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ON PLURIGENERA OF NORMAL ISOLATED SINGULARITIES, I

by

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§0. Introduction

In this work we show the analogous invariants of plurigenera of compact complex manifolds can also be defined for normal isolated singularities.

Our presentation goes as follows. In Sect. 1, we give the definition of plurigenera δ_m, m a positive integer, of an n-dimensional normal isolated singularity and calculate them in the typical three cases. In Sect. 2, we study the normal surface singularities and prove some theorems about δ_m. In the last section of this paper we classify the surface singularities such that $0 \leqq \delta_m \leqq 1$ for $m \geqq 1$.

Let x be a normal singularity of two-dimensional analytic space X. In [2], Artin introduced a definition for x to be rational. A point x is rational if $R^1\pi_*\mathcal{O}_{\tilde{X}} = 0$ where $\pi:\tilde{X} \to X$ is a resolution of the singularity. Laufer [13] derived a necessary and sufficient criterion for x to be rational that does not involve a priori knowledge of what a resolution of x looks like. Yau [27] generalized Laufer's result to higher dimensions. Let (X,x) be a normal n-dimensional isolated singularity. It follows from Hironaka's work [7] that a resolution $\pi:\tilde{X} \to X$ always exists. The geometric genus of the singularity is defined as

$$p_g(X,x) = \dim (R^{n-1}\pi_*\mathcal{O}_{\tilde{X}})_x .$$

Assume that V is a Stein neighborhood of x in X. Let K be the canonical line bundle of V-{x}. Then

$$p_g(X,x) = \dim \Gamma(V-\{x\},\mathcal{O}(K))/L^2(V-\{x\}) .$$

(Here $L^2(V-\{x\})$ denotes the set of all square integrable holomorphic n-forms on V-{x}, see p.601 of [13].) Following the m-genus of a complex manifold

[22], the plurigenera of an n-dimensional isolated singularity is defined as

$$\delta_m(X,x) = \dim \Gamma(V-\{x\},\mathcal{O}(mK))/L^{2/m}(V-\{x\})$$

where $L^{2/m}(V-\{x\})$ denotes the set of all $L^{2/m}$-integrable m-ple holomorphic n-forms on $V-\{x\}$. $\delta_m(X,x)$ can be described in terms of cohomologies of the resolution. These integers are determined independently to the choice of the Stein neighborhoods. Hence δ_m can be an invariant attached to the singularity. We consider the asymptotic behavior of δ_m when $m \to +\infty$, and calculate the value $\delta = \limsup_{m\to\infty} \delta_m/m^n$ in some cases.

Let (X,x) be defined by a quasihomogeneous polynomial $f(z_0,z_1,\ldots,z_n)$ with weights $r_0,r_1,\ldots,r_n$:

$$X = \{ (z_0,z_1,\ldots,z_n)\in\mathbb{C}^{n+1} \mid f(z_0,z_1,\ldots,z_n) = 0 \} .$$

Let $r(f) = r_0+r_1+\ldots+r_n$. Then (Example 1.15),

(1) $r(f) > 1 \Longrightarrow \delta_m(X,x) = 0$ for $m \geqq 1$,

(2) $r(f) = 1 \Longrightarrow \delta_m(X,x) = 1$ for $m \geqq 1$

and

(3) $r(f) < 1 \Longrightarrow \limsup_{m\to\infty} \delta_m(X,x)/m^n = \frac{1}{n!}\{1-r(f)\}^n \frac{1}{r_0 r_1\cdots r_n}$.

In case (1), (Theorem 1.11) (X,x) is rational by Burns [4], p.239. If (X,x) is a quotient singularity then $\delta_m(X,x) = 0$ for $m \geqq 1$. Suppose that (x,x) is a cusp singularity. Then (Theorem 1.16) $\delta_m(X,x) = 1$ for $m \geqq 1$.

When (X,x) is two-dimensional, we prove two fundamental theorems. One is $\limsup_{m\to\infty} \delta_m/m^2 < \infty$. The other is as follows. Let $\pi:\tilde{X} \to X$ be the minimal resolution of X. Let $A = \pi^{-1}(x)$ be the exceptional set. Suppose A' is a connected proper subvariety of A. A' is also an exceptional set. Let (X',x') be the singularity obtained by blowing down A'. Then (Theorem 2.8) $\delta_m(X,x) \geqq \delta_m(X',x')$. Let $p_a(X,x)$ be the arithmetic genus introduced by Wagreich [24]. Then the latter fundamental theorem allows us to introduce the notion of minimality of a singularity. (X,x) is minimal if $p_a(X,x) \geqq 1$ and $p_g(X,x) > p_g(X',x')$ for every connected proper subvariety A' of A. For instance (Corollary 2.9) Gorenstein singularities have the minimality if $p_a \geqq 1$. Moreover (Theorem 2.13) a minimal singularity with $p_a = 1$ is Gorenstein. When the dual graph of a minimal good resolution is star-shaped, (Theorem 2.21) it becomes possible to get an estimate, in terms of the associated graph including the genera of the irreducible components and certain data.

In Sect. 3 we study the classification of singularities such that $0 \leqq \delta_m \leqq 1$. δ_m characterize the quotient singularities (Theorem 3.9) : (X,x) is a quotient singularity <==> $\delta_m(X,x) = 0$ for $m \geqq 1$. Knöller [11] proved the analogous theorem : (X,x) is a rational double point <==> $\gamma_m(X,x) = 0$ for $m \geqq 1$. We completely classify all rational singularities with $0 \leqq \delta_m \leqq 1$. This result has the striking resemblance to Wagreich's work [23]. As for the singularity such that $\delta_m = 1$ for all $m \geqq 1$, a few exceptions are left in the classification. In particular (Theorem 3.20), if (X,x) is a Gorenstein singularity such that $\delta_m = 1$ for all $m \geqq 1$ then (X,x) is a simple elliptic singularity or a cusp singularity.

The referee has informed the author that Theorem 2.1 is generalized to the case of arbitrary dimensions $n \geqq 2$. The author wishes to thank the referee for his valuable comments and suggestions.

§1. Plurigenera of Isolated Singularities

Let (X,x) be a normal isolated singularity in the n-dimensional analytic space X. It follows from Hironaka's work [7] that a resolution $\pi : \tilde{X} \to X$ always exists.

Definition 1.1. The geometric genus of a normal isolated singularity (X,x) is $p_g(X,x) = \dim (R^{n-1}\pi_* \mathcal{O}_{\tilde{X}})_x$.

The geometric genus is in fact independent of the choice of the resolution. Yau [27] derived an intrinsic definition of p_g that does not involve a priori knowledge of what a resolution of x looks like, which is a generalization of Laufer's theorem [13] in the 2-dimensional case.

Theorem 1.2 (Yau [27]). Let x be a normal n-dimensional isolated singularity of X. Suppose that V is a (sufficiently small) Stein neighborhood of x and K is the canonical line bundle of $V-\{x\}$. Then

$$p_g(X,x) = \dim \Gamma(V-\{x\},\mathcal{O}(K))/L^2(V-\{x\}) ,$$

where $L^2(V-\{x\})$ denotes the set of all square integrable holomorphic n-forms on $V-\{x\}$.

Let $U = \pi^{-1}(V)$ and $A = \pi^{-1}(x)$, then $\Gamma(U,\mathcal{O}(K)) = L^2(U-A)$ by [13, Theorem 3.1, p.601]. Therefore we obtain $p_g(X,x) = \dim \Gamma(U-A,\mathcal{O}(K))/\Gamma(U,\mathcal{O}(K))$.

For convenience, we denote the line bundle $K^{\otimes m}$ by mK. An element of $\Gamma(V-\{x\},\mathcal{O}(mK))$ is considered as an m-ple holomorphic n-form. Let ω be a holomorphic m-ple n-form on U-A. We write ω as

$$\omega = \phi(z)(dz_1 \wedge dz_2 \wedge \dots \wedge dz_n)^m ,$$

using local coordinates $(z_1, z_2, \dots, z_n)$. We associate with ω the continuous (n,n)-form $(\omega\wedge\bar{\omega})^{1/m}$, given locally by

$$|\phi(z)|^{2/m}\left(\frac{\sqrt{-1}}{2}\right)^n dz_1\wedge d\bar{z}_1\wedge dz_2\wedge d\bar{z}_2\wedge\dots\wedge dz_n\wedge d\bar{z}_n .$$

Definition 1.3. ω is called integrable ($L^{2/m}$-integrable) if $\int_{W-A}(\omega\wedge\bar{\omega})^{1/m} < \infty$ for W, any sufficiently small relatively compact neighborhood of A in U.

Let $L^{2/m}(U-A)$ be the set of all integrable holomorphic m-ple n-forms on U-A, which is a subspace of $\Gamma(U-A,\mathcal{O}(mK))$. $L^{2/m}(U-A)$ becomes a vector space $\Gamma(U,\mathcal{O}(mK+(m-1)[A]))$ in the case that A is a divisor which has at most normal crossings by Sakai [19, Theorem 2.1, p.243]. As for $L^{2/m}(V-\{x\})$ we replace U and A with V and $\{x\}$ respectively in the definition of $L^{2/m}(U-A)$.

Following Laufer [13], we consider the sheaf cohomology with support at infinity. The following sequence is exact:

$$0 \to \Gamma(U,\mathcal{O}(mK)) \to \Gamma_\infty(U,\mathcal{O}(mK)) \to H^1_*(U,\mathcal{O}(mK)) \to \dots .$$

By Siu [20], p.374, any section of mK defined near the boundary of U has an analytic continuation to U-A. Therefore there is a natural isomorphism

$$\Gamma_\infty(U,\mathcal{O}(mK)) \xrightarrow{\sim} \Gamma(U-A,\mathcal{O}(mK)) .$$

By Serre duality,

$$H^1_*(U,\mathcal{O}(mK)) \tilde{\to} H^{n-1}(U,\mathcal{O}(K-mK)) .$$

Since U is strongly pseudoconvex, $H^{n-1}(U,\mathcal{O}(K-mK))$ is finite dimensional. Hence by the inequality

$$\dim \Gamma(U-A,\mathcal{O}(mK))/L^{2/m}(U-A) \leqq \dim \Gamma(U-A,\mathcal{O}(mK))/\Gamma(U,\mathcal{O}(mK))$$

$$\leqq \dim H^1_*(U,\mathcal{O}(mK)) = \dim H^{n-1}(U,\mathcal{O}(K-mK)),$$

we have $\dim \Gamma(U-A,\mathcal{O}(mK))/L^{2/m}(U-A) < +\infty$. If $V \supset V'$, with V' another Stein neighborhood of x, then we have the following commutative diagram of exact sequences.

$$\begin{array}{ccccccccc} 0 \to \Gamma(U\ ,\mathcal{O}(mK)) & \to & \Gamma(U-A\ ,\mathcal{O}(mK)) & \to & H^1_*(U\ ,\mathcal{O}(mK)) & \to & H^1(U\ ,\mathcal{O}(mK)) & \to & \cdots \\ \downarrow \beta_0 & & \downarrow \gamma_0 & & \uparrow \alpha_1 & & \downarrow \beta_1 & & \\ 0 \to \Gamma(U',\mathcal{O}(mK)) & \to & \Gamma(U'-A,\mathcal{O}(mK)) & \to & H^1_*(U',\mathcal{O}(mK)) & \to & H^1(U',\mathcal{O}(mK)) & \to & \cdots , \end{array}$$

where β_0, γ_0 and β_1 are the restriction maps which are induced by the inclusion map $j : U' \to U$ and α_1 is the zero extension map of the cohomology. The restriction map β_1 is an isomorphism by Lemma 3.1 of [13]. It follows from an easy diagram chase that

$$\Gamma(U-A,\mathcal{O}(mK))/\Gamma(U,\mathcal{O}(mK)) \to \Gamma(U'-A,\mathcal{O}(mK))/\Gamma(U',\mathcal{O}(mK))$$

is an isomorphism. Thus

$$\Gamma(U-A,\mathcal{O}(mK))/L^{2/m}(U-A) \tilde{\to} \Gamma(U'-A,\mathcal{O}(mK))/L^{2/m}(U'-A).$$

Definition 1.4. The plurigenus (m-genus), m a positive integer, of a normal isolated singularity (X,x) is

$$\delta_m(X,x) = \dim \Gamma(V-\{x\},\mathcal{O}(mK))/L^{2/m}(V-\{x\}) .$$

Theorem 1.5. Let (X,x) be a quotient singularity. Then $\delta_m(X,x) = 0$ for $m \geqq 1$.

Proof. (X,x) is a quotient singularity. Hence we can assume that there exist a ball $B \subset \mathbb{C}^n$ of radius ε, sufficiently small, and a finite group G of unitary linear transformations, no element of which fixes, poitwise, a hyperplane in $\mathbb{C}^n$, so that $(X,x) \cong (B/G,p(0))$ where p is the quotient map $B \to B/G$. If θ is an m-ple n-form on $X-\{x\}$,

$$f = p^*\theta/(dz_1\wedge dz_2\wedge\ldots\wedge dz_n)^m$$

is a holomorphic function on $B-\{0\}$ and hence extends to be holomorphic also at 0, the origin in $\mathbb{C}^n$. Then

$$\int_{X-\{x\}}(\theta\wedge\bar{\theta})^{1/m} = \frac{1}{g}\int_{B-\{0\}}(p^*\theta\wedge\overline{p^*\theta})^{1/m} ,$$

where $g = \mathrm{ord}(G)$. Since $p^*\theta = f(z)(dz_1\wedge dz_2\wedge\ldots\wedge dz_n)^m$ is holomorphic, the integral in question is finite and so $\theta \in L^{2/m}(X-\{x\})$. Thus $\delta_m(X,x) = 0$. □

Theorem 1.6. Let ω be a holomorphic n-form defined on a deleted neighborhood of $x \in X$, which is nowhere vanishing on this neighborhood. If ω is square integrable in a neighborhood of x, then $\delta_m(X,x) = 0$ for all $m \geqq 1$.

Proof. Let V be a sufficiently small Stein neighborhood of x. If θ is any holomorphic m-ple n-form on $V-\{x\}$, $f = \theta/\omega^m$ is a holomorphic function on $V-\{x\}$ and hence extends to be holomorphic also at x. Thus $\theta = f\omega^m$ is $L^{2/m}$-integrable. □

Let M be a compact complex (n-1)-dimensional manifold, and let F be a complex analytic line bundle over M. Assume that F is positive in the sense of [16]. We denote the total space of the dual line bundle F* by $\tilde{X}$. The zero section of $\tilde{X}$ is contractible. Then we get an n-dimensional normal isolated singularity (X,x) by blowing down $\pi : \tilde{X} \to X$.

The Leray spectral sequence of π shows

$$H^i(\tilde{X},\mathcal{O}_{\tilde{X}}) = H^0(X,R^i\pi_*\mathcal{O}_{\tilde{X}}) = (R^i\pi_*\mathcal{O}_{\tilde{X}})_x ,$$

and the Leray spectral sequence for $p : \tilde{X} \to M$, p the projection of F*, shows

$$H^i(\tilde{X},\mathcal{O}_{\tilde{X}}) = H^i(M,R^0p_*\mathcal{O}_{\tilde{X}}) = \bigoplus_{k\geqq 0} H^i(M,\mathcal{O}(kF)) .$$

Let K_M be the canonical line bundle of M. Then all these groups vanish if K_M is negative. In the case that K_M is a trivial line bundle, $H^{n-1}(M,\mathcal{O}_M)$ is one-dimensional. In particular

$$p_g(X,x) = \sum_{k\geqq 0} \dim H^{n-1}(M,\mathcal{O}(kF)) = \sum_{k\geqq 0} \dim H^0(M,\mathcal{O}(K_M-kF)) .$$

Theorem 1.7. Let (X,x) be an (n+1)-dimensional normal isolated singularity. Assume there exists a resolution $\pi : \tilde{X} \to X$ such that $A = \pi^{-1}(x)$ is an n-dimensional compact complex manifold. Then

$$\delta_m(X,x) \leqq \sum_{k\geqq 0} \dim \Gamma(A,\mathcal{O}(mK_A+kN)) ,$$

where N is the normal bundle of A in $\tilde{X}$.

Proof. Since A is compact, we can cover A by finite number of coordinate neighborhoods $\{U_\alpha\}$ with holomorphic coordinates $(z_\alpha^1,\dots,z_\alpha^n,t_\alpha)$ with $A \cap U_\alpha = \{ p\in U_\alpha \mid t_\alpha(p) = 0 \}$. Choose a Stein neighborhood V of x so that $\bigcup_\alpha U_\alpha \supset U = \pi^{-1}(V)$. Then we have an injective homomorphism

$$\Gamma(U-A,\mathcal{O}(mK))/L^{2/m}(U-A) \to \sum_{k\geqq 0} \Gamma(A,\mathcal{O}(mK_A+kN)) .$$

Let $\phi \in \Gamma(U-A,\mathcal{O}(mK))$. $H^1_*(U,\mathcal{O}(mK))$ is a $\Gamma(U,\mathcal{O})$-module of finite dimension over $\mathbb{C}$. So ϕ is meromorphic on U with possible poles on A. ϕ is given by meromorphic function ϕ_α in U_α such that

$$\phi_\alpha\{dz_\alpha^1\wedge\dots\wedge dz_\alpha^n\wedge(dt_\alpha/t_\alpha)\}^m .$$

Expand ϕ_α in a Laurent series

$$\phi_\alpha = \sum_{-\infty<k<\infty} \phi_\alpha^{(k)}(z)t_\alpha^{-k} .$$

In the case where $k \geqq 0$,

$$\left(\frac{1}{2\pi\sqrt{-1}}\right)^m \int\dots\int_{|t_\alpha|=\varepsilon_\alpha} (t_\alpha)^k\phi_\alpha\{dz_\alpha^1\wedge\dots\wedge dz_\alpha^n\wedge(dt_\alpha/t_\alpha)\}^m = \phi_\alpha^{(k)}(dz_\alpha^1\wedge\dots\wedge dz_\alpha^n)^m ,$$

for ε_α suffciently small. Let $\{f_{\alpha\beta}\}$ be transition functions of the line bundle [A]. On $U_\alpha\cap U_\beta$, $t_\alpha = f_{\alpha\beta}t_\beta$, then

$$\phi_\alpha^{(k)}(dz_\alpha^1\wedge\dots\wedge dz_\alpha^n)^m = (f_{\alpha\beta}|A)^k\phi_\beta^{(k)}(dz_\beta^1\wedge\dots\wedge dz_\beta^n)^m .$$

Clearly $[A]|_A$ is the (complex analytic) normal bundle N. It follows that $\{\phi_\alpha^{(k)}\} \in \Gamma(A,\mathcal{O}(mK_A+kN))$ and the homomorphism has been constructed. By the definition this homomorphism is injective. □

Remark. In the case where (X,x) admits a $\mathbb{C}^*$-action, the above homomorphism is surjective. Given $\{\phi_\alpha^{(k)}\} \in \Gamma(A,\mathcal{O}(mK_A+kN))$, we can form the global section of $\Gamma(U-A,\mathcal{O}(mK))$ as follows. Since $t_\alpha = f_{\alpha\beta}(z)t_\beta$,

$$dz_\alpha^1\wedge\dots\wedge dz_\alpha^n\wedge(dt_\alpha/t_\alpha) = dz_\alpha^1\wedge\dots\wedge dz_\alpha^n\wedge(dt_\beta/t_\beta)$$

on $U_\alpha \cap U_\beta$. Then

$$\{\phi_\alpha^{(k)}/(t_\alpha)^k\}\{dz_\alpha^1\wedge\ldots\wedge dz_\alpha^n\wedge(dt_\alpha/t_\alpha)\}^m = \{\phi_\beta^{(k)}/(t_\beta)^k\}\{dz_\beta^1\wedge\ldots\wedge dz_\beta^n\wedge(dt_\beta/t_\beta)\}^m$$

and hence the homomorphism is surjective.

Let M_1, M_2 be compact complex manifolds of dimensions n_1, n_2, respectively. There exist natural projections p_i from $M = M_1 \times M_2$ to M_i, $i = 1, 2$. Obviously $p_1^*K_1+p_2^*K_2$ is the canonical line bundle of M, which we denote by K. Suppose F_1, F_2 be positive line bundles on M_1, M_2, respectively. $F = p_1^*F_1+p_2^*F_2$ is also positive. The zero section, which is identified with M, of the total space of F* is contractible. By blowing down M, we get an (n_1+n_2+1)-dimensional normal isolated singularity. Using Theorem 1.7, we get

$$\begin{aligned}\delta_m &= \sum_{k\geqq 0} \dim \Gamma(M,\mathcal{O}(mK-kF)) \\ &= \sum_{k\geqq 0} \dim \Gamma(M_1\times M_2,\mathcal{O}(m(p_1^*K_1+p_2^*K_2)-k(p_1^*F_1+p_2^*F_2))) \\ &= \sum_{k\geqq 0} \{\dim \Gamma(M_1,\mathcal{O}(mK_1-kF_1)) \times \dim \Gamma(M_2,\mathcal{O}(mK_2-kF_2))\}\end{aligned}$$

(by Künneth formula) .

Proposition 1.8. If K_2 is trivial and K_1 is positive, then $\delta_m \sim m^{n_1}$. In fact $\limsup_{m\to\infty} \delta_m/m^{n_1} = (1/n_1!)\{c_1(K_1)\}^{n_1}$, where $c_1(K_1)$ is the first Chern class of K_1.

Example 1.9. Let (X,x) be the n-dimensional normal isolated singularity obtained by blowing down the zero section, denoted by M, of a negative line bundle. Then

(1) K_M : negative $\Longrightarrow \delta_m = 0$ ($m \geqq 1$)

(2) K_M : trivial $\Longrightarrow \delta_m = 1$ ($m \geqq 1$)

(3) K_M : positive $\Longrightarrow \limsup_{m \to \infty} \delta_m/m^n > 0$.

Next we consider a normal isolated singularity defined by a quasi-homogeneous polynomial.

Definition 1.10. Suppose that $(r_0,r_1,\ldots,r_n)$ are fixed positive rational numbers. A polynomial $f(z_0,z_1,\ldots,z_n)$ is said to be quasi-homogeneous of type $(r_0,r_1,\ldots,r_n)$ if it can be expressed as a linear combination of monomials $z_0^{i_0}z_1^{i_1}\ldots z_n^{i_n}$ for which $i_0r_0+i_1r_1+\ldots+i_nr_n = 1$.

Let d denote the smallest positive integer so that $r_0d = q_0$, $r_1d = q_1$, ..., $r_nd = q_n$ are integers. Then $f(t^{q_0}z_0,\ldots,t^{q_n}z_n) = t^d f(z_0,\ldots,z_n)$.

Theorem 1.11. Let (X,x) be an n-dimensional normal isolated singularity defined by a quasihomogeneous polynomial f of type $(r_0,r_1,\ldots,r_n)$. Then (X,x) is rational in the sense of Burns [4] if and only if $r(f) = r_0+r_1+\ldots+r_n > 1$.

Proof. By virture of [4, Proposition (3.2), p.239] it suffices to show that $\omega = dz_1\wedge\ldots\wedge dz_n/\frac{\partial f}{\partial z_0}$ is square integrable in a neighborhood of x. Let d denote the smallest positive integer such that there exists, for each i, an integer q_i so that $r_id = q_i$. Let $\phi : \mathbb{C}^{n+1} \to \mathbb{C}^{n+1}$ be defined by $\phi(w_0,\ldots,w_n) = (w_0^{q_0},\ldots,w_n^{q_n})$ and let $X' = \phi^{-1}(X)$. Then X' is defined by a homogeneous polynomial ϕ^*f of degree d with the singular locus $S(X')$. Let

$\underline{A}$ be an irreducible hypersurface defined by ϕ^*f in $\mathbb{P}_n$; X' is a "cone over $\underline{A}$", and let $\rho : A \to \underline{A}$ be a resolution of the singularity. Suppose that $H_A^* \to A$ is the line bundle induced on A by the "tautological" line bundle on $\mathbb{P}_n$ (dual to the hyperplane bundle). Then the "tautological" map $\pi : H_A^* \to X'$ is a resolution of the singularity with $\pi^{-1}(0) = A$. Since $\phi^*\omega$ is locally square integrable at any point $p \in S(X')-\{0\}$, $\pi^*\phi^*\omega$ is holomorphic off A. By easy calculation, $\pi^*\phi^*\omega$ has zero of order $q_0+ q_1+ \cdots + q_n- d - 1$ at A. From [13], ω is square integrable if and only if $q_0+ q_1+ \cdots + q_n- d - 1 \geqq 0$, i.e., $r_0+ r_1+ \cdots + r_n > 1$. □

Corollary 1.12 (Burns [4]). Arnold's singularities [1] are rational.

When (X,x) is defined by a quasihomogeneous polynomial, then δ_m is completely determined by its weights $\{ r_0,r_1,\ldots,r_n \}$.

Theorem 1.13. Let (X,x) be an n-dimensional normal isolated singularity defined by a quasihomogeneous polynomial f of type $(r_0,r_1,\ldots, r_n)$. Let d denote the smallest positive integer such that there exists, for each i, an integer q_i so that $r_i d = q_i$. Then

$$\delta_m(X,x) = \#\{(\lambda_0,\ldots,\lambda_n) \in \mathbb{N}^{n+1} \mid m\{d-(q_0+\cdots+q_n)\} \geqq \lambda_0 q_0+\cdots+\lambda_n q_n \}$$

$$- \#\{(\lambda_0,\ldots,\lambda_n) \in \mathbb{N}^{n+1} \mid m\{d-(q_0+\cdots+q_n)\}-d \geqq \lambda_0 q_0+\cdots+\lambda_n q_n \}.$$

Proof. If θ is any holomorphic m-ple n-form on X-{x}, $g = \theta/\omega^m$ is a holomorphic function on X-{x} and hence extends to be holomorphic also at x. Expand g in a power series : $g = \sum a_{\lambda_0\ldots\lambda_n} z_0^{\lambda_0}\cdots z_n^{\lambda_n}$. Notation being as in Theorem 1.11, $\pi^*\phi^*z^\lambda\omega^m$ has zeros of order

$$\lambda_0 q_0 + \lambda_1 q_1 + \cdots + \lambda_n q_n + m(q_0 + q_1 + \cdots + q_n - d - 1)$$

at A. From [19, Theorem 2.1, p.243], $z^\lambda \omega^m \in L^{2/m}(X-\{x\})$ if and only if

$$0 \leqq (m-1) + \{ \lambda_0 q_0 + \lambda_1 q_1 + \cdots + \lambda_n q_n + m(q_0 + q_1 + \cdots + q_n - d - 1)\}$$

Hence

$$\theta \sim \tilde{g}\omega^m \quad (\bmod L^{2/m}(X-\{x\}))$$

where $\tilde{g} = \sum a_{\lambda_0\lambda_1\ldots\lambda_n} z_0^{\lambda_0} z_1^{\lambda_1} \ldots z_n^{\lambda_n}$ with

$$m\{d-(q_0+q_1+\cdots+q_n)\} \geqq \lambda_0 q_0 + \lambda_1 q_1 + \cdots + \lambda_n q_n \quad .$$

Assume moreover that $\tilde{g} \equiv 0$ on X. Then there exists a polynomial p(z) such that $\tilde{g}(z) = f(z)p(z)$, where $p(z) = \sum b_{\lambda_0\lambda_1\ldots\lambda_n} z_0^{\lambda_0} z_1^{\lambda_1} \ldots z_n^{\lambda_n}$ with

$$m\{d-(q_0+q_1+\cdots+q_n)\} - d \geqq \lambda_0 q_0 + \lambda_1 q_1 + \cdots + \lambda_n q_n \; .$$

Thus we get the desired result. □

Corollary 1.14.

$$p_g(X,x) = \#\{ (\lambda_0,\ldots,\lambda_n) \in \mathbb{N}^{n+1} | \; d-(q_0+\cdots+q_n) \geqq \lambda_0 q_0 + \cdots + \lambda_n q_n \} \; .$$

One can easily check that Theorem 1.13 gives the following example.

Example 1.15. If we are as above, then

$r(f) > 1 \iff \delta_m = 0$, for $m \geqq 1$,

$r(f) = 1 \iff \delta_m = 1$, for $m \geqq 1$,

$$r(f) < 1 \iff \limsup_{m \to \infty} \delta_m/m^n = (1/n!)(1-r(f))^n(1/r_0 r_1 \ldots r_n) .$$

Let k be a totally real field of degree n over the rationals and M an additive subgroup of k which is a free abelian group of rank n. Let U_M^+ be the group of those units ε of k which are totally positive and satisfy $\varepsilon M = M$. For a given pair (M,E) with $E \subset U_M^+$ (where E has rank n-1) one defines

$$G(M,E) = \{ \begin{pmatrix} \varepsilon & \mu \\ 0 & 1 \end{pmatrix} \mid \varepsilon \in E, \mu \in M \} .$$

Let $\mathbb{H}$ be the upper half plane. The group G(M,E) operates freely and properly discontinuously on $\mathbb{H}^n$ by $z_j \mapsto \varepsilon^{(j)} z_j + \mu^{(j)}$, where $x \mapsto x^{(j)}$, $1 \leqq j \leqq n$, denote the n different embeddings of k into the reals. Then $\mathbb{H}^n/G(M,E)$ defined a complex manifold which acquires a normal singularity when an additional point ∞ is added with neighborhoods

$$|\mathrm{Im}(z_1)\mathrm{Im}(z_2)\ldots\mathrm{Im}(z_n)| > c ,$$

where c is a constant. The singularity at ∞ will be called a cusp singularity of type (M,E).

Theorem 1.16. Let (X,x) be a cusp singularity. Then $\delta_m(X,x) = 1$ for any $m \geqq 1$.

The proof will be found in [26]. In the 2-dimensional case the proof was given in [6].

§2. Normal Surface Singularities

We begin by recalling some theorems and definitions in Section 1. Let (X,x) be a (germ of) 2-dimensional normal singularity and $\pi : \tilde{X} \to X$ be a resolution of the singularity. We assume that V is a Stein neighborhood of x in X. Then $U = \pi^{-1}(V)$ is a strongly pseudoconvex neighborhood of $A = \pi^{-1}(x)$. Let K be the canonical line bundle of U. The following integer is defined by Knöller [11] :

$$\gamma_m(X,x) = \dim \Gamma(U-A,\mathcal{O}(mK))/\Gamma(U,\mathcal{O}(mK)) \quad (m \geqq 1) .$$

This integer is determined independently to the choice of the Stein neighborhoods. Hence γ_m is an invariant attached to the singularity. Knöller considered the asymptotic behavior of γ_m when $m \to +\infty$, and showed

Theorem 2.1 (Knöller [11]). There is a positive constant c such that $\gamma_m \leqq cm^2$ for a 2-dimensional normal singularity.

By the definition $\Gamma(U,\mathcal{O}(mK)) \subset L^{2/m}(U-A)$, we have $\delta_m \leqq \gamma_m$. Therefore we obtain the following.

Theorem 2.2. For any 2-dimensional normal singularity, we have

$$\delta = \limsup_{m \to \infty} \delta_m/m^2 < \infty .$$

We call this theorem the first fundamental theorem of $\{ \delta_m \}$ for 2-dimensional normal singularities.

Let $\pi^{-1}(x) = A = \cup A_i$, $1 \leqq i \leqq n$, be the decomposition of the exceptional set A into irreducible components. We associate a weighted graph Γ to π in the following eay.

Definition 2.3. We associate to a resolution π a weighted graph Γ_π with weighted vertices $\gamma_i(b_i, g_i)$, $i = 1,\ldots, n$ where $b_i = A_i \cdot A_i$ and g_i = genus(A_i); γ_i and γ_j are joined by 1-simplex if $A_i \cap A_j \neq \phi$. A vertex of weight (b,g) is denoted by $\underset{[g]}{ⓑ}$ and ⓑ will denote $\underset{[0]}{ⓑ}$.

Definition 2.4. A vertex γ_i of Γ is said to be center if either $g_i >$ 0 or $g_i = 0$ and γ_i is joined to at least three other vertices. We say Γ is star-shaped if there is at most one center.

Definition 2.5. If γ is a vertex of Γ we defined Γ-$\{\gamma\}$ to be the weighted graph obtained from Γ by removing γ and all edges joined to γ. If γ is the center of a star-shaped graph then the components of Γ-$\{\gamma\}$ are called the branches of Γ.

By a cycle, we shall mean an element of the vector space over the rational numbers generated by $\{ A_i \}$. A cycle $Z = \sum r_i A_i$ is called effective if all r_j's are non-negative. A cycle $Z = \sum r_i A_i$ is called integral if all r_j's are rational integers. A cycle $Z = \sum r_i A_i$ is positive if Z is effective and $r_j > 0$ for some j. We let $|Z| = \cup A_i$, $r_i \neq 0$, denote the support of Z. We abbreviate a positive integral cycle to a PI-cycle.

In the following, by a curve we shall mean a compact irreducible 1-dimensional analytoc subset of $\tilde{X}$, i.e., a curve is the one of $\{ A_j \}$.

The intersection number $Z_1 \cdot A_2$ of cycles Z_1 and Z_2 can be naturally defined. Note that this is in general a rational number. Let F be a line bundle on $\tilde{X}$. We define the intersection number $F \cdot C$ of F with a curve C by the degree of the line bundle $F|_C$ restricted to C. Denote by [Z] the line bundle over $\tilde{X}$ defined by the integral cycle Z. Then it is easy to see that $[Z]\cdot C = Z\cdot C$ for any curve C. Since the intersection matrix $S = (A_i \cdot A_j)$ is a non-singular matrix, we can uniquely determine the cycle $Z = \sum r_i A_i$ satisfying

$$Z\cdot C = F\cdot C \qquad (*)$$

for all curves C. In general det(S) is not ± 1, therefore the coefficients r_j may by rational numbers. The cycle Z defined by (*) is said to be numerically equivalent to F. We define the intersection number of line bundles F_1 and F_2 by $F_1\cdot F_2 = Z_1\cdot Z_2$, where Z_i are numerically equivalent cycles to F_i $(i = 1,2)$. $(\det(S))F_1\cdot F_2$ is an integer. It is easy to check that for integral cycles Z_1 and Z_2 we have $[Z_1]\cdot[Z_2] = Z_1\cdot Z_2$.

We define the virtual genus of PI-cycle Z to be

$$p(Z) = (1/2)(Z\cdot Z+K\cdot Z)+1 ,$$

where K is the canonical line bundle on $\tilde{X}$. Now we define

$$p_a(X,x) = \sup p(Z)$$

where Z ranges over all PI-cycle on $\tilde{X}$.

The fundamental cycle is the unique PI-cycle Z_0 on $\tilde{X}$ such that

(1) $Z_0\cdot A_i \leqq 0$ for every component A_i of $\pi^{-1}(x)$,

(2) if Z is a PI-cycle such that $Z\cdot A_i \leqq 0$ for any i, then $Z \geqq Z_0$.

The existence of Z_0 is shown by Artin [2]. Given the intersection matrix $(A_i\cdot A_j)$, one can easily determine Z_0. We define

$$p_f(X,x) = p(Z_0) .$$

In fact p_a and p_f are independent of the choice of the resolution (for details see [24]).

Definition 2.6. We call $p_a(X,x)$ the arithmetic genus of (X,x).

The invariants defined thus far are not independent. One can easily see that $p_g \geqq p_a \geqq p_f$. Furthermore Artin [2] has proven the following theorem :

$$p_g = 0 \iff p_a = 0 \iff p_f = 0 .$$

Moreover Wagreich [24] has proven that $p_a = 1 \iff p_f = 1$.

Definition 2.7. Let (X,x) be a normal surface singularity. We say (X,x) is rational (resp. elliptic) if $p_g(X,x) = 0$ (resp. $p_g(X,x) = 1$).

Remark. For the definirion of elliptic singularities some authors work instead with p_a : (X,x) is elliptic $\iff p_a(X,x) = 1$. In this case they say (X,x) is strongly elliptic if $p_g(X,x) = 1$.

Let A' be any connected proper subvariety of A. Then A' is exceptional in U by [12], Lemma 5.11, p.89. A' has a pseudoconvex neighborhood U'. Blowing down A', we get a normal surface singularity, which is denoted by (X',x'). The singularity which appears in this way will be simpler than the original singularity (X,x), provided that A is the exceptional set of the minimal resolution.

We call the following theorem the second fundamental theorem of $\{ \delta_m \}$ for normal surface singularities.

Theorem 2.8. In the case of a minimal resolution, for any $m \geqq 1$, we have

$$\gamma_m(X,x) \geqq \gamma_m(X',x') ,$$

$$\delta_m(X,x) \geqq \delta_m(X',x') .$$

Proof. Let $A^{0,q}(F)$ be the sheaf of germs of (0,q)-forms with coefficients in a complex analytic line bundle F. Then we have a fine resolution $\{ A^{0,q}(F) \}$ of $\mathcal{O}(F)$.

Following Laufer [13], we have a diagram :

$$\begin{array}{ccccccccc} 0 \to \Gamma(U',\mathcal{O}(mK)) & \to & \Gamma(U'-A',\mathcal{O}(mK)) & \to & H^1_*(U',\mathcal{O}(mK)) & \to & \cdots \\ & & & & \downarrow & & \\ 0 \to \Gamma(U,\mathcal{O}(mK)) & \to & \Gamma(U-A,\mathcal{O}(mK)) & \to & H^1_*(U,\mathcal{O}(mK)) & \to & H^1(U,\mathcal{O}(mK)) . \end{array}$$

Given an element $\omega \in \Gamma(U'-A',\mathcal{O}(mK))$, there exists $\xi \in \Gamma(U',A^{0,0}(mK))$ such that $\xi = \omega$ near the boundary of U'. $\bar\partial\xi = \bar\partial\omega = 0$ near the boundary of U'. Hence $\bar\partial\xi$ has compact support, i.e., $\bar\partial\xi \in \Gamma_*(U',A^{0,1}(mK))$. $\bar\partial(\bar\partial\xi) = 0$, so $\bar\partial\xi$ is a cocycle in $H^1_*(U',\mathcal{O}(mK))$. Let $\widetilde{\bar\partial\xi}$ be the zero extension of $\bar\partial\xi$ from U' to U. Then $\bar\partial(\widetilde{\bar\partial\xi}) = 0$, and $\widetilde{\bar\partial\xi}$ is a cocycle in $H^1_*(U,\mathcal{O}(mK))$. By [10], Vanishing Theorem, p.246, $H^1(U,\mathcal{O}(mK)) = 0$. Therefore $\widetilde{\bar\partial\xi}$ is the $\bar\partial$-image of some $\zeta \in \Gamma(U,A^{0,0}(mK))$: $\bar\partial\zeta = \widetilde{\bar\partial\xi}$. Since $\widetilde{\bar\partial\xi}$ is zero near the boundary of U, ζ is holomorphic there. By Siu [20], p.374, there exists $\tilde\omega \in \Gamma(U-A,\mathcal{O}(mK))$ such that $\tilde\omega = \zeta$ near the boundary of U. It is easy to check that the map $\omega \mapsto \tilde\omega$ induces a well-defined homomorphism

$$\Gamma(U'-A',\mathcal{O}(mK)) \to \Gamma(U-A,\mathcal{O}(mK))/\Gamma(U,\mathcal{O}(mK)) \to \Gamma(U-A,\mathcal{O}(mK))/L^{2/m}(U-A) .$$

ζ is holomorphic outside of some compact set in U'. $\tilde\omega$ has possible poles on A'. Since $\bar\partial(\zeta-\xi) = 0$ on U', $\zeta-\xi = \lambda$, $\lambda \in \Gamma(U',\mathcal{O}(mK))$. Hence $\tilde\omega-\omega = \lambda$ on U'.

Therefore, if $\tilde{\omega} \in \Gamma(U,\mathcal{O}(mK))$, then $\omega \in \Gamma(U',\mathcal{O}(mK))$; besides, if $\tilde{\omega} \in L^{2/m}(U-A)$, then $\omega \in L^{2/m}(U'-A')$. Thus homomorphisms

$$\phi : \Gamma(U'-A',\mathcal{O}(mK))/\Gamma(U',\mathcal{O}(mK)) \to \Gamma(U-A,\mathcal{O}(mK))/\Gamma(U,\mathcal{O}(mK))$$

$$\psi : \Gamma(U'-A',\mathcal{O}(mK))/L^{2/m}(U'-A') \to \Gamma(U-A,\mathcal{O}(mK))/L^{2/m}(U-A)$$

are defined and injective. □

Remark. Note that $\Omega \in \Gamma(U-A,\mathcal{O}(mK))$ having poles exactly on A is not in the image of ϕ and ψ respectively.

Corollary 2.9. Let (X,x) be a Gorenstein singularity, i.e., there is some neighborhood V of x in X and a holomorphic 2-form ω on $V-\{x\}$ such that ω has no zeros on $V-\{x\}$. If $p_g(X,x) \geqq 1$, then $p_g(X,x) > p_g(X',x')$.

Proof. Let $\pi : \tilde{X} \to X$ be the minimal resolution of the singularity. The support of $\pi^*\omega$ is empty or $A = \pi^{-1}(x)$. If $\pi^*\omega \in \Gamma(U,\mathcal{O}(K))$, then (X,x) is a rational singularity, and so $p_g(X,x) = 0$, a contradiction. Hence the support of $\pi^*\omega$ is A. Thus $\pi^*\omega$ is not in the image of $\phi_1 = \psi_1$ for any (U',A') as in the proof of Theorem 2.8, where ϕ_1 (resp. ψ_1) denotes ϕ (resp. ψ) with m = 1. □

Arnold defined the inner modality for quasihomogeneous isolated singularities. In the 2-dimensional case, the following theorem is proved in [29].

Theorem. Let μ_0 be the inner modality of a quasihomogeneous isolated singularity of dimension 2. Then $p_g \leqq \mu_0 \leqq \delta_2$. Furthermore,

$$p_g = 1 \implies \mu_0 \leq 4 \quad ; \quad \mu_0 = 1 \text{ or } 2 \implies p_g = 1 .$$

Definition 2.10. A normal surface singularity (X,x) with $p_a \geq 1$ is minimal if $p_g(X,x) > p_g(X',x')$ for any (X',x'). Moreover (X,x) is minimally elliptic if $p_a = 1$ and $p_g(X',x') = 0$ for any (X',x') (see Laufer [15, Theorem 3.4 (3)]).

Remark. The definition of "minimally elliptic" is equivalent to that of Laufer [15, p.1263].

Theorem 2.11. Let (X,x) be a normal surface singularity. If (X,x) is Gorenstein and $p_a \geq 1$, then (X,x) is minimal.

Proof. Obvious by Corollary 2.9. □

Corollary 2.12 (Laufer [15]). If (X,x) is Gorenstein and $p_g(X,x) = 1$, then (X,x) is a minimally elliptic singularity.

Theorem 2.13. Let (X,x) be a normal surface singularity. If (X,x) is minimal and $p_a = 1$, then (X,x) is Gorenstein.

Proof. Suppose $p_g = n$, then $n = p_g \geq p_a = 1$. Choose $\omega_1, \omega_2, \ldots, \omega_n \in \Gamma(U-A,\mathcal{O}(K))$ to be a basis for $\Gamma(U-A,\mathcal{O}(K))/\Gamma(U,\mathcal{O}(K))$, where U is a strongly pseudoconvex neighborhood of A. Let C_i (resp. D_i) be the pole (resp. zero) locus of ω_i, then $K = (\omega_i) = -C_i + D_i$, and C_i is a positive cycle. By the definition of p_a,

$$1 = p_a \geq p(C_i) = (1/2)(C_i \cdot C_i + C_i \cdot K) + 1 = (1/2)C_i \cdot D_i + 1 .$$

Since $C_i \cdot D_i \geq 0$, $p(C_i) = 1$.

Should $|C_i| \cap |C_j| = \phi$ for some i and j, then there would exist a positive cycle Z such that $p(Z) \geqq 0$, $Z \cdot C_i \geqq 1$ and $Z \cdot C_j \geqq 1$, so

$$1 = p_a \geqq p(C_i + Z + C_j) = p(C_i) + p(Z) + p(C_j) + C_i \cdot Z + C_j \cdot Z + C_i \cdot C_j - 2 \geqq 2 ,$$

which is a contradiction. Hence $\cup |C_i|$ is a connected analytic subvariety of A. Since (X,x) is minimal, $A = \cup |C_i|$. Now consider linear combinations of $\{ \omega_i \}$ with coefficients $\sum \alpha_i \omega_i$. In those forms there exists a meromorphic 2-form ω such that the supprot of the pole of ω is A. We write the divisor (ω) as $(\omega) = -C + D$ where C is the pole locus and D is the zero locus of ω respectively. From the definition of p_a,

$$1 = p_a \geqq p(C) = (1/2)(C \cdot C + K \cdot C) + 1 = (1/2) C \cdot D + 1 .$$

Since $C \cdot D \geqq 0$, $C \cdot D = 0$. Hence ω has no zeros near A. Thus (X,x) is Gorenstein. □

Lemma 2.14. With the notation being above let f be a holomorphic function on U and non-vanishing off A. Then $f(A) \neq 0$.

Proof. Suppose $f(A) = 0$. Then $(f) = Z$ where $Z = \sum d_i A_i$. Now $(f) \cdot A_i = 0$ for any i, therefore $Z \cdot Z = 0$. It contradicts the fact that the intersection matrix $(A_i \cdot A_j)$ is negative definite. □

In the case where (X,x) is a minimally elliptic singularity we get more information about the singularity.

Theorem 2.15 (Laufer [15]). Let (X,x) be a minimally elliptic singularity. Then (X,x) is Gorenstein and $p_g(X,x) = 1$.

Proof. Let U be a strongly pseudoconvex neighborhood of the exceptional set A. Since $p_g \geqq p_a = 1$, there exists at least one non-zero element in $\Gamma(U-A,\mathcal{O}(K))/\Gamma(U,\mathcal{O}(K))$. Let $\omega \in \Gamma(U-A,\mathcal{O}(K))$ be the representative of the above element. Denote by C (resp. D) the pole (resp. zero) locus of ω, then $K = (\omega) = -C + D$, where C is a positive cycle. So

$$1 = p_a \geqq p(C) = (1/2)(C\cdot C + C\cdot K) + 1 = (1/2)C\cdot D + 1$$

Sinec $C\cdot D \geqq 0$, $C\cdot D = 0$. Now (X,x) is minimally elliptic, therefore $|C| = A$. Hence we may assume that ω does not vanish off A. This implies (X,x) is Gorenstein.

Let ω' be another non-zero element in $\Gamma(U-A,\mathcal{O}(K))/\Gamma(U,\mathcal{O}(K))$. By the similar argument ω' does not vanish off A and the support of its pole locus is A. Then $f = \omega'/\omega$ is a nowhere vanishing holomorphic function on U-A and hence extends to be holomorphic also at A. We claim that

$$(f-f(A))\omega \in \Gamma(U,\mathcal{O}(K)) .$$

Suppose otherwise, i.e., $(f-f(A))\omega \notin \Gamma(U,\mathcal{O}(K))$. By the same argument $(f-f(A))\omega$ does not vanish off A. By Lemma 2.14 $(f-f(A))$ does not vanish on A, which is a contradiction. Therefore

$$\omega'-f(A)\omega = (f-f(A))\omega \in \Gamma(U,\mathcal{O}(K)) .$$

Thus $p_g = \dim \Gamma(U-A,\mathcal{O}(K))/\Gamma(U,\mathcal{O}(K)) = 1$. □

Theorem 2.16. Let (X,x) be a normal surface singularity. If (X,x) is Gorenstein and $p_g \geqq 2$, then $1 \leqq p_f < p_g$.

Proof. We assume $\pi : \tilde{X} \to X$ to be the minimal resolution of the singularity. Let U be a strongly pseudoconvex neighborhood of the exceptional set $\pi^{-1}(x) = A$. Suppose that $p_f = p_g$. Then $p(Z_0) = p_g \geqq 2$.

Kato's theorem [10], p.246, says

$$\dim \Gamma(U-A,\mathcal{O}(-Z_0))/\Gamma(U,\mathcal{O}(-Z_0)) + \dim H^1(U,\mathcal{O}(-Z_0)) = (1-p(Z_0)) + p_g(X,x) .$$

By the hypothesis the right hand side is equal to 1. Since $|Z_0| = A$, a non-zero constant function is not a zero element in

$$\Gamma(U-A,\mathcal{O}(-Z_0))/\Gamma(U,\mathcal{O}(-Z_0)) ,$$

so $\dim \Gamma(U-A,\mathcal{O}(-Z_0))/\Gamma(U,\mathcal{O}(-Z_0)) \geqq 1$. Hence $\dim H^1(U,\mathcal{O}(-Z_0)) = 0$. Now consider the sheaf cohomology with support at infinity. Let K be the canonical line bundle of U. Then the following sequence is exact :

$$0 \to \Gamma(U,\mathcal{O}(K+Z_0)) \underset{\phi}{\to} \Gamma(U-A,\mathcal{O}(K+Z_0)) \to H^1_*(U,\mathcal{O}(K+Z_0)) \to \cdots .$$

Serre duality gives $H^1(U,\mathcal{O}(-Z_0))$ as dual to $H^1_*(U,\mathcal{O}(K+Z_0))$. Then ϕ is an isomorphism. As (X,x) is Gorenstein, there is a holomorphic 2-form ω on $U-A$ such that ω has no zeros on $U-A$. Let $(\omega) = \sum_{i=1}^n \lambda_i A_i$ denote the divisor of ω, where A_i $(i = 1,2,\ldots,n)$ are the irreducible components of A. Then we obtain n linear equations

$$K\cdot A_j = (\omega)\cdot A_j = \sum_{i=1}^n \lambda_i A_i \cdot A_j \quad (j = 1,2,\ldots,n)$$

in n unknowns $\lambda_1,\lambda_2,\ldots,\lambda_n$. Since (X,x) is not a rational double point, $K\cdot A_{j_0} > 0$ for some j_0 [cf. 11]. By Lemma 3.2 $\lambda_i < 0$ for all i, i.e., $\lambda_i \leqq -1$ for all i. Then $-(\omega)$ is a PI-cycle. Now (ω) is a cycle on A and $\omega+Z_0 \in \Gamma(U-A,\mathcal{O}(K+Z_0))$. Hence $\omega+Z_0 \in \Gamma(U,\mathcal{O}(K+Z_0))$, so $Z_0 \geqq -(\omega)$. Since π is the minimal resolution, $0 \leqq K\cdot A_j = (\omega)\cdot A_j$ for any j, so $-(\omega) \geqq Z_0$ by the minimality of the fundamental cycle Z_0. Therefore $K = (\omega) = -Z_0$. Thus $p(Z_0) = 1$, which is a contradiction. □

Corollary 2.17 (Yoshinaga-Ohyanagi [28]). Let (X,x) be a normal surface singularity. If (X,x) is Gorenstein and $p_g = 2$, then $p_a = 1$.

Proof. By Theorem 2.16 $p_f = 1$, and which implies $p_a = 1$. □

Thus, by Theorems 2.11 and 2.13 we obtain the following theorems.

Theorem 2.18. Let (X,x) be a normal surface singularity with $p_g = 2$. Then (X,x) is Gorenstein if and only if (X,x) is minimal and $p_a = 1$.

Theorem 2.19. Let (X,x) be a normal surface singularity with $p_a = 1$. Then (X,x) is Gorenstein if and only if (X,x) is minimal.

A resolution $\tilde{X} \to X$ of a normal surface singularity (X,x) is good, if

(i) All the components of the exceptional set of $\tilde{X} \to X$ are smooth and intersect transversely.

(ii) Not more than two components pass through any given point.

(iii) Two different components intersect at most once.

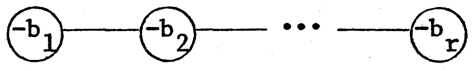
It is well-known (and easy to see) that there is a minimal resolution having these properties.

Now we give δ_m-formula for the normal surface singularity whose minimal good resolution is star-shaped. In what follows we consider the normal surface singularity whose minimal good resolution is star-shaped. Let A_0 be the center of the weighted graph. The branches of the star-shaped graph are indexed by i, $1 \leqq i \leqq n$. The curves of the i-th branch are denoted by A_{ij}, $1 \leqq j \leqq r_i$, where A_{i1} intersects A_0 and A_{ij} intersects $A_{i,j+1}$. Let $-b = A_0 \cdot A_0$, and $-b_{ij} = A_{ij} \cdot A_{ij}$. Then $b_{ij} \geqq 2$ and $b \geqq 1$. Finally, set

$$d_i/e_i = b_{i1} - \cfrac{1}{b_{i2} - \cfrac{1}{\ddots \cfrac{}{} - \cfrac{1}{b_{ir_i}}}}$$

$= [b_{i1}, b_{i2}, \ldots, b_{ir_i}]$ with $e_i < d_i$, and e_i and d_i are relatively prime.

Lemma 2.20 (Brieskorn [3]). Let $\tilde{X}$ be the minimal good resolution of a normal surface singularity (X,x) such that the weighted (dual) graph is



Let $d/e = [b_1, b_2, \ldots, b_r]$, e and d relatively prime. Then X is analytically isomorphic to the quotient of $\mathbb{C}^2$ by the cyclic group G of order d, acting by $(z_1, z_2) = (\zeta z_1, \zeta^e z_2)$, where ζ is a d-th root of unity.

We call this singularity the cyclic quotient singularity of type (d,e).

For any $k \geqq 0$ and $m \geqq 1$ let $D_m^{(k)}$ be the divisor on A_0 :

$$D_m^{(k)} = kD - \sum_i [\{ke_i + m(d_i - 1)\}/d_i]P_i ,$$

where D is any divisor such that $\mathcal{O}_{A_0}(D)$ is the conormal sheaf of A_0, $P_i = A_0 \cap A_{i1}$, and for any $a \in \mathbb{R}$, $[a]$ is the grestest integer less than, or equal to a.

Theorem 2.21. In case the minimal good resolution of (X,x) is star-shaped, the plurigenus δ_m is not more than

$$\sum_{k \geqq 0} \dim \Gamma(A_0, \mathcal{O}_{A_0}(mK_{A_0} - D_m^{(k)})) .$$

Corollary 2.22. In the above situation the geometric genus $p_g(X,x)$ is not more than

$$\sum_{k \geqq 0} \dim H^1(A_0, \mathcal{O}_{A_0}(D_1^{(k)})) .$$

Under the condition that (X,x) admits a $\mathbf{C}^*$ action

$$\delta_m(X,x) = \sum_{k \geqq 0} \dim \Gamma(A_0, \mathcal{O}_{A_0}(mK_{A_0} - D_m^{(k)})) ,$$

and,

$$p_g(X,x) = \sum_{k \geqq 0} \dim H^1(A_0, \mathcal{O}_{A_0}(D_1^{(k)})) ,$$

which was proved by Pinkham [17].

Proof of the Theorem. Let $\{ U_i \}$ be a cover of A such that $P_i \in U_i$ for i = 1,..., n, and $P_j \notin U_i$, $i \neq j$. Assume moreover there exist local coordinates (z_i, t_i) with $A_0 \cap U_i = \{ P \in U_i \mid t_i(P) = 0 \}$ and $P_i = \{ P \in U_i \mid t_i(P) = z_i(P) = 0 \}$. Take a (sufficiently small) Stein neighborhood V of x so that $\cup U_i \supset \pi^{-1}(V) = U$. Let ω be any m-ple holomorphic 2-form on U-A : $\omega \in \Gamma(U-A, \mathcal{O}(mK))$. On U_i ω is written as $\omega|_{U_i} = \phi_i \{dz_i \wedge (dt_i/t_i)\}^m$. Expand ϕ_i in Laurent series on U_i: $\phi_i = \sum \phi_i^{(k)} t_i^{-k}$. The same argument as Theorem 1.7 works in this case, and so $\{ \phi_i^{(k)} (dz_i)^m \}$ becomes a meromorphic section of $mK_{A_0} + kN$ where N is the normal bundle of A_0 in U. Let ν_i be the order of the pole of $\phi_i^{(k)}$ at P_i. Then

$$\nu_i \leqq [\{ke_i + m(d_i - 1)\}/d_i] .$$

In order to prove this, it is sufficient to prove the following lemma, since each branch is the cyclic quotient singularity.

Lemma. Given $b_1, b_2, \ldots, b_r$ with the b_i integers such that $b_i \geqq 2$, the manifold $M = M(b_1,b_2,\ldots,b_r)$ will be covered by r+1 coordinate patches, $W_i = (u^{(i)},v^{(i)}) = \mathbb{C}^2$, $0 \leqq i \leqq r$, joined as follows.

$$\begin{array}{lll} W_0 \cap W_1 = \{ u \neq 0 \} & u' = 1/u & v' = u^{b_1}v \\ W_1 \cap W_2 = \{ v' \neq 0 \} & v'' = 1/v' & u'' = (v')^{b_2}u' \\ W_2 \cap W_3 = \{ u'' \neq 0 \} & u''' = 1/u'' & v''' = (u'')^{b_3}v'' \\ \vdots & \vdots & \vdots \end{array}$$

Let $A_0 = \{ u = 0 \}$, $A_1 = \{ v = 0 \} \cup \{ v' = 0 \}$, $A_2 = \{ u' = 0 \} \cup \{ u'' = 0 \}$, $A_3 = \{ v'' = 0 \} \cup \{ v''' = 0 \}$, ... , $A_r = \{ u^{(r-1)} = 0 \} \cup \{ u^{(r)} = 0 \}$, $A_{r+1} = \{ v^{(r)} = 0 \}$ if r is even and $A_r = \{ v^{(r-1)} = 0 \} \cup \{ v^{(r)} = 0 \}$, $A_{r+1} = \{ u^{(r)} = 0 \}$ if r is odd. Then $A' = A_1 \cup A_2 \cup A_3 \cup \cdots \cup A_r$ is a compact analytic subset of M.

We define positive integers $\{ \lambda_r, \lambda_{r-1}, \ldots, \lambda_0 \}$ as follows :

$$\begin{array}{ll} \lambda_r = 1 & \vdots \\ \lambda_{r-1} = b_r\lambda_r & \lambda_2 = b_3\lambda_3 - \lambda_4 \\ \lambda_{r-2} = b_{r-1}\lambda_{r-1} - \lambda_r & e = \lambda_1 = b_2\lambda_2 - \lambda_3 \\ \vdots & d = \lambda_0 = b_1\lambda_1 - \lambda_2 \; . \end{array}$$

Now consider a meromorphic m-ple 2-form

$$\omega = (du \wedge dv)^m / u^{k+m} v^{\nu}$$

on W_0. Then ω is holomorphic on $M-(A_0 \cup A')$ if and only if

$$\nu \leqq [\{ke+m(d-1)\}/d] \; .$$

Proof. Let $(\omega) = \sum_{i=0}^{r+1} -a_iA_i$ be the divisor of ω on M, where $a_0 = k + m$ and $a_1 = \nu$. Since $p(A_i) = 0$ for i = 1, ... , r,

$$m(-2 + b_i) = -a_{i-1} + b_i a_i - a_{i+1} .$$

Hence, by the definition of $\{ \lambda_i \}$

$$\begin{aligned} -a_{r+1} &= a_{r-1} - b_r a_r + m(-2 + b_r) \\ &= \lambda_r(a_{r-1} - m) + m(\lambda_{r-1} - 1) - a_r\lambda_{r-1} \\ &= \lambda_{r-1}(a_{r-2} - m) + m(\lambda_{r-2} - 1) - a_{r-1}\lambda_{r-2} \\ &\vdots \\ &= \lambda_1(a_0 - m) + m(\lambda_0 - 1) - a_1\lambda_0 \\ &= ek + m(d - 1) - \nu d . \end{aligned}$$

Thus ω is holomorphic on $M-(A_0 \cup A')$ if and only if $ek+m(d-1)-\nu d \geqq 0$, i.e.,

$$\nu \leqq [\{ek + m(d - 1)\}/d] . \qquad \square$$

Remark. If $k < 0$, then $a_0 = k + m < m$. Since

$$\lambda_1(a_0 - m) + \lambda_0(m - a_1) - m \geqq 0 ,$$

$\lambda_1(a_0 - m) > \lambda_0(a_1 - m)$, and so $a_1 < m$. Hence by induction, it is true that $a_i < m$ for $i = 2,3, \ldots , r$.

Now we continue our proof of Theorem 2.21. Since $\nu_i \leqq [\{ke_i+m(d_i-1)\}/d_i]$, $\{ \phi_i^{(k)}(dz_i)^m \}$ is a holomorphic section of

$$mK_{A_0} + kN + \sum_i [\{ke_i+m(d_i-1)\}/d_i]P_i \quad .$$

Therefore we have a homomorphism

$$\Gamma(U-A,\mathcal{O}(mK)) \to \bigoplus_{k\geqq 0} \Gamma(A_0,\mathcal{O}_{A_0}(mK_{A_0}-D_m^{(k)})) .$$

By the above remark the kernel of this mapping is

$$\Gamma(U,\mathcal{O}(mK+(m-1)A)) \cong L^{2/m}(U-A) \ .$$

Thus

$$\Gamma(U-A,\mathcal{O}(mK))/L^{2/m}(U-A) \to \bigoplus_{k\geq 0} \Gamma(A_0,\mathcal{O}_{A_0}(mK_{A_0}-D_m^{(k)}))$$

is injective.

Next consider the case where (X,x) admits a $\mathbb{C}^*$-action. For a holomorphic section $\{ \phi_i^{(k)}(dz_i)^m \}$ of $mK_{A_0}-D_m^{(k)}$,

$$\phi_i^{(k)} t_i^{-k}\{dz_i \wedge (dt_i/t_i)\}^m$$

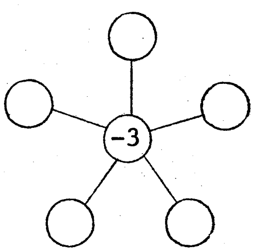
becomes a meromorphic section of $\mathcal{O}(mK)$, which is defined in the neighborhood of A_0, and extends to be a holomorphic section of $\mathcal{O}(mK)$ on U-A. Thus

$$\Gamma(U-A,\mathcal{O}(mK))/L^{2/m}(U-A) \cong \bigoplus_{k\geq 0} \Gamma(A_0,\mathcal{O}_{A_0}(mK_{A_0}-D_m^{(k)})) \ . \qquad \square$$

Example 2.23. Let (X,x) be defined by a quasihomogeneous polynomial $z_0^2+z_1^5+z_2^5$:

$$X = \{ (z_0,z_1,z_2) \in \mathbb{C}^3 \mid z_0^2+z_1^5+z_2^5 = 0 \} \ .$$

Then δ_m can be calculated by Theorem 1.13. The minimal resolution of (X,x) is as follows, so δ_m can also be computed directly from the above theorem.



Since $c_1(mK_{A_0}-D_m^{(k)}) = -2m-3k+5[(k+m)/2]$,

$$\delta_m = (m^2 + 3)/4, \quad \text{if m is odd,}$$

$$\delta_m = (m^2 + 8)/4, \quad \text{if m is even.}$$

by the Riemann-Roch theorem. In particular

$$\delta = \limsup_{m \to \infty} \delta_m/m^2 = 1/4 .$$

Using Theorem 2.21, T. Tomaru classified, by the behavior of δ_m, the singularities with $\mathbb{C}^*$ action.

Theorem 2.24 (Tomaru [21]). Normal surface singularities with $\mathbb{C}^*$ action are classified as follows : $g = \text{genus}(A_0)$, $L = \text{L.C.M.}(d_1, d_2, \ldots, d_n)$.

δ	δ_m	structure
> 0	δ_m diverges with second order as $m \to \infty$	(i) $g \geqq 2$ (ii) $g = 1$ and $n \geqq 1$ (iii) $g = 0$ and $\sum_{i=1}^{n} (d_i - 1)/d_i > 2$
0	(I) $\delta_m = 1$ for any $m \geqq 1$	$g = 1$ and $n = 0$
	(II) $\delta_m = \begin{cases} 0 \text{ if } m \not\equiv 0 \pmod{L} \\ 1 \text{ if } m \equiv 0 \pmod{L} \end{cases}$	$g = 0$ and $\sum_{i=1}^{n} (d_i - 1)/d_i = 2$
	(III) $\delta_m = 0$ for any $m \geqq 1$	$g = 0$ and $\sum_{i=1}^{n} (d_i - 1)/d_i < 2$, or cyclic quotient singularities

Corollary 2.25. If $\limsup_{m \to \infty} \delta_m/m^2 > 0$, then

$$\limsup_{m \to \infty} \delta_m/m^2 = (1/2)\{2g-2 + \sum_{i=1}^{n}(d_i - 1)/d_i\}^2/(b - \sum_{i=1}^{n} e_i/d_i) .$$

§3. Classification

Next we study the normal surface singularities such that δ_m is either 0 or 1 for all $m \geqq 1$.

Lemma 3.1. Let (a_{ij}) be negative definite matrix of rank n. Assume $a_{ij} \geqq 0$ for $i \neq j$ and $a_{ii} < 0$ for all i. Consider n linear equations

$$b_j = \sum_{i=1}^{n} x_i a_{ij} \quad \text{for } j = 1, \dots , n$$

in n unknowns $x_1, x_2, \dots , x_n$. If $b_j \geqq 0$ for all j, then any $x_i \leqq 0$.

Proof. The proof will be by induction on n. When n = 1, the equation is

$$b_1 = x_1 a_{11} \quad ,$$

and hence the lemma trivially holds. Therefore assume that the lemma holds for n-1. By the negative definiteness of (a_{ij}), for any $(x_j) \neq 0$,

$$\sum_{j=1}^{n} b_j x_j = \sum_{j=1}^{n} \sum_{i=1}^{n} x_i a_{ij} x_j < 0 \quad .$$

Therefore, if $x_i > 0$ for all i, then $b_{j_0} < 0$ for some j_0. This contradicts the assumption. Thus we can assume $x_{i_0} \leqq 0$ for some i_0. Consider n-1 linear equations

$$b_j + (-x_{i_0}) a_{i_0 j} = \sum_{i \neq i_0} x_i a_{ij} \quad \text{for } j \neq i_0 \quad .$$

By the induction hypothesis, $x_i \leqq 0$ for $i \neq i_0$, thus proving the theorem. □

Lemma 3.2. In the above situation we assume moreover that for any i there exists j such that $a_{ij} > 0$ in the case $n \geqq 2$. If $b_j \geqq 0$ for all j and $b_{j_0} > 0$ for some j_0, then any $x_i < 0$.

Proof. Quite similar to the above case. □

Let $\pi : \tilde{X} \to X$ be the minimal resolution of the normal surface singularity (X,x). The exceptional set $A = \pi^{-1}(x)$ is decomposed into irreducible components ; $A = \cup_{i=1}^{n} A_i$.

Proposition 3.3. Let $\sum_{i=1}^{n} k_i A_i$ be the cycle which is numerically equivalent to the canonical line bundle K of $\tilde{X}$. Then $k_i \leqq 0$ for $i = 1, \ldots, n$. If, moreover, K is not numerically trivial (i.e., (X,x) is not a rational double point), then $k_i < 0$ for $i = 1, \ldots, n$.

Proof. The virtual genus of A_j is

$$p(A_j) = (1/2)(A_j \cdot A_j + K \cdot A_j) + 1 .$$

Then we obtain n linear equations

$$-2 + 2p(A_j) - A_j \cdot A_j = (\textstyle\sum_{i=1}^{n} k_i A_i) \cdot A_j, \quad 1 \leqq j \leqq n ,$$

in n unknowns $k_1, k_2, \ldots, k_n$. Since π is the minimal resolution,

$$-2 + 2p(A_j) - A_j \cdot A_j \geqq 0 \quad \text{for all } j .$$

Moreover the intersection matrix is negative definite. Hence the rest part of the proof is obvious by Lemma3.1 and Lemma 3.2. □

Let A' be a connected proper analytic subset of A. Then A' is also exceptional ; and so there exists a strongly pseudoconvex neighborhood U' of A'. We may assume, without loss of generality, that $A' = \bigcup_{i=1}^{m} A_i$, $m < n$.

Theorem 3.4. Let A be the exceptional set of the minimal resolution. Let $\sum_{i=1}^{n} k_i A_i$ (resp. $\sum_{i=1}^{m} k_i' A_i$) be the cycle which is numerically equivalent to the canonical line bundle K of $\tilde{X}$ (resp. U'). Suppose that K is not numerically trivial. Then $k_i < k_i'$ for i = 1, ... , m. If K is numerically trivial, then $k_i' = 0$ for all i.

Proof. First suppose that K is not numerically trivial. Then we have two systems of linear equations. One is

$$-2 + 2p(A_j) - A_j \cdot A_j = (\textstyle\sum_{i=1}^{n} k_i A_i) \cdot A_j \quad \text{for } j = 1, \dots, n, \qquad (*)$$

and the other is

$$-2 + 2p(A_j) - A_j \cdot A_j = (\textstyle\sum_{i=1}^{m} k_i' A_i) \cdot A_j \quad \text{for } j = 1, \dots, m. \qquad (**)$$

From (*) and (**)

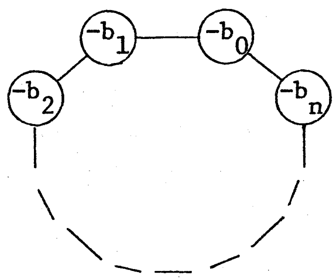
$$(\textstyle\sum_{i=m+1}^{n} k_i A_i) \cdot A_j = \textstyle\sum_{i=1}^{m} (k_i' - k_i) A_i \cdot A_j \quad \text{for } j = 1, \dots, m .$$

Since $k_i < 0$ for all i by Proposition 3.3, $A_i \cdot A_j \geqq 0$ for all $i \geqq m+1 > j$, and $A_i \cdot A_j > 0$ for some $i \geqq m+1 > j$ by the connectivity of A, we have $k_i' - k_i > 0$ for i = 1, ... , m, by Lemma 3.2. The latter statemant is clear. □

Proposition 3.5. Let (X,x) be a normal surface singularity with the minimal resolution $\pi : \tilde{X} \to X$. Let $\pi^{-1}(x) = A$. Denote by A_i (i = 1, ... , n) the irreducible component of A. Let $\sum_{i=1}^{n} k_i A_i$ be the cycle which is numerically equivalent to the canonical line bundle of $\tilde{X}$. We have $\delta_m = 0$ for all $m \geqq 1$ if and only if $-1 < k_i \leqq 0$ for all i.

Proof. Since $p_g = \delta_1 = 0$, (X,x) is a rational singularity. Then hK is defined by an integral cycle for some h, say $hK = \sum_{i=1}^{n} \mu_i A_i$. Hence there exists a meromorphic h-ple 2-forms ω such that $(\omega) = \sum_{i=1}^{n} \mu_i A_i$. ω is an element of $\Gamma(U-A, \mathcal{O}(hK))$ and $0 = \delta_h = \dim \Gamma(U-A, \mathcal{O}(hK))/L^{2/h}(U-A)$. Thus ω is an element of $L^{2/h}(U-A)$. (X,x) is rational, and so A is an integral cycle of normal crossings. Hence ω is the element of $\Gamma(U, \mathcal{O}(hK+(h-1)A))$, i.e., $-h < \mu_i$ for all i. By Proposition 3.3, $k_i \leqq 0$. Therefore $-1 < \mu_i/h = k_i \leqq 0$. □

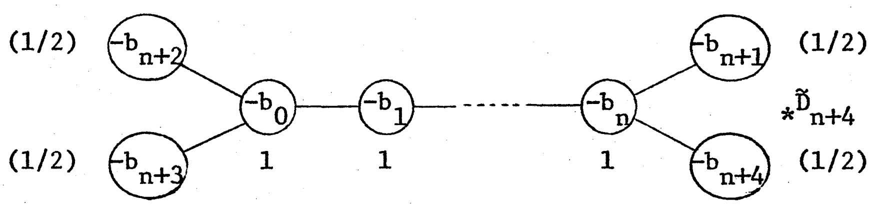
Example 3.6. Suppose G is of the form



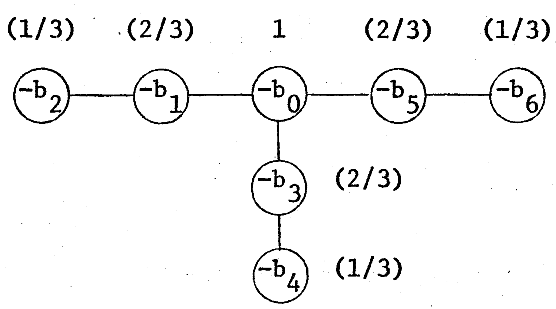
$_*\tilde{A}_n$ $(n \geqq 1)$

Then $K \sim \sum_{i=0}^{n} -A_i$. In fact K is linearly equivalent to $\sum_{i=0}^{n} -A_i$, i.e., (X,x) is Gorenstein (cf. Theorem 2.15).

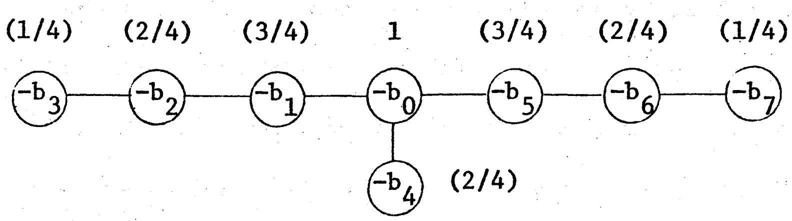
Proposition 3.7. Suppose G =



$_*\tilde{D}_{n+4}$ $(n \geqq 0)$,

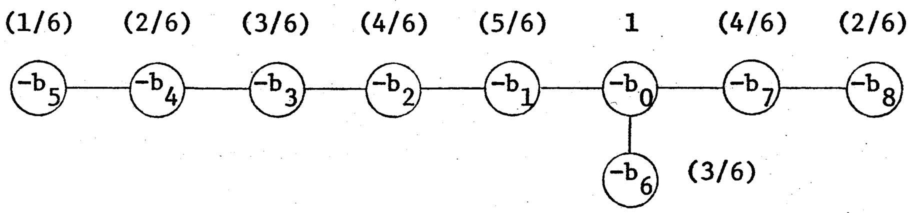


$_*\tilde{E}_6$,



$*\tilde{E}_7$,

or



$*\tilde{E}_8$.

Let $\sum k_iA_i$ be the cycle which is numerically equivalent to the canonical line bundle. Then $\sum k_iA_i \leqq \sum -\lambda_iA_i$, where λ_i's are the numbers written by the side of each vertex.

Proof. Let Z be defined by $Z = \sum \lambda_iA_i$. Then $Z\cdot A_j = \lambda_j(2+A_j\cdot A_j)$ for any j. Since A_j is a non-singular rational curve, $-2 - A_j\cdot A_j = K\cdot A_j$, and hence

$$(\lambda_j-1)(2+A_j\cdot A_j) = (Z+K)\cdot A_j = \sum (\lambda_i+k_i)A_i\cdot A_j \quad .$$

$\lambda_j \leqq 1$ and $A_j\cdot A_j \leqq -2$ and so $(\lambda_j-1)(2+A_j\cdot A_j) \geqq 0$. Thus it follows from Lemma 3.1 that $\lambda_i+k_i \leqq 0$, i.e., $k_i \leqq -\lambda_i$ for any i. □

Theorem 3.8. Suppose that (X,x) is a normal surface singularity and G is the weighted dual graph associated with the minimal resolution $\pi : \tilde{X} \to X$. If $\delta_m = 0$ for all $m \geqq 1$, then G is either chain-shaped or star-shaped with three branches.

Proof. Let $A = \pi^{-1}(x) = \bigcup_{i=1}^{n} A_i$. Since $p_g = \delta_1 = 0$, $A_i \cong \mathbb{P}_1$ and π is a good resolution. Let $\sum_{i=1}^{n} k_iA_i$ be the cycle which is numerically equivalent to the canonical line bundle of $\tilde{X}$. Then by Proposition 3.5,

$-i < k_i \leqq 0$ for all i. By Example 3.6 and Theorem 3.4, G can not contain ${}_*\tilde{A}_n$ as a subgraph. Hence G is tree-shaped. Similarly G does not contain ${}_*\tilde{D}_{n+4}$ as a subgraph. Thus there is at most one center in G, and so G is either chain-shaped or star-shaped with three branches. Suppose not, then G would contain ${}_*\tilde{D}_4$ as a subgraph. This is a contradiction. □

Let (X,x) be a normal surface singularity. Suppose that the weighted dual graph of the minimal resolution of $\tilde{X}$ is star-shaped as follows :

$$\begin{array}{ccccccccc} A_\ell & - \cdots - & A_1 & - & D & - & C_1 & - \cdots - & C_n \\ & & & & | & & & & \\ & & & & B_1 & & & & \\ & & & & | & & & & \\ & & & & \vdots & & & & \\ & & & & | & & & & \\ & & & & B_m & & & & \end{array}$$

where A_i, B_j, C_k and D denote non-singular rational curves. Let

$$E = dD + \sum_{i=1}^{\ell} \lambda_i A_i + \sum_{j=1}^{m} \mu_j B_j + \sum_{k=1}^{n} \nu_k C_k$$

be the cycle which is numerically equivalent to the canonical line bundle K of $\tilde{X}$:

$$(*) \quad \begin{cases} -2-D\cdot D = K\cdot D = E\cdot D & \\ -2-A_i\cdot A_i = K\cdot A_i = E\cdot A_i & i = 1, \ldots , \ell , \\ -2-B_j\cdot B_j = K\cdot B_j = E\cdot B_j & j = 1, \ldots , m , \\ -2-C_k\cdot C_k = K\cdot C_k = E\cdot C_k & k = 1, \ldots , n . \end{cases}$$

Let $Z_A = \sum_{i=1}^{\ell} \alpha_i A_i$ be the cycle which is uniquely determined by the following equation :

$$(**)\quad \begin{cases} 1 + (-2-A_1\cdot A_1) = 1 + K\cdot A_1 = Z_A\cdot A_1 \\ -2-A_i\cdot A_i = K\cdot A_i = Z_A\cdot A_i \quad 2 \leq i \leq \ell . \end{cases}$$

Then (**) is equivalent to

$$\begin{cases} 0 = (Z_A + A)\cdot A_i \quad 1 \leq i \leq \ell-1 , \\ -1 = (Z_A + A)\cdot A_\ell \end{cases}$$

where $A = \sum_{i=1}^{\ell} A_i$. So by Lemma 3.2, $0 < \alpha_i + 1$, i.e., $-1 < \alpha_i$. (In the case where $\ell = 1$, $-1 = \alpha_1$). Moreover, (**) is equivalent to

$$\begin{cases} 1 + d = (Z_A - E|_A)\cdot A_1 \\ 0 = (Z_A - E|_A)\cdot A_i \quad 2 \leq i \leq \ell . \end{cases}$$

By Lemma 3.2, hence

$$1 + d > 0 \implies \alpha_i - \lambda_i < 0$$

$$1 + d = 0 \implies \alpha_i - \lambda_i = 0$$

$$1 + d < 0 \implies \alpha_i - \lambda_i > 0 ,$$

i.e.,

$$d > -1 \implies \alpha_i < \lambda_i$$

$$d = -1 \implies \alpha_i = \lambda_i$$

$$d < -1 \implies \alpha_i > \lambda_i .$$

It is the same with Z_B and Z_C. So we obtain a cycle $Z = -D + Z_A + Z_B + Z_C$, and (*) is equivalent to

$$\begin{cases} -2 - \alpha_1 - \beta_1 - \gamma_1 = (E - Z)\cdot D \\ 0 = (E - Z)\cdot A_i \quad 1 \leqq i \leqq \ell \\ 0 = (E - Z)\cdot B_j \quad 1 \leqq j \leqq m \\ 0 = (E - Z)\cdot C_k \quad 1 \leqq k \leqq n \end{cases} .$$

Then, by Lemma 3.2,

$$-2 - \alpha_1 - \beta_1 - \gamma_1 < 0 \implies d > -1$$

$$-2 - \alpha_1 - \beta_1 - \gamma_1 = 0 \implies d = -1$$

$$-2 - \alpha_1 - \beta_1 - \gamma_1 > 0 \implies d < -1 .$$

We must still express α_1 in terms of the selfintersection number of the A_i, $1 \leqq i \leqq \ell$. Let $-a_i = A_i \cdot A_i$, then (**) is equivalent to

$$\begin{cases} -1 + a_1 = -a_1\alpha_1 + \alpha_2 \\ -2 + a_i = \alpha_{i-1} - a_i\alpha_i + \alpha_{i+1} \quad 2 \leqq i \leqq \ell-1, \\ -2 + a_\ell = \alpha_{\ell-1} - a_\ell\alpha_\ell . \end{cases}$$

Hence

$(\alpha_2+1)/(\alpha_1+1) = a_1$, $(\alpha_3+1)/(\alpha_1+1) = a_1a_2-1$, ... , $(\alpha_{i+1}+1)/(\alpha_1+1) = a_i(\alpha_i+1)/(\alpha_1+1) - (\alpha_{i-1}+1)/(\alpha_1+1)$, ... , $1/(\alpha_1+1) = a_\ell(\alpha_\ell+1)/(\alpha_1+1) - (\alpha_{\ell-1}+1)/(\alpha_1+1)$.

Let $p_\ell = 1/(\alpha_1+1)$. Moreover, an easy induction proof shows that

$$p_\ell = 1/(\alpha_1+1) \geqq \ell+1 .$$

Then the first few p_ℓ's are

$$p_1 = a_1 \geq 2 \quad ,$$

$$p_2 = a_1 a_2 - 1 \geq 3 \quad ,$$

$$p_3 = a_1 a_2 a_3 - a_1 - a_3 \geq 4 \quad ,$$

$$p_4 = a_1 a_2 a_3 a_4 - a_1 a_4 - a_3 a_4 - a_1 a_2 + 1 \geq 5 \quad ,$$

$$p_5 = a_1 a_2 a_3 a_4 a_5 - a_1 a_4 a_5 - a_3 a_4 a_5 - a_1 a_2 a_5 - a_1 a_2 a_3 + a_1 + a_3 + a_5 \geq 6 \quad .$$

Remark. Setting $p_0 = 1$, then $p_\ell / p_{\ell-1}$ has the continued fraction expansion :

$$p_\ell / p_{\ell-1} = [a_\ell, a_{\ell-1}, \ldots, a_1] \, .$$

Thus, letting $q_m = 1/(\beta_1 + 1)$ and $r_n = 1/(\gamma_1 + 1)$, we obtain the following theorem.

Theorem 3.8.B. Let (X,x) be a normal surface singularity. Let G be the weighted graph which is associated with the minimal resolution. Suppose that $\delta_m = 0$ for all $m \geq 1$ and G has three branches, then G is of the form

$A_\ell - \cdots - A_1 - D - C_1 - \cdots - C_n$
$\quad\quad\quad\quad\quad\quad\quad |$
$\quad\quad\quad\quad\quad\quad\quad B_1$
$\quad\quad\quad\quad\quad\quad\quad \vdots$
$\quad\quad\quad\quad\quad\quad\quad B_m$

such that

$$(*) \qquad 1 < 1/p_\ell + 1/q_m + 1/r_n \quad .$$

The systems of positive intergers p_ℓ, q_m, r_n which satisfy the condition (*) of Theorem 3.8.B are the following four types :

$$(2,2,n),\ n \geqq 2,\ (2,3,3),\ (2,3,4),\ (2,3,5)\ .$$

Hence from the result of Brieskorn [3], these singularities are quotient singularities. Thus we have the following.

Theorem 3.9. Let (X,x) be a normal surface singularity. Then (X,x) is a quotient singularity if and only if $\delta_m(X,x) = 0$ for all $m \geqq 1$ (see [25]).

Theorem 3.10. Let (X,x) be a normal surface rational singularity with the minimal resolution $\pi : \tilde{X} \to X$. Let $\pi^{-1}(x) = A$. Denote by A_i $(i = 1,\dots, n)$ the irreducible component of A. Let $\sum_{i=1}^n k_i A_i$ be the cycle which is numerically equivalent to the canonical line bundle of $\tilde{X}$. Suppose that $0 \leqq \delta_m \leqq 1$ for all $m \geqq 1$. Then $-1 \leqq k_i \leqq 0$ for $i = 1, \dots , n$.

Proof. Suppose not, then there would exist k_{i_0} with $k_{i_0} < -1$. Take a holomorphic function f on U which vanishes on A_{i_0}, and let α_{i_0} be its order. Then there is a positive integer m such that $-m-mk_{i_0} > \alpha_{i_0}$ and such that all mk_i are integers. (X,x) is rational, and so mK is linearly equivalent to the integral cycle $\sum mk_iA_i$; there exists $\omega \in \Gamma(U-A,\mathcal{O}(mK))$ so that $(\omega) = \sum mk_iA_i$. Since $\alpha_{i_0} + mk_{i_0} + (m-1) < -1$, it follows that

$$(f\omega) + (m-1)A \ngeq 0,$$

and so ω and $f\omega$ are not $L^{2/m}$-integrable. Thus $\delta_m \geqq 2$, a contradiction. □

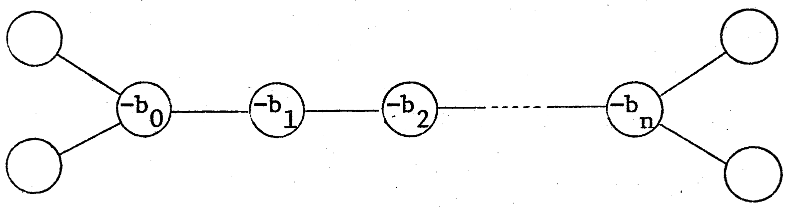
By Theorem 3.4 and 3.10, we have

Corollary. Let (X,x) be a normal surface rational singularity with the minimal resolution $\pi : \tilde{X} \to X$. Let $A = \pi^{-1}(x)$ be the exceptional set. If $\delta_m(X,x)$ is either 0 or 1 for $m \geqq 1$, then every connected proper subvariety of A is the exceptional set for a quotient singularity.

Lemma 3.11. Let (X,x) be a normal surface rational singularity. Let G be the weighted graph associated with the minimal resolution of (X,x). Then G does not contain ${}_*\tilde{A}_n$ as a subgraph.

Proof. In general $p_a(X,x) \leqq p_g(X,x)$ and $p(Z) \leqq p_a(X,x)$ for a positive integral cycle Z. Since $p({}_*\tilde{A}_n) = 1$, ${}_*\tilde{A}_n$ can not be a subgraph of G. □

Proposition 3.12. Let (X,x) be a normal surface rational singularity. Let G be the weighted graph which is associated with the minimal resolution. Suppose that $0 \leqq \delta_m \leqq 1$ and G has at least two centers. Then G is of the form



Proof. From Lemma 3.11 G is tree-shaped and so G contains ${}_*\tilde{D}_{n+4}$ as a subgraph. Let $\sum \kappa_i A_i$ be the cycle which is numerically equivalent to the canonical line bundle of the minimal resolution $\tilde{X}$ of X. Let $\sum_{i=0}^{n+4} k_i A_i$ and $\{ \lambda_i \}$ be as in Proposition 3.7. Then by Theorem 3.4, $-1 \leqq \kappa_i \leqq k_i \leqq -\lambda_i$. If ${}_*\tilde{D}_{n+4}$ is a proper subgraph of G, then $\kappa_i < k_i$ by Lemma 3.2. This contradicts the fact that a certain k_i is equal to -1, and hence $G = {}_*\tilde{D}_{n+4}$. Moreover if $\lambda_i = 1$, then $\kappa_i = k_i = -1$. Let Z be defined by $Z = \sum_{i=0}^{n} A_i$. Since Z is a reduced cycle, $p(Z) \geqq 0$. Hence $0 = p_a(X,x) \geqq p(Z) \geqq 0$, i.e., $0 = p(Z) = (1/2)(Z \cdot Z + K \cdot Z) + 1$. Therefore

$$-2 = Z \cdot Z + (-Z + \sum_{i=1}^{4} k_{n+i} A_{n+i}) \cdot Z$$

and

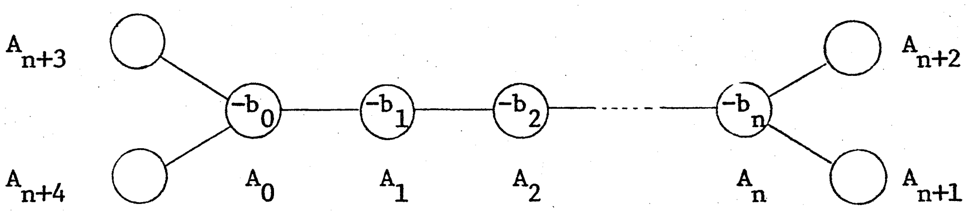
$$-2 = \sum_{i=1}^{4} k_{n+i} .$$

$k_{n+i} \leqq -(1/2)$ for i = 1, 2, 3 and 4, then $k_{n+i} = -(1/2)$ for i = 1, 2, 3 and 4. Thus, for i = 1, 2, 3 and 4

$$0 = p(A_{n+i}) = (1/2)(A_{n+i} \cdot A_{n+i} + K \cdot A_{n+i}) + 1$$

and $-2 = A_{n+i} \cdot A_{n+i}$; the result follows. □

Finally we show that δ_m of G =



is either 0 or 1 for all $m \geqq 1$. Consider A_0 as the center of G. Then we get the analogous formula as in Theorem 2.21.

Theorem 3.13. Let P_0, P_1 and P_2 be the intersection points on A_0 which is defined respectively by $P_0 = A_0 \cap A_1$, $P_1 = A_0 \cap A_{n+3}$ and $P_2 = A_0 \cap A_{n+4}$. Set $d/e = [b_1, b_2, \ldots, b_n - 1]$ with $1 \leqq e \leqq d$, and e and d relatively prime. Let

$$\sum_{k\geqq 0} \dim \Gamma(A_0,\mathcal{O}_{A_0}(mK_{A_0}+kN+[(ke+md)/d]P_0+[(k+m)/2]P_1+[(k+m)/2]P_2))$$

be denoted by Δ_m. Then $\delta_m \leqq \Delta_m$.

Proof. The same argument as in Theorem 2.21 works in the almost part of the proof. Following Theorem 2.21 we define a homomorphism

$$\phi : \Gamma(U-A,\mathcal{O}(mK)) \to \bigoplus_{k\geqq 0} \Gamma(A_0,\mathcal{O}_{A_0}(mK_{A_0}+kN+[(ke+md)/d]P_0$$
$$+[(k+m)/2]P_1+[(k+m)/2]P_2))$$

Let U' be a strongly pseudoconvex neighborhood of $\cup_{i=1}^{n+2} A_i$. Set $A_0' = A_0 \cap U'$. Suppose $\omega \in \Gamma(U-A,\mathcal{O}(mK))$. Then $(\omega|_{U'}) = (-\mu_0)A_0' + \sum_{i=1}^{n+2} (-\mu_i)A_i + B$ where B is effective and does not involve any of the A_i, $i = 1, 2, \ldots, n+2$. Let $Z = (-\mu_0)A_0' + \sum_{i=1}^{n+2} (-k_i)A_i$ be the divisor which is numerically equivalent to $mK_{U'}$, i.e., $mK_{U'}\cdot A_i = Z\cdot A_i$ for $i = 1, 2, \ldots, n+2$. By easy computation $k_1 = \{(\mu_0-m)e+dm\}/d$. $Z\cdot A_j = mK_{U'}\cdot A_j = (\omega|_{U'})\cdot A_j$, and so

$$\sum_{i=1}^{n+2} (\mu_i-k_i)A_i\cdot A_j = B\cdot A_j \ .$$

Since $B\cdot A_j \geqq 0$, $\mu_i-k_i \leqq 0$ by Lemma 3.1. In particular

$$\mu_1 \leqq k_1 = \{(\mu_0-m)e+dm\}/d \ .$$

Hence ϕ is well-defined. If $\mu_0 < m$, then $\mu_1 < m$, and by induction $\mu_i < m$ for $i = 2, 3, \ldots, n+2$. Therefore the kernel of ϕ is

$$\Gamma(U,\mathcal{O}(mK+(m-1)A)) \cong L^{2/m}(U-A) \ .$$

Thus the proof is complete. □

Lemma 3.14. If we are as above,

$$\Delta_m = \begin{cases} 0 & \text{if } m \text{ is odd,} \\ 1 & \text{if } m \text{ is even.} \end{cases}$$

Proof. We compute the degree of the line bundle ;

$$\text{degree}(mK_{A_0}+kN+[(ke+md)/d]P_0+[(k+m)/2]P_1+[(k+m)/2]P_2)$$

$$= -2m-b_0k+[(ke+md)/d] + [(k+e)/2] + [(k+e)/2]$$

$$\leqq -2m-b_0k+(ke+md)/d+(k+e)/2+(k+e)/2 = k(-b_0+ 1 + e/d) .$$

By the definition of e/d, $e/d \leqq 1$ and $e/d = 1$ if and only if $b_i = 2$ for $i = 1, 2, \ldots , n$. Then $-b_0 + 1 + e/d \leqq 0$. If $-b_0 + 1 + e/d = 0$, $b_i = 2$ for all i, and so G is not negative definite, a contradiction. Hence degree $= 0$ if and only if $k = 0$ and $-2m+[m/2]+[m/2]+m = 0$, i.e., $k = 0$ and m is even. Thus our result follows from the Riemann-Roch theorem. □

Let Z be defined by

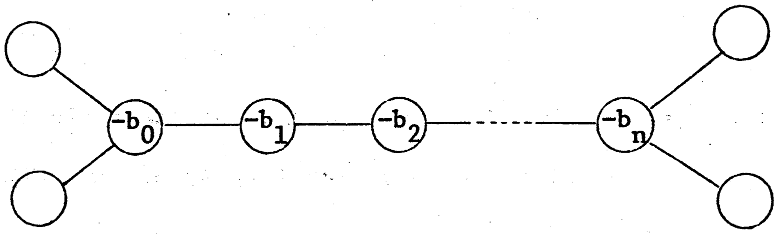
$$Z = 2(A_0+ A_1+ \cdots + A_n) + A_{n+1}+ A_{n+2}+ A_{n+3}+ A_{n+4} .$$

Then $-Z$ is numerically equivalent to $2K$. By the above lemma the singularity with graph G is rational, and so $-Z$ is linearly equivalent to $2K$. Hence $\delta_{2m} \geqq 1$. Then it follows from Theorem 3.13 and Lemma 3.14 that

$$\delta_m = \begin{cases} 0 & \text{if } m \text{ is odd,} \\ 1 & \text{if } m \text{ is even.} \end{cases}$$

Thus we obtain the following.

Proposition 3.15. Suppose G is of the form

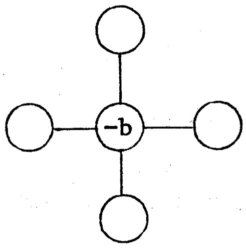


Then

$$\delta_m = \begin{cases} 0 & \text{if } m \text{ is odd,} \\ 1 & \text{if } m \text{ is even.} \end{cases}$$

Similarly for n = 0 we have

Proposition 3.15.B. Let (X,x) be a normal surface rational singularity. Let G be the weighted graph which is associated with the minimal resolution. Suppose $0 \leqq \delta_m \leqq 1$ ($\delta_{m_0} = 1$ for some $m_0 \geqq 2$) and G is star-shaped with at least four branches. Then G is of the form



In fact δ_m of the singularity with the above graph is

$$\delta_m = \begin{cases} 0 & \text{if } m \text{ is odd,} \\ 1 & \text{if } m \text{ is even.} \end{cases}$$

Moreover, from Theorem3.10 and the proof of Theorem 3.8.B we obtain the following.

Proposition 3.15.C. Let (X,x) be a normal surface rational singularity. Let G be the weighted graph which is associated with the minimal resolution. Suppose that $0 \leqq \delta_m \leqq 1$ ($\delta_{m_0} = 1$ for some $m_0 \geqq 2$) and G is star-shaped with three branches. Then G is of the form

```
A_ℓ - ··· - A_1 - D - C_1 - ··· - C_n
                  |
                  B_1
                  |
                  :
                  |
                  B_m
```

such that

$$(*) \qquad 1 = 1/p_\ell + 1/q_m + 1/r_n \ .$$

The possible solutions of (*) are easily enumerated. They are depicted as follows :

$$(p_\ell, q_m, r_n) \in \{ (2,3,6), (2,4,4), (3,3,3) \} \ .$$

Hence it follows from Theorem 2.21, Theorem 2.24 and Theorem 3.9 that $0 \leqq \delta_m \leqq 1$ for the singularities with the condition (*).

Next we recall a few results about minimally elliptic singularities, which was examined by Laufer [15]. Karras [9] and Saito [18] have studied some of particular elliptic singularities.

A normal surface singularity (X,x) is called a simple elliptic singularity if the exceptional set of the minimal resolution consists of a single non-singular elliptic curve A. (X,x) up to analytic isomorphism is uniquely determined by the analytic structure of the curve A, j(A) =

$g_2^3/(g_2^3 - 27g_3^2)$, where $w^2 = 4z^3 - g_2 z - g_3$ is the equation of A in $\mathbb{C}^2$, see [5] and [18].

Cusp singularities are characterized as follows. Let (X,x) be a normal surface singularity and $\pi : \tilde{X} \to X$ be the minimal resolution of (X,x). Let $A = \pi^{-1}(x)$ be the exceptional set. Then (X,x) is a cusp singularity if and only if A is an irreducible rational curve with a node singularity or A is a "cycle" of non-singular rational curve A_i. The configuration is illustrated in Example 3.6. Moreover, the associated cycle

$$\{ (-b_0, -b_1, \dots , -b_n) \}$$

of selfintersection numbers determines the singularity (X,x) up to complex-analytic equivalence (see [8, 9, 14]).

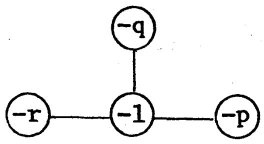
Then by Example 1.9 and Theorem 1.16 we have

Theorem 3.16. Let (X,x) be a simple elliptic singularity or a cusp singularity. Then $\delta_m(X,x) = 1$ for all $m \geqq 1$.

Definition 3.17. Let (X,x) be a normal surface singularity. (X,x) is purely elliptic if $\delta_m(X,x) = 1$ for $m \geqq 1$.

In the following we shall review the resolutions of minimally elliptic singularities and with a few exceptions classify those graphs which can arise from the purely elliptic singularities.

Lemma 3.18. Let (X,x) be a normal surface singularity. If one of the good resolutions of (X,x) has the following weighted dual graph :



then $\limsup_{m \to \infty} \delta_m/m^2 > 0$.

Proof. Since (X,x) is minimally elliptic (see Theorem 3.19, (3)-(5)), there is some neighborhood V of x in X and a holomorphic 2-form ω on V-{x} such that ω has no zeros on V-{x}. Let $\pi : \tilde{X} \to X$ be the minimal good resolution. Then the weighted dual graph of $\tilde{X}$ is of the form mentioned above. Let ($\pi^*\omega$) be the divisor of $\pi^*\omega$. An easy computation shows that $-(\pi^*\omega)$ is not reduced. In fact the multiplicity of $-(\pi^*\omega)$ at the central curve ia equal to two. Let $\rho : X \to \Delta \subset \mathbb{C}^2$ be an admissible representation and let $f = \rho^*z$ and $g = \rho^*w$, where (z,w) is a coordinate system for Δ. Denote by a (resp. b) the order of zeros of π^*f (resp. π^*g) at the central curve . If $m \geqq \lambda a + \mu b$, then

$$(\pi^*(f^\lambda g^\mu \omega^m)) + (m-1)A \ngeq 0 ,$$

i.e., $f^\lambda g^\mu \omega^m$ is not $L^{2/m}$-integrable. Hence

$$\delta_m \geqq \#\{ (\lambda,\mu) \in \mathbb{N}^2 \mid m \geqq \lambda a + \mu b \} .$$

Thus $\limsup_{m \to \infty} \delta_m/m^2 > 0$. □

About the resolutions of the minimally elliptic singularities the following fact was proved by Laufer [15].

Theorem 3.19 (Laufer [15]). Let $\pi : \tilde{X} \to X$ be the minimal resolution for a minimally elliptic singularity (X,x). Let $\pi' : \tilde{X} \to X$ be the minimal reresolution such that A_i' are non-singular and have normal crossings, i.e., the A_i' meet transversely and no three meet at a point. Then $\pi = \pi'$ and all the A_i are rational curves except for the following cases :

(1) A is an elliptic curve. $\pi = \pi'$.

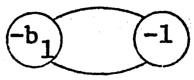
(2) A is a rational curve with a node singularity.

(3) A is a rational curve with a cusp singularity.

(4) A is two non-singular rational curves which have first order tangential contact at one point.

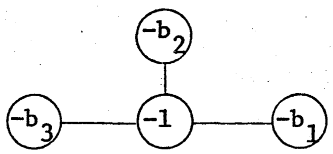
(5) A is three non-singular rational curves all meetiong transversely at the same point.

In case (2), π' has the following weighted dual graph :



with $b_1 \geqq 5$.

In case (3) - (5), π' has the following weighted dual graph :



with $b_i \geqq 2$, $1 \leqq i \leqq 3$.

Remark. In case (1), (X,x) is a simple elliptic singularity. In case (2), (X,x) is one of the simplest cusp singularity.

Theorem 3.20. If a purely elliptic singularity (X,x) is Gorenstein, then (X,x) is a simple elliptic singularity or a cusp singularity.

Proof. Let $\pi : \tilde{X} \to X$ be the minimal resolution of the singularity. Denote $\pi^{-1}(x)$ by A. Let $A = \cup A_i$, $1 \leqq i \leqq n$, be the decomposition of the exceptional set A into irreducible components. We assume that U is a strongly pseudoconvex neighborhood of A. Let K be tha canonical line bundle of U. Since $p_g = \delta_1 = 1$ and (X,x) is Gorenstein, there exists $\omega \in \Gamma(U-A,\mathcal{O}(K))$ such that

$$K = (\omega) = \sum -\lambda_i A_i$$

with $\lambda_i \geqq 1$, that is, K is defined by an integral cycle. By Corollary 2.12 (X,x) is a minimally elliptic singularity. Lemma 3.18 implies that (X,x) is none of (3), (4) and (5) of Theorem 3.19. In case (1), (X,x) is a simple elliptic singularity. In case (2), (X,x) is one of the simplest cusp singularity. Thus we may assume that π is the minimal good resolution and any A_i is a non-singular rational curve. For any holomorphic function f which vanishes at x

$$(\pi^*f) + m(\omega) + (m-1)A \geqq 0$$

as $\delta_m = 1$ for any $m \geqq 1$. Let α_i be the order of zeros of f at A_i. Then $\alpha_i \geqq 1$ and

$$\alpha_i + m(-\lambda_i) + (m-1) \geqq 0 \quad \text{for} \quad m \geqq 1 .$$

Hence $\lambda_i = 1$ and so K = -A. Since $p(A_j) = 0$,

$$0 = (1/2)\{A_j \cdot A_j + (\sum -A_i)\cdot A_j\} + 1 .$$

Then $2 = (\sum_{i \neq j} A_i)\cdot A_j$. This implies that A_j meets two other irreducible components of A. Thus (X,x) is a cusp singularity. □

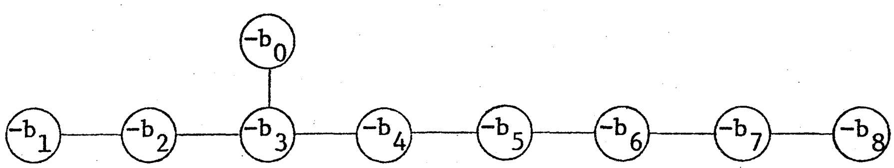
Let $\pi : \tilde{X} \to X$ be the minimal resolution for a purely elliptic singularity (X,x). Let $A = \pi^{-1}(x)$. Suppose that A' is a connected proper analytic subvariety of A. Then the singularity (X',x') obtained by blowing down A' is also a purely elliptic if p(A') = 1. For $\delta_1(X',x') = p_g(X',x')$ $\geqq p(A') = 1$ and $1 = \delta_m(X,x) \geqq \delta_m(X',x') \geqq 1$ by the second fundamental theorem If any connected proper subvariety of A is the exceptional set of a rational singularity, then (X,x) is minimally elliptic. Therefore (X,x) is Gorenstein and so (X,x) is a simple elliptic singularity or a cusp singularity by Theorem 3.20.

Now suppose that there exists a connected proper subvariety A' of A such that p(A') = 1. Let A_0 be the minimal one. Such a cycle always exists by [15, Proposition 3.2, p.1261]. Since A_0 has the minimality, A_0 is the exceptional set of the minimal resolution for a simple ellipric singularity or a cusp singularity. Hence, applying the second fundamental theorem we get the followings.

Theorem 3.21. Suppose that (X,x) is a purely elliptic, $\pi : \tilde{X} \to X$ is a minimal resolution of the singularity, $A = \pi^{-1}(x)$. If (X,x) is not Gorenstein and there exists a connected proper analytic subvariety A_0 of A such that A_0 is the exceptional set of a cusp singularity, then any connected proper analytic subvariety of A, not containing A_0, is the exceptional set of a quotient singularity.

Corollary 3.22. In the above situation the number of the irreducible components of A_0 is at most eight.

Proof. Suppose not, then A would contain



$*\tilde{E}_8$

as a proper connected subvariety of A. Let (X',x') be the singularity obtained by blowing down ${}_*\tilde{E}_8$, which is not a quotient singularity. Then $\delta_{m_0}(X',x') \geqq 1$ for some $m_0 \geqq 1$ by Theorem 3.9. Hence the second fundamental theorem says that $\delta_{m_0}(X,x) \geqq 2$, a contradiction. □

§4. Appendix

In this section we prove that Theorem 2.1 is generalized to the case of arbitrary dimensions $n \geqq 2$, which was suggested to us by the referee. Then the first fundamental theorem of $\{ \delta_m \}$ holds in the case of arbitrary dimensions $n \geqq 2$.

Theorem 4.1. There is a positive constant c such that $\gamma_m \leqq cm^n$ for an n-dimensional normal isolated singularity (X^n,x).

Proof. By a theorem of M. Artin [30], (X^n,x) can be realized as a Zariski open subset $\underline{U}$ of a projective variety $\underline{V}$ with $x \in \underline{U}$ as its only singularity. Let $\pi : U \to \underline{U}$ be a resolution of the singular point. Then, in a natural manner, we get a desingularization $p : V \to \underline{V}$ of $\underline{V}$ by letting V to be $(\underline{V} - \{x\}) \cup U$. Let $A = \pi^{-1}(x) = p^{-1}(x)$ and consider local cohomologies on V and U with the support A. Since

$$\Gamma(U-A,\mathcal{O}(mK_U))/\Gamma(U,\mathcal{O}(mK_U)) \hookrightarrow H^1_A(U,\mathcal{O}(mK_U)) \cong H^1_A(V,\mathcal{O}(mK_V)),$$

it suffices to show that $h^1_A(V,\mathcal{O}(mK_V)) \leqq am^n$ for some $a > 0$. By the exact sequence

$$0 \to H^0(V,\mathcal{O}(mK_V)) \to H^0(V-A,\mathcal{O}(mK_V)) \to H^1_A(V,\mathcal{O}(mK_V)) \to H^1(V,\mathcal{O}(mK_V)) \to \cdots ,$$

we have

$$h^1_A(V,\mathcal{O}(mK_V)) \leqq h^0(V-A,\mathcal{O}(mK_V)) + h^1(V,\mathcal{O}(mK_V)) \quad .$$

From the compactness of V, $h^1(V,\mathcal{O}(mK_V)) \leqq a_1 m^n$ follows for some $a_1 > 0$. Since V-A is strongly pseudoconcave, we have $h^0(V-A,\mathcal{O}(mK_V)) < \infty$ by a theorem of Andreotti-Grauert [31]. Hence it remains to prove that

$h^0(V-A,\mathcal{O}(mK_V)) \leqq a_2m^n$ holds for some $a_2 > 0$. Let H_1 be a very ample line bundle on V such that $H = K_V + H_1$ is also very ample. Since

$$H^0(V-A,\mathcal{O}(mK_V)) \hookrightarrow H^0(V-A,\mathcal{O}(mH)) ,$$

it is enough to show that $h^0(V-A,\mathcal{O}(mH)) \leqq a_3m^n$ $(a_3 > 0)$ holds for any very ample line bundle H. We shall prove this by the induction on n. Suppose that $n = 2$. Let $m_0 = |\det(A_i \cdot A_j)|$, where $(A_i \cdot A_j)$ is the intersection matrix of the exceptional set $A = \cup A_j$. Let H_1 be another ample line bundle on V and $H' = m_0H + m_0H_1$. We can choose H_1 so that $(m_0-1)H + m_0H_1$ has a global non-zero section, and moreover, the restriction H'_U of H' to the open subset U satisfies $H'_U \cdot A_j \geqq K \cdot A_j$ for any curve A_j of A. Then, by the exact sequence

$$0 \to H^0(U,\mathcal{O}(mH'_U)) \to H^0(U-A,\mathcal{O}(mH'_U)) \to H^1_A(U,\mathcal{O}(mH'_U)) \to H^1(U,\mathcal{O}(mH'_U)) \to \cdots ,$$

and [10, Vanishing Theorem], we have

$$h^1_A(U,\mathcal{O}(mH'_U)) = \dim \Gamma(U-A,\mathcal{O}(mH'_U))/\Gamma(U,\mathcal{O}(mH'_U)) .$$

Note that $mH'_U = mm_0(H + H_1)_U$ is numerically equivalent to an integral divisor for any $m \geqq 1$. Therefore from [10, Theorem 2], it follows that

$$\dim \Gamma(U-A,\mathcal{O}(mH'_U))/\Gamma(U,\mathcal{O}(mH'_U)) \leqq (1/2)(mK_U \cdot H'_U - m^2H'_U \cdot H'_U) - m_0K_U \cdot K_U .$$

Hence we get $h^1_A(U,\mathcal{O}(mH'_U)) \leqq a_3m^2$ $(a_3 > 0)$. It is clear that $h^0(V,\mathcal{O}(mH'))$ $\leqq a_4m^2$ $(a_4 > 0)$, since V is compact. Thus

$$h^0(V-A,\mathcal{O}(mH)) \leqq h^0(V-A,\mathcal{O}(mH+m((m_0-1)H+m_0H_1)))$$

$$= h^0(V-A,\mathcal{O}(mH') \leqq h^0(V,\mathcal{O}(mH')) + h^1_A(V,\mathcal{O}(mH'))$$

$$= h^0(V,\mathcal{O}(mH')) + h^1_A(U,\mathcal{O}(mH'_U)) \leqq (a_3 + a_4)m^2 .$$

Thus the case $n = 2$ is proved. Next suppose that $n \geqq 3$. Let $V_1 = (s)$ be the non-singular divisor for a general element $s \in H^0(V, \mathcal{O}(H))$. We can assume that $A_1 = V_1 \cap A \neq \phi$. On V-A, consider the exact sequence

$$0 \to \mathcal{O}((k-1)H) \to \mathcal{O}(kH) \to \mathcal{O}_{V_1-A_1}(kH) \to 0$$

for $k = 1, \ldots, m$. Then summing up the inequalities

$$h^0(V-A, \mathcal{O}(kH)) \leqq h^0(V-A, \mathcal{O}((k-1)H)) + h^0(V_1-A_1, \mathcal{O}_{V_1-A_1}(kH))$$

for $k = 1, \ldots, m$, we have

$$h^0(V-A, \mathcal{O}(mH)) \leqq h^0(V-A, \mathcal{O}) + \sum_{k=1}^{m} h^0(V_1-A_1, \mathcal{O}_{V_1-A_1}(kH)) \ .$$

Since V_1-A_1 is strongly pseudoconcave, $h^0(V_1-A_1, \mathcal{O}_{V_1-A_1}(kH)) \leqq a_5 k^{n-1}$ holds by the induction assumption. Hence

$$h^0(V-A, \mathcal{O}(mH)) \leqq h^0(V-A, \mathcal{O}) + a_5 \sum_{k=1}^{m} k^{n-1} \leqq a_6 m^n$$

holds for some $a_6 > 0$. □

Thus we obtain the first fundamental theorem of $\{ \delta_m \}$:

Theorem 4.2. For any n-dimensional normal isolated singularity, We have

$$\delta = \limsup_{m \to \infty} \delta_m / m^n < \infty \quad .$$

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