arrangement by a combinatorial method. Our main tool for it is the Möbius function on the lattice associated with the n-arrangement. We shall geve a characterization of the Möbius function (1.5). For this purpose we need a notion called i-cumulativeness which plays a main role in the proof of Main Theorem. At the end of Sect. 1, we state Proposition A concerning the cumulativeness of product of Möbius functions.

In Sect. 2, we try to compute the Hilbert polynomial $H(\mathcal{O}/J(X);\nu)$, where $J(X)$ stands for the Jacobian ideal of the defining equation Q of $|X|$. Assume that X is a free n-arrangement. Then we have an explicit formula (2.9) for $H(\mathcal{O}/J(X);\nu)$ by using the exponents of

X. This formula and Proposition B in Sect. 2, which asserts the cumulativeness of the coefficients of $H(\theta/J(X);\nu)$, lead us to the proof of Main Theorem which is in Sect. 3.

Our key results for the proof are a characterization of the Möbius function (1.5), Proposition A, B and the explicit formula (2.9) for $H(\theta/J(X);\nu)$.

Let X be a finite family of hyperplanes in $\mathbb{C}^{n+1}$ or $\mathbb{P}^{n+1}(\mathbb{C})$. The intersection of all hyperplanes belonging to X may be void. We can define the notion of the freeness for X also in this case. Moreover we can define the exponents of X if X is free and prove that

$$P_M(t) = \prod_{i=0}^{n}(1+d_i t).$$

($M = \mathbb{C}^{n+1} \setminus \bigcup_{H\in X} H$ or $\mathbb{P}^{n+1}(\mathbb{C}) \setminus \bigcup_{H\in X} H$ and $(d_0,\ldots,d_n)$ are the exponents of X.) This gives a generalization of Main Theorem. For the full explanation on this generalization, see [9].

§1. Combinatorial study of an n-arrangement

Let X be an n-arrangement in this section.

(1.1) Definition. Let

$$L(X) := \left\{ \bigcap_{H \in A} H;\ A \subset X \right\},$$

where we interpret that

$$\mathbb{C}^{n+1} = \bigcap_{H \in \phi} H.$$

Define the join and meet operations in L(X) by

$$s \vee t = s \cap t,$$

and $s \wedge t = \bigcap H$ (H runs over a set $\{L \in X;\ L \supset s \cup t\}$) for $s, t \in L(X)$.

Then L(X) becomes a lattice which is called the lattice associated with an n-arrangement X.

Write $s \prec t$ if $s \vee t = t$ $(s, t \in L(X))$.

(1.2) Definition. Define the Möbius function μ on L(X) inductively defined by

$$\mu(\mathbb{C}^{n+1}) = 1$$

$$\mu(s) = -\sum_{\substack{t \prec s \\ t \neq s}} \mu(t).$$

(1.3) Definition. The rank of $s \in L(X)$, denoted by $r(s)$, is the length of the longest chain in $L(X)$ below s. Thus

$$r(s) = \operatorname{codim}_{\mathbb{C}^{n+1}} s.$$

For any integer $i \geq 0$, put

$$\mu_i(L(X)) := \sum_{\substack{s \in L(X) \\ r(s)=i}} |\mu(s)| .$$

For any $s \in L(X)$, define a new n-arrangement

$$X_s := \{H \in X;\ s \subset H\}.$$

Put $\mathcal{A}(X) := \{X_s;\ s \in L(X)\}$. Consider the mappings

$$\mu_i \circ L : \mathcal{A}(X) \longrightarrow \mathbb{Z} \quad (i \geq 0)$$

corresponding $Y \in \mathcal{A}(X)$ to $\mu_i(L(Y))$.

We will give a characterization of these mappings $\mu_i \circ L$ $(i \geq 0)$. For this purpose we need

(1.4) Definition. For a mapping

$$q : \mathcal{A}(X) \longrightarrow \mathbb{Z},$$

define a new mapping

$$r_i q : \mathcal{A}(X) \longrightarrow \mathbb{Z}.$$

by $$(r_i q)(Y) = q(Y) - \sum_{\substack{s \in L(Y) \\ r(s)=i}} q(Y_s)$$

for any $Y \in \mathcal{A}(X)$ and any integer $i \geq 0$. Denote $r_i r_{i-1} \cdots r_0 q$ by $R_i q$.

We say that q is i-<u>cumulative</u> ($i \geq 0$) on X if

$$(R_i q)(X) = 0.$$

(1.5) <u>Theorem</u>. (A characterization of $\mu_i \circ L$ ($i \geq 0$))
Assume that the mappings

$$q_j : \mathcal{A}(X) \longrightarrow \mathbb{Z} \quad (j = 0,1,2,\ldots)$$

satisfy the following conditions:

I. $q_0(\phi) = 1$.

II. $q_j(X_s) = 0$ if $s \in L(X)$ and $r(s) < j$ ($j \geq 0$).

III. The alternating sum of $q_j(Y)$ ($j = 0,1,2,\ldots$) is zero if $Y \in \mathcal{A}(X) \setminus \{\phi\}$.

IV. q_j is j-cumulative on any $Y \in \mathcal{A}(X)$ ($j = 0,\ldots,i$).

Then $q_j = \mu_j \circ L$ ($j = 0,\ldots,i$) on $\mathcal{A}(X)$.

<u>Proof</u>. see [8].

Define the mappings

$$q_j : \mathcal{A}(X) \longrightarrow \mathbb{Z} \quad (j \geq 0)$$

by $\quad q_j(Y) = b_j(\mathbb{C}^{n+1} \setminus |Y|) \quad (Y \in \mathcal{A}(X)),$

where the right handside stands for the j-th Betti number of $\mathbb{C}^{n+1} \setminus |Y|$. Then it is not too difficult to show that the conditions I-IV in (1.5) hold true for any $i \geq 0$ (cf. [1] Lemma 3). Thus we have

(1.6) Theorem. For any n-arrangement, we have

$$b_j(\mathbb{C}^{n+1} \setminus |X|) = \mu_j \circ L(X) \quad (j = 0,1,2,\ldots).$$

This theorem was first proved by Orlik-Solomon [2].

Let X be a finite family of hyperplanes in $\mathbb{C}^{n+1}$ or $\mathbb{P}^{n+1}(\mathbb{C})$. The intersection of all hyperplaens belonging to X may be void. Put

$$M = \mathbb{C}^{n+1} \setminus \bigcup_{H \in X} H \quad \text{or} \quad \mathbb{P}^{n+1}(\mathbb{C}) \setminus \bigcup_{H \in X} H.$$

We have a formula for $P_M(t)$ by using the Möbius functions also in this case. For further details of this generalization, see [9].

Assume that $Q \in \mathbb{R}[z_0,\ldots,z_n]$, a product of real linear forms, is a defining equation of a free n-arrangement X. By combining Main Theorem with (1.6) and the Zaslavsky's result ([11] p. 18 Theorem A), we have

$$\#\left\{\text{connected component of } \mathbb{R}^{n+1} \setminus \{Q = 0\}\right\}$$
$$= \sum_{i=0}^{n+1} b_i(\mathbb{C}^{n+1} \setminus |X|) = \prod_{i=0}^{n}(1+d_i).$$

This equality was proved when $n = 2$ in [7]. K. Saito proved

$$\#\left\{\text{connected component of } \mathbb{R}^{n+1} \setminus \{Q = 0\}\right\} \leq \prod_{i=0}^{n}(1+d_i)$$

in [4].

For an arbitrary multi-index $I = (I(1),\ldots,I(k))$ composing of k non-negative integers, define

$$\mu_I \circ L : \mathcal{A}(X) \longrightarrow \mathbb{Z}$$

by $\mu_I \circ L(Y) = \prod_{j=1}^{k} \mu_{I(j)} \circ L(Y)$. Define $|I| = \sum_{j=1}^{k} I(j)$.

One reason why the notion of i-cumulativeness plays an important role in our theory is the following

<u>Proposition</u> A. $\mu_I \circ L$ is $|I|$-cumulative.

The proof, which is omitted here, is purely combinatorial (see [8]).

§2. The Hilbert polynomial of $\mathcal{O}/J(X)$

From now on we denote $\mathcal{O}_{\mathbb{C}^{n+1},0}$ simply by $\mathcal{O}$.

Let Q be a defining equation of $|X|$. By ∂Q denote we the Jacobian ideal of Q in $\mathcal{O}$ (i.e., $\partial Q = (\partial Q/\partial z_0,\dots,\partial Q/\partial z_n)\mathcal{O}$). Then ∂Q depends only on X. Define the Jacobian ideal $J(X)$ of X by

$$J(X) = \begin{cases} \partial Q & \text{if } X \neq \phi \\ \mathcal{O} & \text{if } X = \phi . \end{cases}$$

(2.1) <u>Definition</u>. Introduce a decreasing filtration

$$(\mathcal{O}^k)_m = \underbrace{\mathfrak{m}^m \oplus \cdots \oplus \mathfrak{m}^m}_{k} \quad (m \geq 0)$$

on an $\mathcal{O}$-module $\mathcal{O}^k$ $(k > 0)$. Then this filtration $((\mathcal{O}^k)_m)_{m\geq 0}$ makes $\mathcal{O}^k$ to be an $\mathfrak{m}$-bonne filtered $\mathcal{O}$-module (see [5]).

By the natural projection $\mathcal{O} \longrightarrow \mathcal{O}/J(X)$, we can introduce an $\mathfrak{m}$-bonne filtration on $\mathcal{O}/J(X)$.

On the other hand, $D(X)$ can be embedded in $\mathcal{O}^{n+1}$ by the correspondence

$$\sum_{i=0}^{n} f_i(\partial/\partial z_i) \longmapsto (f_0,\dots,f_n) \quad (f_i \in \mathcal{O}\ (i = 0,\dots,n)).$$

Denote this mapping by $\alpha : D(X) \longrightarrow \mathcal{O}^{n+1}$. So one can induce an $\mathfrak{m}$-bonne filtration on $D(X)$.

From now on we regard $\mathcal{O}^{n+1}$, $\mathcal{O}$, $\mathcal{O}/J(X)$ and $D(X)$ as

$\mathfrak{m}$-bonne filtered $\mathcal{O}$-modules in the above manners.

(2.2) <u>Definition</u>. Let $M = (M_n)_{n\geq 0}$ be an $\mathfrak{m}$-bonne (decreasingly) filtered $\mathcal{O}$-module. A polynomial $H(M;\nu)$ is characterized by the property that:

$H(M;\nu) \in \mathbb{Q}[\nu]$ equals the dimension of $\mathcal{O}/\mathfrak{m} \simeq \mathbb{C}$-vector space $M_\nu/M_{\nu+1}$ for sufficiently large ν.

We call $H(M;\nu)$ <u>the Hilbert polynomial of</u> $M = (M_n)_{n\geq 0}$.

(2.3) <u>Definition</u>. Let $M = (M_n)_{n\geq 0}$ be a filtered $\mathcal{O}$-module. Then $M(k) = (M(k)_n)_{n\geq 0}$ is another filtered $\mathcal{O}$-module defined by $M(k)_n = M_{k+n}$ for $k\in\mathbb{Z}$, $k\geq 0$. Then it is easy to see that

$$H(M(k);\nu) = H(M;k+\nu)$$

for $k\in\mathbb{Z}$, $k\geq 0$.

Let $m = \# X = \deg Q$. Then we have an exact sequence

$$\text{(2.4)}\quad 0 \longrightarrow D(X) \xrightarrow{\alpha} \mathcal{O}^{n+1} \xrightarrow{\beta} (\mathcal{O}/Q\cdot\mathcal{O})(m-1) \xrightarrow{\gamma} (\mathcal{O}/J(X))(m-1) \longrightarrow 0,$$

where

$$\beta(f_0,\ldots,f_n) = \sum_{i=0}^{n} f_i(\partial Q/\partial z_i) \quad (f_i\in\mathcal{O}\ (i = 0,\ldots,n))$$

and γ is the natural projection. Each mapping above is strictly compatible with each filtration. Thus we have

$$
\begin{aligned}
&H(\mathcal{O}/J(X);\nu+m-1) \\
&= H(\mathcal{O}/Q\cdot\mathcal{O};\nu+m-1) - H(\mathcal{O}^{n+1};\nu) + H(D(X);\nu).
\end{aligned}
$$

For our convenience, put

$$f^{(m)} = \frac{(f+1)\cdots(f+m)}{m} \text{ and } f^{(0)} = 1$$

for any polynomial f and $m > 0$. Then

$$H(\mathcal{O};\nu) = \nu^{(n)},$$

and thus

$$H(\mathcal{O}^{n+1};\ ) = (n+1)\nu^{(n)}.$$

It is easy to see that

$$
\begin{aligned}
&H(\mathcal{O}/Q\cdot\mathcal{O};\nu+m-1) \\
&= (\nu+m-1)^{(n)} - (\nu-1)^{(n)} \\
&= m\cdot\nu^{(n-1)} + \sum_{i=2}^{n}\binom{m+i-2}{i}\nu^{(n-i)}.
\end{aligned}
$$

Let X be free with its exponents $(d_0,\ldots,d_n)$ throughout this section. Then we have

$$H(D(X);\nu) = \sum_{i=0}^{n} (\nu - d_i)^{(n)},$$

and thus

(2.5) $H(\mathcal{O}/J(X);\nu+m-1)$

$$= m\cdot\nu^{(n-1)} + \sum_{i=2}^{n}\binom{m+i-2}{i}\cdot\nu^{(n-i)} - (n+1)\cdot\nu^{(n)} + \sum_{i=0}^{n}(\nu-d_i)^{(n)}$$

$$= \left(m-\sum_{i=0}^{n} d_i\right)\cdot\nu^{(n-1)} + \sum_{i=2}^{n}\left\{\binom{m+i-2}{i} + (-1)^i\sum_{j=0}^{n}\binom{d_j}{j}\right\}\cdot\nu^{(n-i)}.$$

On the other hand we know that

$$\deg H(\mathcal{O}/J(X);\nu) = \deg(\mathcal{O}/\partial Q;\nu) = \dim \mathrm{Spec}(\mathcal{O}/\partial Q)-1 \leq n-2$$

if $X \neq \phi$. If $X = \phi$, then

$$H(\mathcal{O}/J(X);\nu) = 0.$$

Thus we have proved

(2.6) <u>Proposition</u>. $m = \sum_{i=0}^{n} d_i$.

Define $P_i(X)$ $(i = 2,\ldots,n) \in \mathbb{Z}$ by

$$H(\mathcal{O}/J(X);\nu) = \sum_{i=2}^{n} P_i(X)\cdot\nu^{(n-i)}.$$

Then we can explicitly compute

(2.7) $P_i(X)$

$$= \sum_{j=0}^{i-2}\left\{(-1)^j\binom{d_0+\cdots+d_n+i-j-2}{i-j} + (-1)^i\sum_{k=0}^{n}\binom{d_k}{i-j}\right\}\cdot\binom{d_0+\cdots+d_n-1}{j}$$

because of (2.5) and (2.6).

(2.8) <u>Definition</u>. Let $k \geq 1$. Let $I = (I(1),\ldots,I(k))$ be a multi-index composing of k non-negative integers. Define

$$\sigma_I(X) = \prod_{i=1}^{k}\sigma_{I(i)}(d_0,\ldots,d_n),$$

where $\sigma_j \in \mathbb{C}[t_0,\ldots,t_n]$ ($j \geq 0$) is the elementary symmetric polynomial of degree j. When $k = 1$, we write $\sigma_j(X)$ instead of $\sigma_{(j)}(X)$ ($j \geq 0$). Thus (2.6) asserts that $\#X = \sigma_1(X)$.

The following key lemma is not difficult to be verified:

(2.9) <u>Lemma</u>. For each integer i ($2 \leq i \leq n$), there exist real numbers $c(I;i)$ ($I \in I[i]$), which are independent of X, such that

$$P_i(X) + \frac{1}{(i-1)!}\sigma_i(X) = \sum_{I\in I[i]} c(I;i)\sigma_I(X).$$

Here

$$I[i] := \left\{ I = (I(1),\dots,I(k));\ 0 \leq I(j) < i\ (j = 1,\dots,k),\ \sum_{j=1}^{k} I(j) \leq i \right\}.$$

Since X is free, any element in $\mathcal{A}(X)$ is also free (see[8] (5.5)). Thus we can define the mappings

$$\begin{array}{ccccc} P_j & : \mathcal{A}(X) & \longrightarrow & \mathbb{Z} & (2 \leq j \leq n) \\ & \Psi & & \Psi & \\ & Y & \longmapsto & P_j(Y). & \end{array}$$

The following is the most important proposition for the proof of Main Theorem:

<u>Proposition</u> B. P_j is j-cumulative $(2 \leq j \leq n)$.

Our proof is difficult and long. See [8](5.10).

§3. Proof of Main Theorem

In this section we shall prove Main Theorem. The crucial results for our proof are (1.5), Proposition A (§1), Proposition B (§2) and (2.9).

The following is stronger than Main Theorem:

(3.1) Theorem. Let $i \geq 0$. Then we have

$1)_i$ $\sigma_i(X) = \mu_i \circ L(X)$ for any free n-arrangement X,

$2)_i$ $\sigma_i : \mathcal{A}(X) \longrightarrow \mathbb{Z}$ is i-cumulative for any free n-arrangement X.

Proof. When $i \leq 1$, we can verify $1)_i$ and $2)_i$ because of (2.6).

Let $i \geq 2$. Assume that $1)_j$ $(j = 0,1,\ldots,i-1)$ hold true. Let X be a free n-arrangement. Recall (2.9), then we have

$$P_i(X) + \frac{1}{(i-1)!}\,{}_i(X) = \sum_{I \in I[i]} c(I;i)(\mu_I \circ L)(X).$$

By Proposition A, we know that $\mu_I \circ L$ is $|I|$-cumulative. Since $|I| \leq i$ for $I \in I[i]$, we can see that $\mu_I \circ L$ is i-cumulative. Thus we have the i-cumulativeness of μ_i because the sum of two i-cumulative mappings is also i-cumulative. This is $2)_i$.

Next assume $2)_j$ $(j = 0,1,\ldots,i)$. Let X be a free n-arrangement. Then the assumption implies that the mappings

$$\sigma_j : \mathcal{A}(X) \longrightarrow \mathbb{Z} \quad (j \geq 0)$$

satisfy the condition IV in (1.5). Moreover it is not too difficult to see that the mappings σ_j $(j \geq 0)$ also satisfy the conditions I, II and III in (1.5). Thus we can apply (1.5) and have

$$\sigma_i = \mu_i \circ L$$

on $\mathcal{A}(X)$. This is $1)_i$. Q.E.D.

(3.2) The observation so far shows that the following four data concerning a free n-arrangement X are equivalent:

(1) The set of the exponents $(d_0, \dots, d_n)$ of X, which is equivalent to the polynomial

$$\sum_{i=0}^{n} \sigma_i(X) t^i = \prod_{i=0}^{n} (1+d_i t),$$

(2) The Hilbert polynomial $H(\mathcal{O}/J(X);\nu)$ together with $\#X$, which is equivalent to the data

$$(\#X,\ P_2(X), \dots, P_n(X)),$$

(3) The polynomial $\sum_{i=0}^{n} (\mu_i \circ L(X)) t^i$,

(4) The Poincaré polynomial of $M = \mathbb{C}^{n+1} \setminus |X|$, which is equivalent to the data

$$(b_0(M), b_1(M), \ldots, b_{n+1}(M)).$$

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