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On Two-dimensional Normal Singularities of Type ${}_*A_n$, ${}_*D_n$ and ${}_*E_n$

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ABSTRACT. Let G be the weighted dual graph associated with a contractible curve $A=\cup A_i$. There are many combinations of the weights $A_i \cdot A_i$ which make the graph to be contractible. If G is a graph which is the weighted dual graph for a rational singularity with any combination of the weights, then G is either ${}_*A_n$, ${}_*D_n$ or ${}_*E_n$.

1. Introduction. Let $A=\cup A_i$, where A_i are its irreducible components, lie on a nonsingular complex surface $\tilde{X}$. The curve A is said to be contractible (exceptional) if there exists a holomorphic mapping $\pi:\tilde{X}\to X$ of the surface $\tilde{X}$ into a complex space X that maps the whole curve A into one point $x\in X$ and is biholomorphic on $\tilde{X}-A$. In [5], it was proved that a curve A is contractible if and only if the intersection matrix $(A_i \cdot A_j)$ is negative-definite. The point x in X is an isolated singular point, in general, and the mapping $\pi:\tilde{X}\to X$ is a resolution of this singularity. The topological nature of the embedding of A in $\tilde{X}$ is described by the weighted dual graph G (see [9]). The vertices of G correspond to the A_i. The edges of G connecting the vertices corresponding to A_i and $A_j, i\neq j$, correspond to the points $A_i \cap A_j$. Finally, associated to each A_i is its genus as a Riemann surface, its singularities, and its weight, $A_i \cdot A_i$, the topological self-intersection number. The G will denote the graph, along with the genera, the singularities and the weights.

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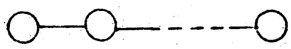
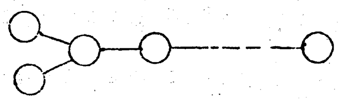
The geometric genus of (X,x) which is obtained from $(\tilde{X},A)$ by blowing down $\pi:\tilde{X}\to X$, is defined by $p_g(X,x)=\dim_{\mathbb{C}}(R^1\pi_*\mathcal{O}_{\tilde{X}})_x$. The geometric genus is in fact a finite integer.

Let $G(a_1,\ldots,a_n)$ denote the weighted dual graph associated with a contractible curve $A=\cup A_i$, where a_i represents the weight of the correspomding component A_i. There are many other combinations of the weights $(a_1',\ldots,a_n')$ which make the graph $G(a_1',\ldots,a_n')$ to be contractible. We shall denote by $p_g(a_1,\ldots,a_n)$ the geometric genus of the (X,x) obtained from a contractible graph $G(a_1,\ldots,a_n)$.

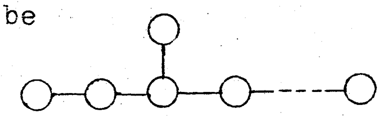
Then, we want to prove that there will be an integer m such that $p_g(a_1,\ldots,a_n)\leq m$, for any $(a_1,\ldots,a_n)$. Let M be the smallest one among such integers. In this paper, we decide all those graphs which have M=0. Namely, we enumerate all the graphs which are the weighted dual graphs for rational singularities, with any combination of the weights.

"Preliminaries and Main Theorems." In [2], Artin has studied the rational double points. He has shown that if x is a rational double point, then the weighted dual graph associated to (X,x) is one of the graphs $A_n, n\geq 1$; $D_n, n\geq 4$; E_6; E_7; E_8 as follows.

Let $A_n, n\geq 1$ be  (n-vertices),

$D_n, n\geq 4$ be  (n-vertices)

and $E_n, n=6,7$ and 8 be

 (n-vertices).

where each vertex represents a nonsingular rational curve with self-intersection number -2.

The quotient singularities are well-known examples of rational singularities, which are defined as follows. For a two-dimensional normal singularity (X,x), there is a finite subgroup of $GL(2,\mathbb{C})$ such that the quotient space of $\mathbb{C}^2$ by this group with a singular point at the origin is analytically isomorphic to (X,x).

In [3], Brieskorn has enumerated the weighted dual graphs for quotient singularities. There are seventeen types of graphs, and each of them has the same type as the rational double points up to the weights.

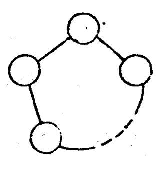
In section 2, we consider those graphs each of which has the same type as the rational double points up to the weights. Our graphs consist of only nonsingular rational curves. We may assume that each weight of our graphs is less than, or equal to -2, since we may assume that $\pi:\widetilde{X}\to X$ is the minimal resolution of (X,x). We can say that our graphs are those ones which are generalized from the quotient singularities' graphs.

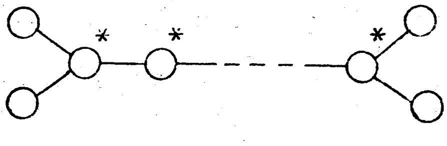
One can easily check that our graphs are always contractible. Hence we need not feel concern for contractibility of our graphs. We shall denote them by ${}_*A_n$, ${}_*D_n$, ${}_*E_6$, ${}_*E_7$ and ${}_*E_8$. For these graphs, the following theorem is proved.

Theorem A. Each of the graphs ${}_*A_n, n\geq 1$; ${}_*D_n, n\geq 4$; ${}_*E_n, n=6,7$ and 8 is the weighted dual graph for a rational singularity.

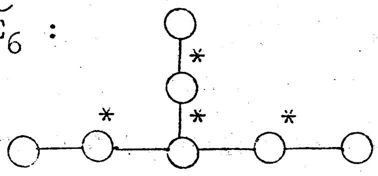
In sections 3 and 4, we recall the method in determinating the graphs A_n, D_n, E_n. If G is the weighted dual graph for a rational double point then each vertex of G corresponds to a nonsingular rational curve with self-intersection number -2. Since each of the graphs $\widetilde{A}_n$, $\widetilde{D}_n$, $\widetilde{E}_n$ (described as follows) is not contractible, G cannot have them as its proper subgraphs. And then, G is either A_n, D_n or E_n.

"Figures"

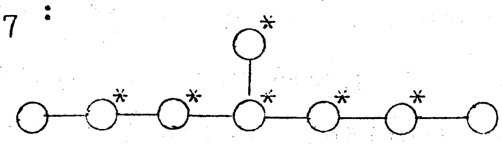
$\widetilde{A}_n, n\geq 1$:  (n+1)-vertices,

$\widetilde{D}_n, n\geq 4$:  (n+1)-vertices,

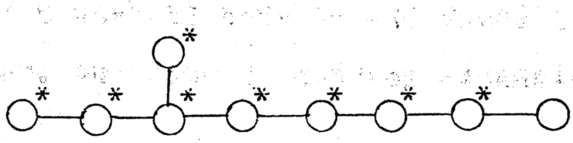
$\widetilde{E}_6$:



$\widetilde{E}_7$:



$\widetilde{E}_8$:



where each vertex corresponds to a nonsingular rational curve with self-intersection number -2.

We consider those graphs, contractible, each of which has the same type as the graphs $\widetilde{A}_n$, $\widetilde{D}_n$, $\widetilde{E}_n$ up to the weights. We may assume that each weight is less than, or equal to -2. Such graph, however, is not always contractible. But one can easily check that such graph is contractible if and only if there is at least one vertex whose weight is less than, or equal to -3. We denote these contractible graphs by ${}_*\widetilde{A}_n, n\geq 1$; ${}_*\widetilde{D}_n, n\geq 4$; ${}_*\widetilde{E}_n, n=6,7$ and 8.

There is a characterization of these graphs as follows. Let G be a graph which is not of type ${}_*A_n$, ${}_*D_n$, ${}_*E_n$. Suppose that any connected proper subgraph of G is one of the graphs ${}_*A_n$, ${}_*D_n$, ${}_*E_n$. Then G is one of the graphs ${}_*\widetilde{A}_n$, ${}_*\widetilde{D}_n$, ${}_*\widetilde{E}_n$.

For these graphs, the following theorem is proved.

Theorem B. For each of the graphs ${}_*\widetilde{D}_n, n\geq 4$; ${}_*\widetilde{E}_n, n=6,7$ and 8, the associated two-dimensional normal singularity is as follows.

(1) if all of vertices $\bigcirc^*$ in the graphs have the weights -2, then it is a minimally elliptic singularity (see [12]),

(2) otherwise, it is a rational singularity.

Remark. The graph of type ${}_*\widetilde{A}_n$ is the weighted dual graph of a cusp singularity, for any combination of the weights, and this is always minimally elliptic.

In section 5, we decide all those graphs with M=0.

Theorem C. They are of types ${}_*A_n, n\geq 1$; ${}_*D_n, n\geq 4$; ${}_*E_n, n=6,7,8$.

Remark. The graph of type ${}_*A_n$ is the weighted dual graph of a cyclic quotient singularity, for any combination of the weights.

I wish to thank the referee Professor X X X for pointing out the relationship between my work and that of Dolgachev.

2. Geometric genera of ${}_*D_n$, ${}_*E_6$, ${}_*E_7$ and ${}_*E_8$.

For ${}_*D_n$, ${}_*E_6$, ${}_*E_7$ and ${}_*E_8$, we can use the following Theorem, proved by Watanabe [16, Theorem 2.21], to calculate the geometric genera of these normal singularities, since they are star-shaped.

Let A_0 be the center of the star-shaped weighted dual graph. The branches of the graph are indexed by i, $1\leq i\leq m$. The curves of the i-th branche are denoted by A_{ij}, $1\leq j\leq r_i$, where A_{i1} intersects A_0 and A_{ij} intersects $A_{i,j+1}$. Let $-b=A_0\cdot A_0$ and $-b_{ij}=A_{ij}\cdot A_{ij}$. Finally, set $\alpha_i/\beta_i=[b_{i1},b_{i2},\ldots,b_{ir_i}]$; continued fraction; with $\alpha_i>\beta_i$, and α_i and β_i are relatively prime integers.

For any $k\geq 0$, let $D^{(k)}$ be the divisor on A_0 :

$$D^{(k)}= kD - \sum_{i=1}^{m} \left[(k\beta_i+\alpha_i-1)/\alpha_i\right] P_i \quad ,$$

where D is any divisor such that $\mathcal{O}_{A_0}(D)$ is the conormal sheaf of A_0, $P_i=A_0\cap A_{i1}$, and for any $a\in\mathbb{R}$, $[a]$ is the greatest integer less than, or equal to a.

Theorem 2.1. The geometric genus of such singularity is

$$\sum_{k\geq 0} \dim_{\mathbb{C}}\Gamma(A_0,\mathcal{O}_{A_0}(K_{A_0}-D^{(k)})) \quad ,$$

where K_{A_0} is the canonical line bundle of A_0.

As for ${}_*D_n$, ${}_*E_6$, ${}_*E_7$ and ${}_*E_8$, any vertex corresponds to a nonsingular rational curve, so we have the following by the Riemann-Roch Theorem.

$$\dim_{\mathbb{C}}\Gamma(A_0,\mathcal{O}_{A_0}(K_{A_0}-D^{(k)})) = (c(k)+1+|c(k)+1|)/2 \quad ,$$

where c(k) denotes the first Chern class of $K_{A_0} - D^{(k)}$, i.e.,

$$c(k)=-2-bk+\sum_{i=1}^{m}[(k\beta_i+\alpha_i-1)/\alpha_i],$$

and for any $a\in\mathbb{R}$, $|a|$ is the absolute value of a.

We say (X,x) is rational if $p_g(X,x)=0$.

Theorem 2.2. Let (X,x) be a two-dimensional normal singularity whose minimal resolution is of type either $_*D_n$, $_*E_6$, $_*E_7$ or $_*E_8$. Then (X,x) is rational.

Proof. In order to prove this, it is sufficient to prove that $c(k)\leq -1$ for all $k\geq 0$. Each graph has three branches

$$c(k)=-2-bk+\sum_{i=1}^{3}[(k\beta_i+\alpha_i-1)/\alpha_i]=1-bk+\sum_{i=1}^{3}[(k\beta_i-1)/\alpha_i],$$

since $[(k\beta_i+\alpha_i-1)/\alpha_i]=[1+(k\beta_i-1)/\alpha_i]=1+[(k\beta_i-1)/\alpha_i]$, $1\leq i\leq 3$.

The following Lemma is trivial.

Lemma 2.3. Let $p_i\geq 2$, $1\leq i\leq r$, be integers. Then the continued fraction $[p_1,p_2,\dots,p_r]\geq[\underbrace{2,2,\dots,2}_{r\text{-times}}]=(r+1)/r$.

By this Lemma, we can see as follows.

(i) For $_*D_n$, $c(k)\leq 1-2k+2\cdot[(k-1)/2]+[(k(n-3)-1)/(n-2)]$

$$\leq 1-2k+k-1+(k(n-3)-1)/(n-2)=-(k+1)/(n-2)<0,$$

since $n\geq 4$. Thus $c(k)\leq -1$ for all $k\geq 0$.

(ii) For $_*E_6$, $c(k)\leq 1-2k+[(k-1)/2]+2\cdot[(2k-1)/3]$

$$\leq 1-2k+(k-1)/2+(4k-2)/3=-(k+1)/6<0.$$

Thus $c(k)\leq -1$ for all $k\geq 0$.

(iii) For $_*E_7$, $c(k)\leq 1-2k+[(k-1)/2]+[(2k-1)/3]+[(3k-1)/4]$

$$\leq 1-2k+(k-1)/2+(2k-1)/3+(3k-1)/4=-(k+1)/12<0.$$

Thus $c(k)\leq -1$ for all $k\geq 0$.

(iv) For $_*E_8$, $c(k)\leq 1-2k+[(k-1)/2]+[(2k-1)/3]+[(4k-1)/5]$

$$\leq 1-2k+(k-1)/2+(2k-1)/3+(4k-1)/5=-(k+1)/30<0.$$

Thus $c(k)\leq -1$ for all $k\geq 0$.

Therefore $p_g(X,x)=0$.

Q.E.D.

3. $_*\widetilde{A}_n$, $n\geq 1$; $_*\widetilde{D}_n$, $n\geq 4$; $_*\widetilde{E}_6$; $_*\widetilde{E}_7$; $_*\widetilde{E}_8$. Recall that if an weighted dual graph G consists of only such vertices that correspond to nonsingular rational curves with self-intersection number -2, then G does not contain proper subgraphs of type $\widetilde{A}_n$, $\widetilde{D}_n$, $\widetilde{E}_6$, $\widetilde{E}_7$ or $\widetilde{E}_8$. And then G is either A_n, D_n, E_6, E_7 or E_8. This is proved by the contractibility for weighted dual graphs.

In this section, we consider those contractible graphs which are of type $\widetilde{A}_n$, $\widetilde{D}_n$, $\widetilde{E}_6$, $\widetilde{E}_7$ and $\widetilde{E}_8$. We shall denote them by $_*\widetilde{A}_n$, $_*\widetilde{D}_n$, $_*\widetilde{E}_6$, $_*\widetilde{E}_7$ and $_*\widetilde{E}_8$. One can easily check that $_*\widetilde{A}_n$ ($_*\widetilde{D}_n$, $_*\widetilde{E}_6$, $_*\widetilde{E}_7$ and $_*\widetilde{E}_8$, too) is contractible if and only if there is at least one vertex whose weight is less than, or equal to -3. These graphs have the following properties.

Let G be a connected graph. Suppose that any connected proper subgraph of G is of type either $_*A_n$, $_*D_n$, $_*E_6$, $_*E_7$ or $_*E_8$.

Theorem 3.1. We assume that G is neither $_*A_n$, $_*D_n$, $_*E_6$, $_*E_7$ nor $_*E_8$. Then G is necessarily one of the $_*\widetilde{A}_n$, $_*\widetilde{D}_n$, $_*\widetilde{E}_6$, $_*\widetilde{E}_7$ and $_*\widetilde{E}_8$.

Proof. (I) If G has a cyclic chain $_*\widetilde{A}_n$ as its subgraph, then G cannot have other vertex. Because if there exists other vertex, then the cyclic chain is a proper subgraph of G. Thus in this case, G is a cyclic chain, i.e., G is of type $_*\widetilde{A}_n$.

(II) When G contains no cyclic chain, G is tree-shaped. If there are more than two branching vertices then there is a proper subgraph with two branching vertices, which is a contradiction. Thus there are at most two branching vertices. If there is no branching vertex then G is $_*A_n$, a contradiction. Thus there is at least one branching vertex.

(i) In the case of two branching vertices, there is $_*\widetilde{D}_n$ as a subgraph of G. Similarly to (I), G is $_*\widetilde{D}_n$.

(ii) In the case of one branching vertex, if there are more than 4 branches in G then there is ${}_*\widetilde{D}_4$ as a subgraph of G. Hence G is ${}_*\widetilde{D}_4$, in this case. If there are at most two branches then G is ${}_*A_n$. So we have that there are 3 branches in G. In each branch, if there are more than two vertices then there is ${}_*\widetilde{E}_6$ as a proper subgraph of G. Hence there is at least one branch which has less than 3 vertices. Let (m_1,m_2,m_3) denote the numbers of vertices in the 3 branches. We may assume that $1\leq m_1\leq m_2\leq m_3$.

If $m_1=2$, $m_2\geq 3$, then $G\supsetneqq {}_*\widetilde{E}_7$.

If $m_1=m_2=2$, $m_3\geq 3$, then $G\supsetneqq {}_*\widetilde{E}_6$.

If $m_1=1$, $m_2=2$, $m_3\geq 6$, then $G\supsetneqq {}_*\widetilde{E}_8$.

Therefore , $(m_1,m_2,m_3)=\begin{cases}(1,3,3) \;\ldots\; {}_*\widetilde{E}_7, \\ (1,2,5) \;\ldots\; {}_*\widetilde{E}_8, \\ (2,2,2) \;\ldots\; {}_*\widetilde{E}_6.\end{cases}$ Q.E.D.

Remark. $(m_1,m_2,m_3)=(1,2,4)$ ... ${}_*E_8$, $(1,2,3)$ ... ${}_*E_7$, $(1,2,2)$ ... ${}_*E_6$ and $(1,1,m)$ ... ${}_*D_{m+1}$ $(m\geq 1)$.

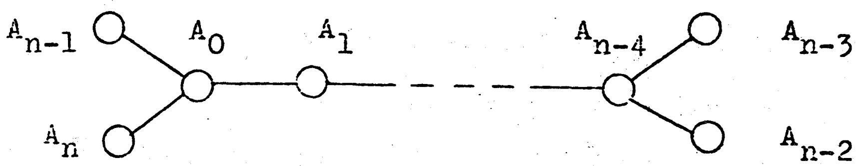
4. Geometric genera of ${}_*\widetilde{D}_n$, ${}_*\widetilde{E}_6$, ${}_*\widetilde{E}_7$ and ${}_*\widetilde{E}_8$.

Let G be a cyclic chain ${}_*\widetilde{A}_n$. Such graph is the weighted dual graph for a cusp singularity; see [7],[8] and [11]. Cusp singulariti s are elliptic (those for which $p_g=1$), furthermore they are minimally elliptic; see [12].

In this section, we investigate those graphs of type ${}_*\widetilde{D}_n$, $n\geq 4$; ${}_*\widetilde{E}_n$, n=6,7 and 8.

"Notations and definitions"

Let ${}_*\widetilde{D}_n$, $n\geq 4$, be



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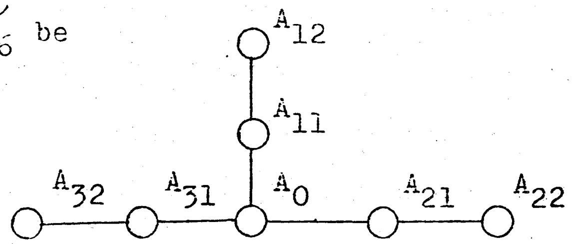
and $a_i=-A_i\cdot A_i$, $0\leq i\leq n$.

Let $_*\widetilde{E}_6$ be



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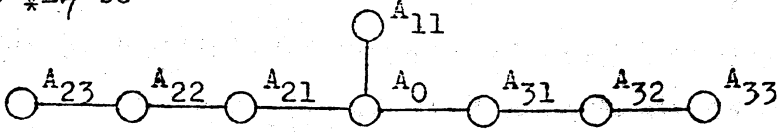
Let $_*\widetilde{E}_7$ be



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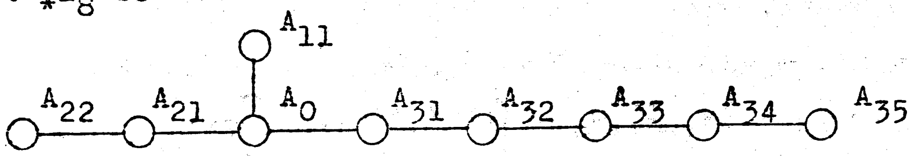
Let $_*\widetilde{E}_8$ be



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4.1. For $_*\widetilde{D}_n$, the followings are proved.

Lemma 4.1.1. (Wagreich [15]) If $a_{n-3}=a_{n-2}=a_{n-1}=a_n=2$, then the associated singularity is rational.

Lemma 4.1.2. If $a_0=a_1=\cdots=a_{n-4}=2$, then the associated singularity is minimally elliptic.

Proof. Let $a_0=a_1=\ldots=a_{n-4}=2$. Then we can simply calculate the fundamental cycle Z over the graph by using a computation sequence; see [10]. It is $Z=2A_0+2A_1+\cdots+2A_{n-4}+A_{n-3}+A_{n-2}+A_{n-1}+A_n$. Then $p(Z)=1+(Z\cdot Z+Z\cdot K)/2=1$, since $Z\cdot Z=\sum_{j=n-3}^{n}(A_j\cdot A_j+2)=-Z\cdot K$. Any connected proper subgraph of $_*\widetilde{D}_n$ is of type $_*A_k$ or $_*D_k$. So by Theorem 2.2, the associated singularity is minimally elliptic.

Q.E.D.

The rest is the following.

(i) $(a_0,a_1,\ldots,a_{n-4})\neq(2,2,\ldots,2)$ and

(ii) $(a_{n-3},a_{n-2},a_{n-1},a_n)\neq(2,2,2,2)$.

Let G denote a weighted dual graph of type $_*\widetilde{D}_n$ with (i) and (ii). Then we define another weighted dual graph of type $_*\widetilde{D}_n$ such that $A_i'\cdot A_i'=A_i\cdot A_i$, for $0\leq i\leq n-4$ and $A_j'\cdot A_j'=-2$, for $n-3\leq j\leq n$.

We shall denote this graph by G'. Since G' succeeds the condition (i), G' is contractible. We know that the associated singularity with G' is rational by Lemma 4.1.1.

Let $Z=\sum_{i=0}^{n} z_i A_i$ be the fundamental cycle over G (the existence and uniqueness are certified). Then there is a positive cycle Z' over G', defined by $\sum_{i=0}^{n} z_i A_i'$. Now, by the definition of G', we have that $A_i'\cdot Z'=A_i\cdot Z$ and $A_i\cdot K=-A_i\cdot A_i-2=-A_i'\cdot A_i'-2=A_i'\cdot K'$ for $0\leq i\leq n-4$, where K (K' resp.) is the canonical line bundle of G (G' resp.).

On the other hand, for $n-3\leq j\leq n$,

$$A_j\cdot(Z+K)=z_jA_j\cdot A_j + z_0 \text{ (or } z_{n-4}) -A_j\cdot A_j-2 .$$

By the hypothesis (ii), there is A_{j_0} at least one, such that $A_{j_0}\cdot A_{j_0}\leq -3$. Set $A_{j_0}\cdot A_{j_0}=-2-\alpha_{j_0}$, then $\alpha_{j_0}\geq 1$.

$$\begin{aligned} A_{j_0}\cdot(Z+K) &= z_0 \text{ (or } z_{n-4}) -2z_{j_0} -(z_{j_0}-1)\alpha_{j_0} \\ &\leq z_0 \text{ (or } z_{n-4}) -2z_{j_0} = A_{j_0}'\cdot(Z'+K'), \text{ since } (z_{j_0}-1)\alpha_{j_0}\geq 0. \end{aligned}$$

$$\begin{aligned} \text{So}\quad p(Z) &= 1+\Big(\sum_{i=0}^{n-4} z_iA_i\cdot(Z+K) + \sum_{j=n-3}^{n} z_jA_j\cdot(Z+K)\Big)/2 \\ &= 1+\Big(\sum_{i=0}^{n-4} z_iA_i'\cdot(Z'+K') + \sum_{j=n-3}^{n} z_jA_j\cdot(Z+K)\Big)/2 \\ &\leq 1+\Big(\sum_{i=0}^{n-4} z_iA_i'\cdot(Z'+K') + \sum_{j=n-3}^{n} z_jA_j'\cdot(Z'+K')\Big)/2 \\ &= p(Z'). \end{aligned}$$

As G' is the weighted dual graph for a rational singularity, we have $p(D)\leq 0$ for any positive cycle D on G'.

Hence $p(Z)\leq p(Z')\leq 0$, and then $p(Z)=0$. Namely the associated singularity with G is rational.

Therefore, we have proved the following Theorem.

Theorem 4.1.3. For $_*\widetilde{D}_n$, $n\geq 4$,

(1) $a_0=a_1=\cdots=a_{n-4}=2 \Longrightarrow$ (X,x) is minimally elliptic,

(2) otherwise $\Longrightarrow$ (X,x) is rational.

Corollary 4.1.4. $\widetilde{{}_*D_n}$ is a weighted dual graph for a hypersurface isolated singularity if and only if $a_0=a_1=\cdots=a_{n-4}=2$ and $9\leq a_{n-3}+a_{n-2}+a_{n-1}+a_n\leq 11$.

Proof. A hypersurface isolated singularity is a Gorenstein singularity. A Gorenstein rational singularity is a rational double point. Thus if $\widetilde{{}_*D_n}$ is a hypersurface isolated singularity then $a_0=a_1=\cdots=a_{n-4}=2$. By Laufer [12], a minimally elliptic singularity is hypersurface isolated singularity if and only if $Z\cdot Z$ is restricted by $-3\leq Z\cdot Z\leq -1$. Now, we have $Z\cdot Z=8-a_{n-3}-a_{n-2}-a_{n-1}-a_n$. Hence the Corollary is proved. Q.E.D.

4.2. For $\widetilde{{}_*E_6}$, we can apply Theorem 2.1 since it is star-shaped. So $p_g(\widetilde{{}_*E_6})=\sum_{k\geq 0}(c(k)+1+|c(k)+1|)/2$, where $c(k)=-2-bk+\sum_{i=1}^{3}[(k\beta_i+\alpha_i-1)/\alpha_i]=1-bk+\sum_{i=1}^{3}[(k\beta_i-1)/\alpha_i]$; see the proof of Theorem 2.2.

Theorem 4.2.1. For $\widetilde{{}_*E_6}$,

(1) $b=b_{11}=b_{21}=b_{31}=2$ $\Longrightarrow$ (X,x) is minimally elliptic,

(2) otherwise $\Longrightarrow$ (X,x) is rational.

Proof. Suppose that $b\geq 3$. By Lemma 4.1.2, $\alpha_i/\beta_i\geq[2,2]=3/2$ for $1\leq i\leq 3$, so we have $c(k)\leq 1-bk+3\cdot[(2k-1)/3]\leq -bk+2k=(2-b)k$. One can easily check that $c(0)=-2$ and $c(k)<0$ for all $k\geq 1$. Hence in this case, $p_g(\widetilde{{}_*E_6})=0$.

When $b=2$,

$$c(k)\underset{①}{\leq} 1-2k+3\cdot[(2k-1)/3]\underset{②}{\leq} 1-2k+3\cdot(2k-1)/3=0 .$$

Hence we have that

$$p_g(\widetilde{{}_*E_6}) = \#\{k\geq 0 : c(k)=0\}.$$

Therefore we must decide the integer k and types of branches (α_i,β_i), $1\leq i\leq 3$, which simultaneously satisfy the two equalities. The equality ② holds when $2k\equiv 1(\mathrm{mod}.3)$, i.e., $k=3l+2$, $l\geq 0$.

The next Lemma is very useful to prove our Theorems.

We recall the lexicographical order for n-ple integers.

Definition 4.2.2. Let p_i, q_j be integers which are greater than, or equal to 2, $1 \leq i, j \leq n$. Then $(p_1, \ldots, p_n) \underset{\text{lex}}{<} (q_1, \ldots, q_n)$ if there is an index i_0, $1 \leq i_0 \leq n$, such that (i) $p_i = q_i$ for all $i < i_0$, and (ii) $p_{i_0} < q_{i_0}$.

Lemma 4.2.3. If $(p_1, \ldots, p_n) \underset{\text{lex}}{<} (q_1, \ldots, q_n)$, then the associated continued fractions hold $[p_1, \ldots, p_n] < [q_1, \ldots, q_n]$.

Proof. By the definition, it is sufficient to prove that if $p_1 < q_1$ then $[p_1, \ldots, p_n] < [q_1, \ldots, q_n]$.

As $[p_1, \ldots, p_n] = p_1 - [p_2, \ldots, p_n]^{-1}$ and $[p_2, \ldots p_n] \geq n/(n-1) > 1$ by Lemma 2.3, we have $p_1 - 1 < [p_1, \ldots, p_n] \leq p_1$. Similarly $q_1 - 1 < [q_1, \ldots, q_n] \leq q_1$. Now $p_1 \leq q_1 - 1$. Thus we have the Lemma.

Q.E.D.

According to the contractibility, we cannot have $\alpha_1/\beta_1 = \alpha_2/\beta_2 = \alpha_3/\beta_3 = [2,2] = 3/2$. So we may assume that $\alpha_1/\beta_1 \geq [2,3] = 5/3$ since there is at least one branch $(b_{i1}, b_{i2}) \underset{\text{lex}}{>} (2,2)$.

If $k = 3\ell + 2$, $\ell \geq 1$, then

$$[(k\beta_1 - 1)/\alpha_1] \leq [(3k-1)/5] = [(9\ell+5)/5] = 1 + [9\ell/5] < 2\ell + 2 = [(2k-1)/3].$$

So the equality ① does not hold.

Set k=2. We have following lexicographical order

$$(2,3) \underset{\text{lex}}{<} \text{---} \underset{\text{lex}}{<} (2,m) \underset{\text{lex}}{<} \text{---} \underset{\text{lex}}{<} (3,2) \underset{\text{lex}}{<} \text{---}.$$

If $(b_{11}, b_{12}) \underset{\text{lex}}{\geq} (3,2)$, i.e., $\alpha_1/\beta_1 \geq [3,2] = 5/2$, then

$$[(2\beta_1 - 1)/\alpha_1] = [(2 \cdot 2 - 1)/5] = 0 < 1 = [(2 \cdot 2 - 1)/3].$$

Thus the equality does not hold.

If $(b_{11}, b_{12}) = (2, m)$, i.e., $\alpha_1/\beta_1 = (2m-1)/m$, $m \geq 3$, then

$$[(2\beta_1 - 1)/\alpha_1] = [(2m-1)/(2m-1)] = 1 = [(2 \cdot 2 - 1)/3].$$

Therefore the equalities ① and ② hold at the same time if and only if $k = b_{11} = b_{21} = b_{31} = 2$. And then, we have the Theorem.

One can easily check that $_*\widetilde{E}_6$ with $p_g=1$ is minimally elliptic by Theorem 2.2.

Corollary 4.2.4. $_*\widetilde{E}_6$ is the weighted dual graph for a hypersurface isolated singularity if and only if $b=b_{11}=b_{21}=b_{31}=2$ and $7 \leq b_{12}+b_{22}+b_{32} \leq 9$.

The proof of this is similar to the proof of Corollary 4.1.4.

4.3.

Theorem 4.3.1. For $_*\widetilde{E}_7$,

(1) $b=b_{11}=b_{21}=b_{22}=b_{31}=b_{32}=2 \Longrightarrow$ (X,x) is minimally elliptic,

(2) otherwise $\Longrightarrow$ (X,x) is rational.

Proof. Suppose that $b \geq 3$. Since α_2/β_2, $\alpha_3/\beta_3 \geq [2,2,2]=4/3$ and $\alpha_1/\beta_1=b_{11}\geq 2$, we have that $c(k) \leq 1-bk+[(k-1)/2]+2\cdot[(3k-1)/4] \leq 1-bk+(k-1)/2+(3k-1)/2=(2-b)k$. So $c(k) \leq (2-b)k < 0$ for $k \geq 1$ and $c(0)=-2$. Hence $p_g(_*\widetilde{E}_7)=0$.

Let $b=2$. Then $c(k) \underset{①}{\leq} 1-2k+[(k-1)/2]+2\cdot[(3k-1)/4]$

$\underset{②}{\leq} 1-2k+(k-1)/2+(3k-1)/2=0$ holds.

Similarly to the proof of Theorem 4.2.1, we should find when the equalities ① and ② are simultaneously satisfied.

The equality ② hold when $3k \equiv 1 (\mathrm{mod}.4)$, i.e., $k=4\ell+3$, $\ell \geq 0$.

If we suppose that $b_{11} \geq 3$ then $[((4\ell+3)-1)/b_{11}] \leq [(4\ell+2)/3] \leq (4\ell+2)/3 < 2\ell+1=[((4\ell+3)-1)/2]$. Thus we must have $b_{11}=2$ in order to hold the equality ①. Then, by the contractibility, there is at least one integer which is greater than, or equal to 3 among $\{b_{ij}\}$. We may assume that $(b_{21},b_{22},b_{23}) \neq (2,2,2)$ without loss of the generality, i.e., $\alpha_2/\beta_2 \geq [2,2,3]=7/5$.

At first, we consider the equality ① for $k=4\ell+3$, $\ell \geq 1$. Then we have that

$[(k\beta_2-1)/\alpha_2] \leq [(5k-1)/7]=[(20\ell+14)/7] \leq (20\ell+14)/7 < 3\ell+2$.

On the other hand, $[(3k-1)/4]=[(12\ell+8)/4]=3\ell+2$.

So the equality ① does not hold in this case. Let k=3. If $\alpha_2/\beta_2 \geq [2,3,2]=8/5$, then $[(3\beta_2-1)/\alpha_2] \leq [(3\cdot 5-1)/8]=1<2=[(3\cdot 3-1)/4]$. So the equality ① does not hold. Let $\alpha_2/\beta_2=[2,2,m]$ with $m\geq 3$, i.e., $\alpha_2/\beta_2=(3m-1)/(2m-1)$. Then we have the equality ① since $[(3\beta_2-1)/\alpha_2]=[(6m-4)/(3m-2)]=2$. Therefore we have the equalities ① and ② if and only if k=3 and $b=b_{11}=b_{21}=b_{22}=b_{31}=b_{32}=2$, and then complete the proof of Theorem, by Theorem 2.2.

Q.E.D.

Corollary 4.3.2. $_*\widetilde{E}_7$ is the weighted dual graph for a hypersurface isolated singularity if and only if $b=b_{11}=b_{21}=b_{22}=b_{31}=b_{32}=2$ and $5\leq b_{23}+b_{33}\leq 7$.

4.4. Finally we consider the graph of type $_*\widetilde{E}_8$.

Theorem 4.4.1. For $_*\widetilde{E}_8$,

(1) $b=b_{11}=b_{21}=b_{22}=b_{31}=b_{32}=b_{33}=b_{34}=2$

$\Longrightarrow$ (X,x) is minimally elliptic,

(2) otherwise $\Longrightarrow$ (X,x) is rational.

Proof. Suppose $b\geq 3$. Since $\alpha_1/\beta_1=b_{11}\geq 2$, $\alpha_2/\beta_2\geq[2,2]=3/2$ and $\alpha_3/\beta_3\geq[2,2,2,2,2]=6/5$, we have that

$$c(k)\leq 1-bk+[(k-1)/2]+[(2k-1)/3]+[(5k-1)/6]$$
$$\leq 1-bk+(k-1)/2+(2k-1)/3+(5k-1)/6=(2-b)k.$$

So $c(k)<0$ for all $k\geq 1$ and $c(0)=-2$. Hence $p_g(_*\widetilde{E}_8)=0$.

Let b=2. Then $c(k) \underset{①}{\leq} 1-2k+[(k-1)/2]+[(2k-1)/3]+[(5k-1)/6]$

$\underset{②}{\leq} 1-2k+(k-1)/2+(2k-1)/3+(5k-1)/6=0.$

The equality ② hold when k satisfies that $2k\equiv 1(\mathrm{mod}.3)$ and $5k\equiv 1(\mathrm{mod}.6)$, i.e., $k=6\ell+5$, $\ell\geq 0$.

If we suppose that $b_{11}\geq 3$ then we have the following equalities and inequalities.

$$[(k\beta_1-1)/\alpha_1]=[(k-1)/b_{11}]\leq[(k-1)/3]=[(6\ell+4)/3]<2\ell+2$$
$$\leq 3\ell+2=[(6\ell+4)/2].$$

Thus we have $b_{11}=2$ in order to hold the equality ①.
If we suppose that $(b_{21},b_{22})\neq(2,2)$ then

$$[(k\beta_2-1)/\alpha_2]\leqq[(3k-1)/5]=[(18\ell+14)/5]\leqq(18\ell+14)/5$$
$$< 4\ell+3=[(2\cdot(6\ell+5)-1)/3].$$

So we get $b_{21}=b_{22}=2$ for the equality ①.
Then the contractibility implies

$$(b_{31},b_{32},b_{33},b_{34},b_{35})\neq(2,2,2,2,2).$$

By Lemma 4.2.3, $\alpha_3/\beta_3\geqq[2,2,2,2,3]=11/9$. Then $[(k\beta_3-1)/\alpha_3]\leqq[(9k-1)/11]$ $=[(54\ell+44)/11]\leqq 5\ell+4=[(5k-1)/6]$.
If $\ell\geqq 1$, then $[(54\ell+44)/11]< 5\ell+4$. Hence we need to consider only the case of $k=5$. And then, for $\alpha_3/\beta_3\geqq[2,2,2,3,2]=14/11$,

$$[(5\beta_3-1)/\alpha_3]\leqq[54/14]=3<4.$$

Let $\alpha_3/\beta_3=[2,2,2,2,m]=(5m-4)/(4m-3)$, $m\geqq 3$. In this case, we have $[(5\beta_3-1)/\alpha_3]=[(20m-16)/(5m-4)]=4$, and the equality holds.
Therefore we can complete the proof of Theorem by Theorem 2.2.

Q.E.D.

Corollary 4.4.2. $_*\widetilde{E}_8$ is the weighted dual graph for a hypersurface isolated singularity if and only if $b=b_{11}=b_{21}=b_{22}$ $=b_{31}=b_{32}=b_{33}=b_{34}=2$ and b_{35} is restricted as follows; $3\leqq b_{35}\leqq 5$.

5. Proof of Theorem C. In this section, we consider those contractible graphs G(----) with M=0, where M is defined in section 1. Since G associates to a rational singularity, it consists of only nonsingular rational curves and has only normal crossings.

Lemma 5.1. Let G(----) be such a graph, and G′(----) be a connected proper subgraph of G(----). Then G′(----) is also such a graph.

This is an application of the following.

Theorem [16, Theorem 2.8]. Let A be a contractible curve. Let A′ be a connected proper subvariety of A. Then for the associated (X,x) to A, and (X′,x′) to A′,

$p_g(X'.x') \leq p_g(X,x)$ holds.

From this Lemma and Theorem B, we have the following Lemma.

Lemma 5.2. Let G be such a graph. Then it cannot contain graphs ${}_*\widetilde{A}_n$, ${}_*\widetilde{D}_n$, ${}_*\widetilde{E}_6$, ${}_*\widetilde{E}_7$ or ${}_*\widetilde{E}_8$ as its subgraphs.

Such graph is either ${}_*A_n$, ${}_*D_n$, ${}_*E_6$, ${}_*E_7$ or ${}_*E_8$. Therefore, by Theorem A, we complete the proof of Theorem C.

Remark. Theorem B gives examples of the graphs with M=1.

6. Bimodular singularities. In this section, we shall note the relationship between our results and that of Dolgachev [4].

In Corollary 4.2.4, 4.3.2 and 4.4.2, we describe 14 distinguished dual graphs of normal surface singularities with embedding dimension 3. These are exactly the graphs of the 14 exceptional bimodular singularities constructed by Dolgachev.

More precisely the relation is described by the following list, where the symbols ${}_*\widetilde{E}_n$ and b_{ij} are those of used in section 4 and others are those of Arnold [1].

${}_*\widetilde{E}_6$				${}_*\widetilde{E}_7$			${}_*\widetilde{E}_8$	
b_{12}	b_{22}	b_{32}		b_{23}	b_{33}		b_{35}	
2	2	3	E_{18}	2	3	E_{19}	3	E_{20}
2	2	4	Z_{17}	2	4	Z_{18}	4	Z_{19}
2	2	5	Q_{16}	2	5	Q_{17}	5	Q_{18}
2	3	3	W_{17}	3	3	W_{18}		
2	3	4	S_{16}	3	4	S_{17}		
3	3	3	U_{16}					

The 6 graphs of type ${}_*\widetilde{D}_4$ described in Corollary 4.1.4 also correspond to an interesting group of 6 bimodal singularities in Arnold's classification.

For graphs ${}_*\widetilde{D}_n$, n>4 the quadriples (2,2,3,3) and (2,2,3,4) will give two types for each quadriples.

In this way, one gets the following 8 series of singularities, where the notations with double indexes are described in [13].

$_*\widetilde{D}_4$					$_*\widetilde{D}_{i+4}$, $1 \leq i$				
a_1	a_2	a_3	a_4		a_{i+1}	a_{i+2}	a_{i+3}	a_{i+4}	
2	2	2	3	E_{16}	2	2	2	3	$J_{3,i}$
2	2	2	4	Z_{15}	2	2	2	4	$Z_{1,i}$
2	2	2	5	Q_{14}	2	2	2	5	$Q_{2,i}$
2	2	3	3	W_{15}	2	2	3	3	$W_{1,i}$
2	2	3	4	S_{14}	2	3	2	3	$W^{\#}_{1,i}$
2	3	3	3	U_{14}	2	2	3	4	$S_{1,i}$
					2	3	2	4	$S^{\#}_{1,i}$
					2	3	3	3	$U_{1,i}$

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