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A naive guide to mixed Hodge theory

Alan H. Durfee

Deligne's mixed Hodge theory is a beautiful, powerful subject which is finding more and more uses. Unfortunately its complexity makes it rather difficult to learn. This short paper consists of a few instructive elementary examples and applications collected as I was learning the subject. After reading this, the reader could go on to a detailed introduction such as [Griffiths and Schmid], and then tackle Deligne's papers.

In this paper, I focus on the weight filtration and the fact that its quotients have a pure Hodge structure. This filtration can easily be described in the special cases when the underlying space is either two smooth transversally-intersecting divisors, a variety with an isolated singular point which resolves in one blowing-up, or the complement in a smooth variety of smooth codimension-one divisor.

I then give two typical applications of mixed Hodge theory. The first is the invariant cycle theorem, whose proof uses the weight filtration alone. The second application shows how the mixed Hodge structure on an elliptic curve with a node determines its analytic type, just as the Hodge structure determines the analytic type of a nonsingular elliptic curve.

The material of this paper was developed in lectures at the University of Washington, the University of British Columbia and Kyoto University.

1. Hodge theory. Let X be a complex manifold. Choose complex local coordinates

$$z_1 = x_1 + iy_1, \ \ldots, \ \ z_n = x_n + iy_n$$

in some open subset of X. Any smooth complex-valued differential m-form may be written in this subset as a sum of forms

$$g\, dz_{i_1} \ldots dz_{i_p}\, d\bar{z}_{j_1} \ldots d\bar{z}_{j_q}$$

of type (p, q), where g is a complex-valued function,

$$dz_h = dx_h + i dy_h$$

$$d\bar{z}_h = dx_h - i dy_h$$

and $p + q = m$.

Now suppose that X is a complex projective algebraic variety, that is, the zero locus in complex projective N-space of a set of homogeneous complex polynomials. Suppose also that X is a smooth complex manifold. Let

$$H^{p,q}(X)$$

denote the cohomology classes in $H^{p+q}(X)$ which can be represented by differential forms of type (p, q). (All cohomology will be with complex coefficients unless otherwise stated.) A theorem of Hodge [Griffiths-Harris] says that there is a direct sum decomposition

$$H^m(X) = \bigoplus_{p+q=m} H^{p,q}(X)$$

for each m, and that

$$\overline{H^{p,q}} = H^{q,p}$$

Such a direct sum decomposition of H^m is called a pure Hodge structure (HS) of weight m.

The Hodge filtration

$$H^m = F^0 \supset F^1 \supset \ldots \supset F^m \supset F^{m+1} = 0$$

is defined by

$$F^p = H^{p,m-p} \oplus H^{p+1,\, m-p-1} \oplus \ldots \oplus H^{m,0}$$

Thus F^p means "at least p dz's". The spaces $H^{p,q}$ can be recovered by

$$H^{p,q} = F^p \cap \overline{F^q}$$

2. The weight filtration. Now suppose that X is a projective variety which is not necessarily smooth, or more generally, a quasi-projective variety (the difference of two such varieties). Deligne's work then tells us what the (p, q)-type of a cohomology class on X should be.

Theorem. Let X be a quasi-projective variety. For each m, there is an increasing weight filtration

$$0 = W_{-1} \subset W_0 \subset \ldots \quad \subset W_{2m} = H^m(X)$$

such that

$$Gr_\ell = W_\ell / W_{\ell-1}$$

for each ℓ has a pure Hodge structure of weight ℓ.

Thus Gr_ℓ looks like the ℓ-th cohomology group of a smooth projective variety. This weight filtration has the following properties:

(i). If X is projective and smooth then

$$0 = W_{m-1} \subset W_m = H^m(X)$$

for each m (so $H^m(X)$ is of pure weight m).

(ii). If X is projective (but not necessarily smooth) then

$$0 = W_{-1} \subset W_0 \subset .. \subset W_m = H^m(X)$$

for each m.

(iii). If X is smooth (but not necessarily projective) then

$$0 = W_{m-1} \subset W_m \subset \dots \subset W_{2m} = H^m(X)$$

for each m. Also, for any smooth compactification

$$i: X \hookrightarrow \overline{X}$$

of X,

$$W_m = i^* H^m(\overline{X}).$$

(iv). It is functorial: For any algebraic map

$$f: X \to Y$$

of varieties X and Y,

$$f^*(W_\ell) \subset W_\ell$$

and f^* preserves the pure Hodge structure of weight ℓ on Gr_ℓ, for each ℓ. Actually, f^* <u>strictly preserves</u> the weight filtration. One way of saying this is that a sequence of algebraic varieties and algebraic maps which gives an exact sequence of cohomology groups remains exact after taking Gr_ℓ, for each ℓ.

3. Mixed Hodge structure. The more precise version of Deligne's theorem is as follows:

Theorem. Let X be a quasi-projective algebraic variety. For each m, there is an increasing weight filtration

$$0 = W_{-1} \subset W_0 \subset \dots \subset W_{2m} = H^m(X)$$

which is defined over the rational numbers, and a decreasing Hodge filtration

$$H^m(X) = F^0 \supset F^1 \supset \dots \supset F^m \supset F^{m+1} = 0$$

such that the filtration induced by $F^\bullet$ on the ℓ-th graded quotient Gr_ℓ of the weight filtration gives a Hodge structure of pure weight ℓ.

This is called a mixed Hodge structure (MHS) on $H^m(X)$. More generally, a mixed Hodge structure on a complex vector space H consists of a rationally defined filtration $W_\bullet$ and a filtration $F^\bullet$ which induces a Hodge structure of pure weight ℓ on the graded quotients of $W_\bullet$.

There are two subtleties to be noticed here. First, the weight filtration is defined over the rational numbers, and hence fixed under complex conjugation, making it possible to have a Hodge structure on its graded quotients. Second, the Hodge filtration induced on the graded pieces Gr_ℓ actually comes from a global filtration $F^\bullet$. This is important in moduli problems (§8).

4. An example with X compact. Suppose that the algebraic variety X is the union of two projective nonsingular varieties X_1 and X_2 intersecting transversally. In this case, the weight filtration for X can be found using the Mayer-Vietoris sequence

$$\xrightarrow{\beta_{m-1}} H^{m-1}(X_1 \cap X_2) \xrightarrow{\delta} H^m(X) \xrightarrow{\alpha} H^m(X_1) \oplus H^m(X_2) \xrightarrow{\beta_m} H^m(X_1 \cap X_2) \rightarrow$$

The first term has pure weight m-1 and the last two terms have pure weight m. The maps β_{m-1} and β_m are constructed from geometric maps, and hence preserve the (p, q) type of a form. Thus, assuming that both δ and α are maps of MHS, the weight filtration on $H^m(X)$ must be defined by

$$\begin{aligned} W_{m-2} &= 0 \\ W_{m-1} &= \mathrm{im}\ \beta \\ W_m &= H^m(X). \end{aligned}$$

Then

$$W_{m-1}/W_{m-2} \simeq \mathrm{im}\ \delta \simeq \mathrm{cok}\ \beta_{m-1}$$

has a pure HS of weight m-1, and

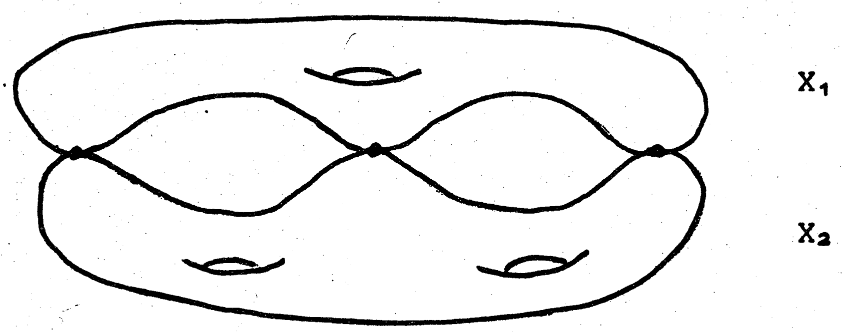
$$W_m/W_{m-1} \simeq \ker \beta_m$$

has a pure HS of weight m, since the kernel and cokernel of a map of Hodge structures have a Hodge structure.

For example, when dim $X_1 = X_2 = 1$, the exact sequence is

$$0 \longrightarrow \tilde{H}^0(X_1 \cap X_2) \longrightarrow H^1(X) \longrightarrow H^1(X_1) \oplus H^1(X_2) \longrightarrow$$

and $H^1(X)$ has classes of two types: Those of weight 1, which lie in one of the X_i's and naturally have type(1, 0) or (0, 1), and those of weight 0 and type (0, 0), which come from the intersection of X_1 and X_2 via the Mayer Vietoris sequence.



More generally, if X of arbitrary dimension is the union of many varieties $X_1, \ldots, X_r$ with mutually transverse intersections, then the above Mayer-Victoris sequence must be replaced by a spectral sequence([Griffiths and Schmid §4], [Cornalba and Griffiths §4]).

5. <u>An example with X singular</u>. Suppose that $\pi: Y \to X$ is a map of algebraic varieties, that $\Sigma \subset X$ is a closed subvariety, and that π restricts to an isomorphism of $X - \Sigma$ to $X - \Sigma'$, where $\Sigma' = \pi^{-1}(\Sigma)$:

$$\begin{tikzcd}
 & \Sigma' \arrow[dl] \arrow[dr, hook] & \\
\Sigma \arrow[dr, hook] & & Y \arrow[dl, "\pi"] \\
 & X &
\end{tikzcd}$$

Under these circumstances there is an exact "Mayer-Vietoris" sequence

$$\to H^i(X) \to H^i(Y) \oplus H^i(\Sigma) \to H^i(\Sigma') \to H^{i+1}(X) \to$$

which may be easily derived by looking at the ladder formed from the exact sequences of the pairs (X, Σ) and (Y, Σ') and the map between them.

Now suppose that X is a variety with an isolated singular point p and that $\Sigma = \{p\}$. Suppose also that Y is smooth and that Σ' is a smooth subvariety D of codimension one in Y. The above exact sequence then becomes $(m > 0)$

$$\to H^{m-1}(Y) \to H^{m-1}(D) \to H^m(X) \xrightarrow{\pi^*} H^m(Y) \to H^m(D) \to$$

which may be used as in the previous section to deduce the weight filtration on $H^m(X)$. The case when X is a curve is treated in §8.

6. <u>An example with X open</u>. Suppose that Z is a smooth projective variety and that $D \subset Z$ is a smooth codimension one subvariety. We will find the MHS on the cohomology groups of the open smooth space

$$X = Z - D.$$

The cohomology of X can be computed using the De Rham complex of smooth forms, so we let

$$F^p \subset H^m(X)$$

be those cohomology classes which can be represented by forms with p or more dz's. Let

$$i\colon \quad X \subset Z$$

be the inclusion map. In the Gysin sequence

$$\to H^{m-2}(D) \xrightarrow{\gamma_m} H^m(Z) \xrightarrow{i^*} H^m(X) \xrightarrow{R} H^{m-1}(D) \xrightarrow{\gamma_{m+1}} H^{m+1}(Z) \to$$

the group $H^{m-2}(D)$ has a pure HS of weight m-2 and the group $H^m(Z)$ has a pure HS of weight m. Furthermore, the Gysin map γ_m takes a form of type (p, q) to a form of type (p + 1, q + 1) [Griffiths and Schmid, p. 45], the restriction map i^* preserves Hodge filtration, and the residue map R has the property that

$$R(F^p) \subset F^{p-1}$$

since it removes a factor of type dz/z from a form. At this point nothing looks right. Now let us change the Hodge structure on $H^{m-2}(D)$ by redefining a class of type (p, q) to be of type $(p+1, q+1)$. Then $H^{m-2}(D)$ has a pure Hodge structure of weight m, γ_m is a morphism of Hodge structures, and R now takes F^p to F^p. As before, we can now define the weight filtration on $H^m(X)$ by

$$W_{m-1} = 0$$
$$W_m = \operatorname{im} i^*$$
$$W_{m+1} = H^m(X).$$

Then

$$W_m/W_{m-1} \simeq \operatorname{im} i^* \simeq \operatorname{cok} \gamma_m$$

has a pure HS of weight m, and

$$W_{m+1}/W_m \simeq \ker \gamma_{m+1}$$

has a pure HS of weight $m + 1$. Furthermore, i^* and R preserve this weight filtration.

For example, when Z is a smooth curve and

$$D = \{p_1, \ldots, p_k\}$$

is a collection of points on Z, the sequence is

$$0 \to H^1(Z) \xrightarrow{i^*} H^1(X) \xrightarrow{R} \tilde{H}^0(D) \to 0.$$

The classes of weight 1 (and type $(1, 0)$ or $(0, 1)$) in $H^1(X)$ come from $H^1(Z)$. The classes of weight 2 in $H^1(X)$ are represented by differences of the forms

$$\frac{dz}{z-p_1}, \ \dots \ \frac{dz}{z-p_k}.$$

and have type (1, 1).

More generally, if D is an arbitrary codimension one divisor on Z, then the Gysin sequence must be replaced by the Gysin spectral sequence [Griffiths and Schmid §5].

7. The invariant cycle theorem.

The invariant cycle theorem is a simple application of the ideas developed so far. Let X be a projective variety, C a projective curve, and

$$\pi : X \to C$$

a surjective algebraic map. Such a map has only a finite number of critical values. Let

$$C^* = C - \{\text{critical values of } \pi\}$$
$$X^* = \pi^{-1}(C^*)$$
$$X_t = \pi^{-1}(t).$$

and let

$$i: \ X_t \subset X^*$$
$$j: \ X^* \subset X$$

be the inclusions. Consider the map

$$H^m(X) \xrightarrow{j^*} H^m(X^*) \xrightarrow{i^*} H^m(X_t).$$

The fundamental group $\pi_1(C^*)$ acts on $H^m(X_t)$. Let

$$H^m(X_t)^{\pi_1(C^*)}$$

denote the elements invariant under this action. Clearly this group is the image of i^* in $H^m(X_t)$. The invariant cycle theorem states that $H^m(X_t)^{\pi_1(C^*)}$ is the image of $i^* j^*$.

To prove this, we note that a class in $H^m(X_t)^{\pi_1(C^*)}$ is of pure weight m, and hence pulls back under i^* to a class of pure weight m (since i^* strictly preserves the filtrations). Now X is a compactification of X^*, so all classes of weight m on X^* are restrictions of classes on X.

This theorem is false when C is a disk and X is not a Kähler manifold; see, for instance, [Clemens p. 229].

8. Mixed Hodge structure and moduli

We will now show, more or less, how the mixed Hodge structure on an elliptic curve with a node determines its analytic type, just as the Hodge structure on a nonsingular elliptic curve determines its analytic type. For further details, see [Carlson].

Let

$$Y = \mathbb{C}/L$$

be a smooth complex curve of genus one defined as the quotient of $\mathbb{C}$ by the lattice L in $\mathbb{C}$ generated by 1 and some complex number y with positive imaginary part.

Let

$$\rho: \mathbb{C} \to Y$$

be the projection map. Let

$$D = \{\rho(0), \rho(x)\}$$

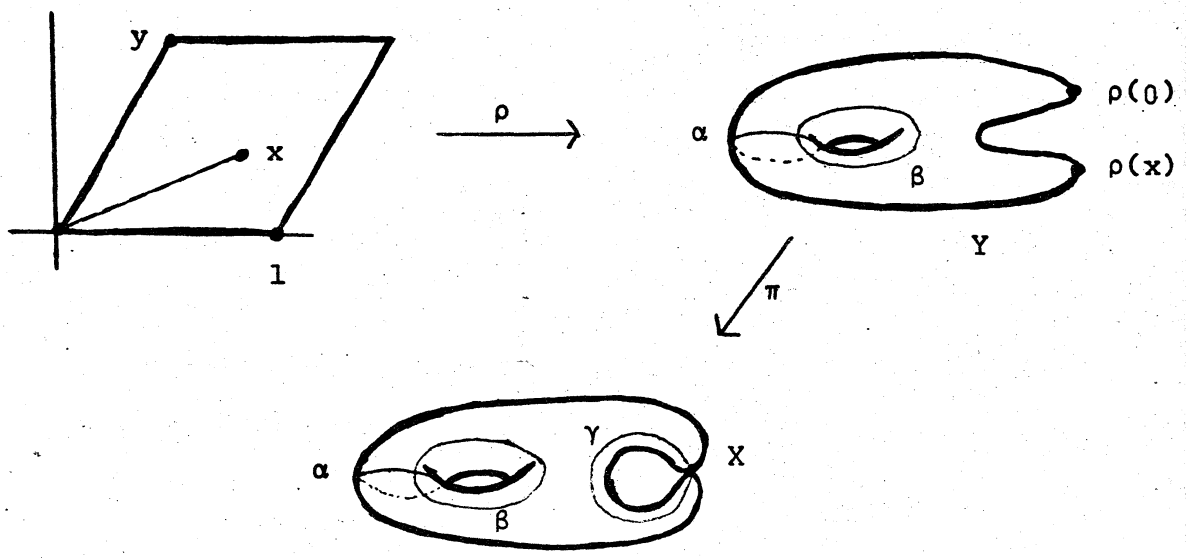
and let

$$X = Y/D$$

be the singular curve defined by identifying the images in Y of the origin and some point x in the parallelogram spanned by 1 and y. Let

$$\alpha,\ \beta,\ \gamma \in H_1(X;\mathbb{Z})$$

be the homology classes defined by the images of the line segments running from the origin to 1, y, and x respectively.



Let

$$\alpha^*,\ \beta^*,\ \gamma^* \in H^1(X;\mathbb{Z})$$

be their duals (so that α^* integrated along α is 1, etc.). These form a basis.

The MHS on X is determined as in §5: The exact sequence is

$$0 \longrightarrow \tilde{H}^0(D) \xrightarrow{\ \iota\ } H^1(X) \xrightarrow{\ \pi^*\ } H^1(Y) \longrightarrow 0.$$

The space $H^0(D)$ has pure weight 0, and $H^1(Y)$ has pure weight one, so the weight filtration is defined by

$$W_0H^1(X) = \operatorname{im} \iota = \text{subspace spanned by } \gamma^*$$

$$W_1H^1(X) = H^1(X)$$

Thus

$$W_0 \simeq \tilde{H}^0(D)$$

has a Hodge structure of pure weight 0, and

$$W_1/W_0 \simeq H^1(Y)$$

has a Hodge structure of pure weight 1.

Here knowledge of the Hodge structure on the quotient W_1/W_0 is not enough; we need the Hodge filtration F^1 on all of $H^1(X;\mathbb{C})$. Let F^1 be the cohomology classes of holomorphic forms on the regular points of X which are square integrable; these are exactly the holomorphic forms ω such that $\pi^*\omega$ extends to a holomorphic form on Y. The one-dimensional spaces F^1 and W_0 in $H^1(X)$ intersect only at the origin. Furthermore π^* preserves the filtration $F^\bullet$.

Let

$$\omega \in F^1 \subset H^1(X)$$

be a generator. Since $(\pi\rho)^*\omega$ is a constant multiple of dz, we may assume without loss of generality that

$$(\pi\rho)^*\,\omega = dz.$$

In terms of the basis α^*, β^*, γ^* of $H^1(X;\mathbb{C})$ we have

$$\omega = \left(\int_\alpha \omega\right)\alpha^* + \left(\int_\beta \omega\right)\beta + \left(\int_\gamma \omega\right)\gamma^*$$

where

$$\int_{\alpha} \omega = \int_{o}^{1} (\pi\rho)^{*}\omega = \int_{o}^{1} dz = 1$$

$$\int_{\beta} \omega = \int_{o}^{y} (\pi\rho)^{*}\omega = \int_{o}^{y} dz = y$$

and

$$\int_{\alpha} \omega = \int_{o}^{x} (\pi\rho)^{*}\omega = \int_{o}^{x} dz = x.$$

Thus the positions of the one-dimensional complex space F^1 and the three dimensional lattice $H^1(X;\mathbb{Z})$ inside $H^1(X;\mathbb{C})$ are related to x and y and hence the analytic type of X.

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Smith College

Northampton, MA 01063