New Exponents and Betti Numbers of Complement of Hyperplanes (Complex Analysis of Singularities)

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New exponents and Betti numbers
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§0. Introduction

The aim of this article is to report the results in
[8][9][10] and to give the outlines of their proofs.
For further details see the original papers.

We define an $n$-arrangement as a finite family of
hyperplanes through the origin $O$ in $\mathbb{C}^{n+1}$. Let $X$ be an
$n$-arrangement. By $|X|$ denote we the union of all hyper-
planes belonging to $X$. Our subject here is the Poincaré
polynomial $P_M(t)$ of $M = \mathbb{C}^{n+1}\setminus|X|$. Let $Q \in \mathbb{C}[z_0, \ldots, z_n]$ be a defining equation of $|X|$.

(0.1) Definition. We say that $X$ is free if

$$D(X) = \left\{ \text{germ } \theta \text{ at } O \text{ of holomorphic vector field such that } \theta \cdot Q \in Q \cdot O \right\}$$

is a free $\mathcal{O}$-module, where $\mathcal{O} = \mathcal{O}_{\mathbb{C}^{n+1}, O}$.

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A germ $\theta$ of holomorphic vector field at $0$ is said to be homogeneous of degree $d$, denoted by $\deg \theta = d$, if $\theta$ has a local expression

$$\theta = \sum_{i=0}^{n} f_i \frac{\partial}{\partial x_i}$$

at the origin such that all $f_i$'s are homogeneous polynomials and all non-zero $f_i$'s have the same degree $d$. A little observation leads us to the existence of a system of homogeneous free basis $\{\theta_0, \ldots, \theta_n\}$ for $D(X)$ if $X$ is a free $n$-arrangement. It is easy to see that the set $\{\deg \theta_0, \ldots, \deg \theta_n\}$ of non-negative integers depends only on $X$.

(0.2) **Definition.** We call $(\deg \theta_0, \ldots, \deg \theta_n)$ the **exponents** of a free $n$-arrangement $X$.

Let $(d_0, \ldots, d_n)$ be the exponents of a free $n$-arrangement $X$. Then our main result here is:

**Main Theorem.** $P_M(t) = \prod_{i=0}^{n} (1+d_i t)$.

Let $G \subset \text{GL}(n+1; \mathbb{C})$ be a finite unitary reflection groups acting on $\mathbb{C}^{n+1}$. Then the set of the reflecting hyperplanes of the unitary reflections in $G$ makes an $n$-arrangement $X$. Such an arrangement is called a **unitary reflection arrangement**. Then we can prove that $X$ is free. Moreover its exponents coincide with the exponents of $G$.
which were recently introduced by Orlik-Solomon ([3]).
In this special case our Main Theorem is nothing other
than the main result in [3]. For details see [10].

Especially when G is real, our Main Theorem was
first proved by Brieskorn ([1] Theorem 6(ii)).

Remark. The class of the free arrangements is far wider
than that of the unitary reflection arrangements. In
fact many examples suggest that the freeness of arrange-
ment is a combinatorial property ([6]).

In Sect. 1, we study an n-arrangement by a
combinatorial method. Our main tool for it is the
Möbius function on the lattice associated with the n-
arrangement. We shall give a characterization of the
Möbius function (1.5). For this purpose we need a
notion called i-cumulativeness which plays a main role
in the proof of Main Theorem. At the end of Sect. 1,
we state Proposition A concerning the cumulativeness
of product of Möbius functions.

In Sect. 2, we try to compute the Hilbert poly-
nomial H(Θ/J(X); ν), where J(X) stands for the Jacobian
ideal of the defining equation Q of |X|. Assume that
X is a free n-arrangement. Then we have an explicit
formula (2.9) for H(Θ/J(X); ν) by using the exponents of
X. This formula and Proposition B in Sect. 2, which asserts the cumulativeness of the coefficients of \( H(\Theta/J(X); \nu) \), lead us to the proof of Main Theorem which is in Sect. 3.

Our key results for the proof are a characterization of the Möbius function (1.5), Proposition A, B and the explicit formula (2.9) for \( H(\Theta/J(X); \nu) \).

Let \( X \) be a finite family of hyperplanes in \( \mathbb{P}^{n+1} \) or \( \mathbb{P}^{n+1}(\mathbb{C}) \). The intersection of all hyperplanes belonging to \( X \) may be void. We can define the notion of the freeness for \( X \) also in this case. Moreover we can define the exponents of \( X \) if \( X \) is free and prove that

\[
\rho_M(t) = \prod_{i=0}^{n} (1+d_it).
\]

(\( M = \mathbb{P}^{n+1}\setminus \bigcup_{H \in X} H \) or \( \mathbb{P}^{n+1}(\mathbb{C})\setminus \bigcup_{H \in X} H \) and \( (d_0, \ldots, d_n) \) are the exponents of \( X \).) This gives a generalization of Main Theorem. For the full explanation on this generalization, see [9].
51. **Combinatorial study of an n-arrangement**

Let $X$ be an $n$-arrangement in this section.

(1.1) **Definition.** Let

$$L(X) := \{ \bigcap_{H \in A; A \subseteq X} H \},$$

where we interpret that

$$\mathbb{R}^{n+1} = \bigcap_{H \in \phi} H.$$

Define the join and meet operations in $L(X)$ by

- $s \vee t = s \wedge t$,
- and $s \wedge t = \bigcap H$ (H runs over a set \{L \in X; L \supseteq s \cup t\}) for $s, t \in L(X)$.

Then $L(X)$ becomes a lattice which is called the **lattice associated with an n-arrangement** $X$.

Write $s \triangledown t$ if $s \vee t = t$ ($s, t \in L(X)$).

(1.2) **Definition.** Define the **Möbius function** $\mu$ on $L(X)$ inductively defined by

$$\mu(\mathbb{R}^{n+1}) = 1$$

$$\mu(s) = -\sum_{t \triangledown s} \mu(t).$$
(1.3) Definition. The rank of $s \in L(X)$, denoted by $r(s)$, is the length of the longest chain in $L(X)$ below $s$. Thus

$$r(s) = \operatorname{codim} \mathcal{C}^{n+1}s.$$  

For any integer $i \geq 0$, put

$$\mu_i(L(X)) := \sum_{\substack{s \in L(X) \\ r(s) = i}} |\mu(s)|.$$  

For any $s \in L(X)$, define a new $n$-arrangement

$$X_s := \{ H \in X; s \subseteq H \}.$$  

Put $\mathcal{A}(X) := \{ X_s; s \in L(X) \}$. Consider the mappings

$$\mu_i \circ L : \mathcal{A}(X) \longrightarrow \mathbb{Z} \quad (i \geq 0)$$

corresponding $Y \in \mathcal{A}(X)$ to $\mu_i(L(Y))$.

We will give a characterization of these mappings $\mu_i \circ L \ (i \geq 0)$. For this purpose we need

(1.4) Definition. For a mapping

$$q : \mathcal{A}(X) \longrightarrow \mathbb{Z},$$

define a new mapping

$$r_1q : \mathcal{A}(X) \longrightarrow \mathbb{Z}.$$
by \( (r_i q)(Y) = q(Y) - \sum_{s \in L(Y)} q(Y_s) \)

for any \( Y \in \mathcal{A}(X) \) and any integer \( i \geq 0 \). Denote 
\[ r_i r_{i-1} \ldots r_0 q \]

by \( R_i q \).

We say that \( q \) is \( i \)-cumulative \( i \geq 0 \) on \( X \) if 
\[ (R_i q)(X) = 0. \]

(1.5) Theorem. (A characterization of \( \mu_{i^*}L \) \( i \geq 0 \))

Assume that the mappings 
\[ q_j : \mathcal{A}(X) \longrightarrow \mathbb{Z} \quad (j = 0, 1, 2, \ldots) \]

satisfy the following conditions:

I. \( q_0(\emptyset) = 1. \)

II. \( q_j(X_s) = 0 \) if \( s \in L(X) \) and \( r(s) < j \) \( j \geq 0 \).

III. The alternating sum of \( q_j(Y) \) \( (j = 0, 1, 2, \ldots) \)

is zero if \( Y \in \mathcal{A}(X) \setminus \{\emptyset\} \).

IV. \( q_j \) is \( j \)-cumulative on any \( Y \in \mathcal{A}(X) \) \( (j = 0, \ldots, i) \).

Then \( q_j = \mu_{j^*}L \) \( (j = 0, \ldots, i) \) on \( \mathcal{A}(X) \).

Proof. see \([3]\).

Define the mappings 
\[ q_j : \mathcal{A}(X) \longrightarrow \mathbb{Z} \quad (j \geq 0) \]
by

\[ q_j(Y) = b_j(\mathcal{G}^{n+1}_Y \setminus |Y|) \quad (Y \in \mathbb{A}(X)), \]

where the right hand side stands for the \(j\)-th Betti number of \(\mathcal{G}^{n+1}_Y \setminus |Y|\). Then it is not too difficult to show that the conditions I-IV in (1.5) hold true for any \(i \geq 0\) (cf. [1] Lemma 3). Thus we have

(1.6) Theorem. For any \(n\)-arrangement, we have

\[ b_j(\mathcal{G}^{n+1}_X \setminus |X|) = \mu_j^*(L) \quad (j = 0, 1, 2, \ldots). \]

This theorem was first proved by Orlik-Solomon [2].

Let \(X\) be a finite family of hyperplanes in \(\mathcal{G}^{n+1}\) or \(\mathbb{P}^{n+1}(\mathbb{C})\). The intersection of all hyperplanes belonging to \(X\) may be void. Put

\[ M = \mathcal{G}^{n+1}_X \setminus \bigcup_{H \in X} H \quad \text{or} \quad \mathbb{P}^{n+1}(\mathbb{C}) \setminus \bigcup_{H \in X} H. \]

We have a formula for \(P_n(t)\) by using the Möbius functions also in this case. For further details of this generalization, see [9].

Assume that \(Q \in \mathbb{P}[z_0, \ldots, z_n]\), a product of real linear forms, is a defining equation of a free \(n\)-arrangement \(X\). By combining Main Theorem with (1.6) and the Zaslavsky's result ([11] p. 10 Theorem A), we have
\#\{(\text{connected component of } \mathbb{R}^{n+1} \setminus \{Q = 0\})\} = \sum_{i=0}^{n+1} b_i(\mathbb{C}^{n+1} \setminus \{X\}) = \prod_{i=0}^{n} (1+d_i).

This equality was proved when \(n = 2\) in \([7]\). K. Saito proved
\[
\#\{(\text{connected component of } \mathbb{R}^{n+1} \setminus \{Q = 0\})\} \leq \prod_{i=0}^{n} (1+d_i)
\]
in \([4]\).

For an arbitrary multi-index \(I = (I(1), \ldots, I(k))\) composing of \(k\) non-negative integers, define
\[
\mu_I \cdot L : \mathfrak{A}(X) \to \mathbb{Z}
\]
by \(\mu_I \cdot L(Y) = \prod_{j=1}^{k} \mu_{I(j)} \cdot L(Y)\). Define \(|I| = \sum_{j=1}^{k} I(j)\).

One reason why the notion of \(i\)-cumulativeness plays an important role in our theory is the following

**Proposition A.** \(\mu_I \cdot L\) is \(|I|\)-cumulative.

The proof, which is omitted here, is purely combinatorial (see \([8]\)).
§2. The Hilbert polynomial of $\mathcal{O}/J(X)$

From now on we denote $\mathcal{O}_{\mathbb{P}^{n+1}, X}$ simply by $\mathcal{O}$.

Let $Q$ be a defining equation of $|X|$. By $\partial Q$ denote we the Jacobian ideal of $Q$ in $\mathcal{O}$ (i.e., $\partial Q = (\partial Q/\partial z_0, \ldots, \partial Q/\partial z_n)(\mathcal{O})$). Then $\partial Q$ depends only on $X$. Define the Jacobian ideal $J(X)$ of $X$ by

$$J(X) = \begin{cases} \partial Q & \text{if } X \neq \emptyset \\ \mathcal{O} & \text{if } X = \emptyset. \end{cases}$$

(2.1) Definition. Introduce a decreasing filtration

$$\mathcal{O}_k^m = \frac{\mathcal{O}^m}{\mathcal{O}^k} \quad (m \geq 0)$$

on an $\mathcal{O}$-module $\mathcal{O}_k^k$ ($k > 0$). Then this filtration $(\mathcal{O}_k^m)_{m \geq 0}$ makes $\mathcal{O}_k^k$ to be an $m$-bonne filtered $\mathcal{O}$-module (see [5]).

By the natural projection $\mathcal{O} \to \mathcal{O}/J(X)$, we can introduce an $m$-bonne filtration on $\mathcal{O}/J(X)$.

On the other hand, $D(X)$ can be embedded in $\mathcal{O}^{n+1}$ by the correspondence

$$\sum_{i=0}^{n} f_i(\partial Q/\partial z_i) \mapsto (f_0, \ldots, f_n) \quad (f_i \in \mathcal{O} \ (i = 0, \ldots, n)).$$

Denote this mapping by $\alpha: D(X) \to \mathcal{O}^{n+1}$. So one can induce an $m$-bonne filtration on $D(X)$.

From now on we regard $\mathcal{O}^{n+1}, \mathcal{O}, \mathcal{O}/J(X)$ and $D(X)$ as
\(m\)-bonne filtered \(\mathcal{O}\)-modules in the above manners.

(2.2) **Definition.** Let \(M = (M_n)_{n \geq 0}\) be an \(m\)-bonne (decreasingly) filtered \(\mathcal{O}\)-module. A polynomial \(H(M; \nu)\) is characterized by the property that:

\(H(M; \nu) \in \mathbb{Q}[\nu]\) equals the dimension of \(\mathcal{O}/M^{M_{\nu}}\) \(\mathbb{C}\)-vector space \(M^{M_{\nu}}\) for sufficiently large \(\nu\).

We call \(H(M; \nu)\) the Hilbert polynomial of \(M = (M_n)_{n \geq 0}\).

(2.3) **Definition.** Let \(M = (M_n)_{n \geq 0}\) be a filtered \(\mathcal{O}\)-module. Then \(M(k) = (M(k)_n)_{n \geq 0}\) is another \(\mathcal{O}\)-module defined by \(M(k)_n = M_{k+n}\) for \(k \in \mathbb{Z}, k \geq 0\). Then it is easy to see that

\[H(M(k); \nu) = H(M; k+\nu)\]

for \(k \in \mathbb{Z}, k \geq 0\).

Let \(m = \#X = \deg \mathcal{O}\). Then we have an exact sequence

(2.4) \(0 \rightarrow D(X) \xrightarrow{\alpha} \mathcal{O}^{n+1} \xrightarrow{\beta} (\mathcal{O}/\mathcal{O})(m-1) \xrightarrow{\gamma} (\mathcal{O}/(X))(m-1) \rightarrow 0\),

where

\[\beta(f_0, \ldots, f_n) = \sum_{i=0}^{n} f_i (\mathcal{O}/\mathcal{O}z_i)\quad (f_i \in \mathcal{O} (i = 0, \ldots, n))\]
and \( \iota \) is the natural projection. Each mapping above is strictly compatible with each filtration. Thus we have

\[
H(\mathcal{O}/J(X); \nu^{m-1}) = H(\mathcal{O}/\mathcal{O}; \nu^{m-1}) - H(\mathcal{O}^{n+1}; \nu) + H(D(X); \nu).
\]

For our convenience, put

\[
f(m) = \frac{(f+1) \cdots (f+m)}{m} \quad \text{and} \quad f(0) = 1
\]

for any polynomial \( f \) and \( m > 0 \). Then

\[
H(\mathcal{O}; \nu) = \nu^{(n)},
\]

and thus

\[
H(\mathcal{O}^{n+1}; \ ) = (n+1)\nu^{(n)}.
\]

It is easy to see that

\[
H(\mathcal{O}/\mathcal{O}; \nu^{m-1}) = (\nu^{m-1})^{(n)} - (\nu^{-1})^{(n)} \quad \text{and} \quad m \nu^{(n-1)} + \sum_{i=2}^{n} \binom{n-i-2}{i} \nu^{(n-i)}.
\]

Let \( X \) be free with its exponents \((d_0, \ldots, d_n)\) throughout this section. Then we have
\[ H(\mathcal{O}(X); \mathcal{U}) = \sum_{i=0}^{n} (\mathcal{U} - d_i)^{(n)}, \]

and thus

\[ (2.5) \quad H(\mathcal{O}/J(X); \mathcal{U} + m - 1) \]

\[ = m \cdot \mathcal{U}^{(n-1)} + \sum_{i=2}^{n} \left( m+i-2 \right) \mathcal{U}^{(n-i)} - (n+1) \mathcal{U}^{(n)} + \sum_{i=0}^{n} (\mathcal{U} - d_i)^{(n)} \]

\[ = \left( m - \sum_{i=0}^{n} d_i \right) \mathcal{U}^{(n-1)} + \sum_{i=2}^{n} \left( \left( m+i-2 \right) + (-1)^i \sum_{j=0}^{i-1} \binom{d_j}{j} \right) \mathcal{U}^{(n-i)}. \]

On the other hand we know that

\[ \text{deg} H(\mathcal{O}/J(X); \mathcal{U}) = \text{deg} (\mathcal{O}/\mathfrak{a}Q; \mathcal{U}) = \dim \text{Spec}(\mathcal{O}/\mathfrak{a}Q) - 1 \leq n - 2 \]

if \( X \not\models \phi \). If \( X = \phi \), then

\[ H(\mathcal{O}/J(X); \mathcal{U}) = 0. \]

Thus we have proved

\[ (2.6) \quad \text{Proposition.} \quad m = \sum_{i=0}^{n} d_i. \]

Define \( P_i(X) (i = 2, \ldots, n) \in \mathbb{Z} \) by

\[ H(\mathcal{O}/J(X); \mathcal{U}) = \sum_{i=2}^{n} P_i(X) \mathcal{U}^{(n-i)}. \]

Then we can explicitly compute
(2.7) \[ P_i(X) = \sum_{j=0}^{i-2} \left( (-1)^j \binom{d_0 + \cdots + d_n + i - j - 2}{i-j} \right) + (-1)^i \sum_{k=0}^{n} \binom{d_k}{i-j} \binom{d_0 + \cdots + d_n - 1}{j} \]

because of (2.5) and (2.6).

(2.8) **Definition.** Let \( k \geq 1 \). Let \( I = (I(1), \ldots, I(k)) \) be a multi-index composing of \( k \) non-negative integers. Define

\[
\sigma_I^1(X) = \prod_{i=1}^{k} \sigma_i^{I(i)}(d_0, \ldots, d_n),
\]

where \( \sigma_j \in \mathbb{C}[t_0, \ldots, t_n] \) (\( j \geq 0 \)) is the elementary symmetric polynomial of degree \( j \). When \( k = 1 \), we write \( \sigma_j(X) \) instead of \( \sigma_{(j)}^1(X) \) (\( j \geq 0 \)). Thus (2.6) asserts that \( \#X = \sigma_i^1(X) \).

The following key lemma is not difficult to be verified:

(2.9) **Lemma.** For each integer \( i \) (\( 2 \leq i \leq n \)), there exist real numbers \( c(I;i) \) (\( I \in I[i] \)), which are independent of \( X \), such that

\[
P_i(X) + \frac{1}{(i-1)!} \sigma_i^1(X) = \sum_{I \in I[i]} c(I;i) \sigma_i^I(X).
\]

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Here

\[ I[i] := \{ I = (I(1), \ldots, I(k)); 0 \leq I(j) < i \ (j = 1, \ldots, k), \]
\[ \sum_{j=1}^{k} I(j) \leq i \} \].

Since \( X \) is free, any element in \( \mathfrak{A}(X) \) is also free (see [8] (5.5)). Thus we can define the mappings

\[ P_j : \mathfrak{A}(X) \to \mathbb{Z} \ (2 \leq j \leq n) \]
\[ \Psi \quad \Psi \]
\[ Y \mapsto P_j(Y). \]

The following is the most important proposition for the proof of Main Theorem:

**Proposition B.** \( P_j \) is \( j \)-cumulative \( (2 \leq j \leq n) \).

Our proof is difficult and long. See [8](5.10).
53. Proof of Main Theorem

In this section we shall prove Main Theorem. The crucial results for our proof are (1.5), Proposition A (§1), Proposition B (§2) and (2.9).

The following is stronger than Main Theorem:

(3.1) **Theorem.** Let \( i \geq 0 \). Then we have

1) \( \sigma_i^1(X) = \mu_i^*L(X) \) for any free \( n \)-arrangement \( X \),

2) \( \sum_i \sigma_i^1 : \mathcal{A}(X) \rightarrow \mathbb{Z} \) is \( i \)-cumulative for any free \( n \)-arrangement \( X \).

**Proof.** When \( i \leq 1 \), we can verify 1)\( _i \) and 2)\( _i \) because of (2.6).

Let \( i \geq 2 \). Assume that 1)\( _j \) (\( j = 0, 1, \ldots, i-1 \)) hold true. Let \( X \) be a free \( n \)-arrangement. Recall (2.9), then we have

\[
P_i^1(X) + \frac{1}{(i-1)!} \sum_{I \subseteq [i]} c(I;i)(\mu_i^*L)(X).
\]

By Proposition A, we know that \( \mu_i^*L \) is \( \mid I \mid \)-cumulative. Since \( \mid I \mid \leq i \) for \( I \subseteq [i] \), we can see that \( \mu_i^*L \) is \( i \)-cumulative. Thus we have the \( i \)-cumulativeness of \( \mu_i \) because the sum of two \( i \)-cumulative mappings is also \( i \)-cumulative. This is 2)\( _i \).

Next assume 2)\( _j \) (\( j = 0, 1, \ldots, i \)). Let \( X \) be a free \( n \)-arrangement. Then the assumption implies that the
mappings

\[ \sigma_j : \mathcal{A}(X) \rightarrow \mathbb{Z} \quad (j \geq 0) \]

satisfy the condition IV in (1.5). Moreover it is not too difficult to see that the mappings \( \sigma_j \quad (j \geq 0) \) also satisfy the conditions I, II and III in (1.5). Thus we can apply (1.5) and have

\[ \sigma_i = \mu_i \circ L \]
on \( \mathcal{A}(X) \). This is \( 1 \rangle_i \).

Q.E.D.

(3.2) The observation so far shows that the following four data concerning a free \( n \)-arrangement \( X \) are equivalent:

1. The set of the exponents \( (d_0, \ldots, d_n) \) of \( X \), which is equivalent to the polynomial

\[ \sum_{i=0}^{n} \sigma_i(X)t^i = \prod_{i=0}^{n} (1 + d_i t), \]

2. The Hilbert polynomial \( H(\Theta/J(X); \mathcal{V}) \) together with \( \#X \), which is equivalent to the data

\( (\#X, p_2(X), \ldots, p_n(X)) \),

3. The polynomial \( \sum_{i=0}^{n} (\mu_i \circ L(X))t^i \),
(4) The Poincaré polynomial of $M = \mathbb{C}^{n+1} \setminus \{x\}$, which is equivalent to the data

$$(b_0(M), b_1(M), \ldots, b_{n+1}(M)).$$

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