On The Symplectic Lazard Ring

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0. Introduction

In 'Elementary proofs of some results of cobordism theory using Steenrod oparations' (Advances in Math., 7 (1971), 29-56.), D.Quillen determined complex cobordism ring $\text{MU}_*$ using the formal group theory. This method is not applicable directly for the symplectic case.

However there are some works along in this line. Especially, Buhstaber-Novikov studied two-valued formal groups and gave some applications to symplectic cobordism ring $\text{MSp}_*$.

We will define symplectic formal system using formal power series like as (two-valued) formal group, and construct a geometrical example of symplectic formal system. To construct this geometrical example, we need some stable maps between the complex (or symplectic) projective and quasiprojective space.

Moreover, we can construct a ring associated with symplectic formal system. We denote the symplectic Lazard ring $\text{LMSp}$ as the associated ring for the universal symplectic formal system.

Then, we can construct a homomorphism $\theta : \text{LMSp} \to \text{MSp}_*/\text{Torsion}$. By some calculations and the result of R. Okita ('On the MSp Hattori-Stong problem', Osaka J. math. 13 (1976), 547-566.), we can conclude that if we apply the rational indecomposable functor $Q(\cdot)$, then $Q(\theta)$ is an isomorphism.
1. Stable maps

There is a symplectification map \( q : \mathbb{CP}^\infty \to \mathbb{HP}^\infty \).

Since \( q \) is a fibre bundle whose fibre is \( S^2 \), there is a Becker-Gottlieb transfer \( t : \mathbb{HP}^\infty \to (s) \mathbb{CP}^\infty \).

Let \( F \) be \( C \) or \( H \) and \( S^n_F \) unit sphere in \( F^n \).

Let \( G_n(C) = U(n) \) and \( G_n(H) = Sp(n) \).

The quasiprojective space \( Q_n(F) \) is defined to be the space of generalized reflections, that is, the image of \( \phi : S^n_F \times S^1_F \to G_n(F) \) where \( \phi(u,q) \) is the automorphism which leaves \( v \) fixed if \( \langle u,v \rangle = 0 \) and sends \( u \) to \( uq \).

We may define \( Q_n(F) \) as the space obtained \( S^n_F \times S^1_F \) by imposing the equivalence relation \((u,q) \sim (ug, g^{-1}qg) \) \((g \in S^1_F)\), and collapsing \( S^n_F \times 1 \) to a point.

By the second definition, we can easily show that \( Q_n(C) \cong \Sigma(\mathbb{CP}^{n-1}) \).

We put \( \tilde{\mathbb{C}}F^n = Q_n(C) \) and \( \tilde{\mathbb{H}}F^n = Q_n(H) \). Clearly we have a symplectification map \( \tilde{q} : \tilde{\mathbb{C}}F^\infty \to \tilde{\mathbb{H}}F^\infty \).

Now we construct a map from \( \tilde{\mathbb{H}}F^n \) to \( \tilde{\mathbb{C}}F^{2n} \).

Let \( z \in \mathbb{H}^n \) and \( z = x + jy \) where \( x,y \in \mathbb{C}^n \).

We denote complexification map \( c : \mathbb{H}^n \to \mathbb{C}^{2n} \) by setting \( c(z) = x \oplus y \in \mathbb{C}^{2n} \).

Let \( q = a + jb \in H \) where \( a,b \in \mathbb{C} \). Since \( S^1_C \) is a maximal torus of \( S^1_H \), there is a \( g \in S^1_H \) such that \( g^{-1}qg \in S^1_C \). If \( g^{-1}qg = e^{i\pi t} \), where \(-1 < t < 0\), then \((gj)^{-1}qgj = e^{-i\pi t} \).

Thus there is a \( g \in S^1_H \) such that \( g^{-1}qg = e^{i\pi t} \) where \( 0 \leq t \leq 1 \).
So a representative element of $\hat{H}^n$ can be taken as $(x + jy, e^{i\pi t})$

where $x, y \in C^n$ and $0 \leq t \leq 1$.

We define $\tilde{c}_n : \hat{H}^n \rightarrow CP^{2n}$ by the equation

$\tilde{c}_n[(x + jy, e^{i\pi t})] = [(x \oplus y, e^{2\pi i t})]$.

Then the following proposition holds.

Proposition. The diagram

\[
\begin{array}{ccc}
\hat{H}^n & \xrightarrow{\tilde{c}_n} & CP^{2n} \\
\downarrow j & & \downarrow j \\
SP(n) & \xrightarrow{c} & U(2n)
\end{array}
\]

commutes up to homotopy.

By the theorem of Becker-Segal,

$Q(HP^n) \simeq BSP \times F$ as an infinite loop space where $Q(\ )$ is a stabilize functor

$\lim_{n} \Omega^n S^n(\ )$.

So we have a map $r : \tilde{c} : \hat{H}^n \rightarrow Q(HP^n)$ such that the diagram

\[
\begin{array}{ccc}
\Sigma \hat{H}^n & \xrightarrow{r} & \Sigma SP \\
\downarrow & & \downarrow \gamma \\
Q(HP^n) & \xrightarrow{\lambda} & BSP
\end{array}
\]

commutes up to homotopy.

We may regard $r$ as a stable map $r : \tilde{c} : \hat{H}^n \rightarrow HP^n$.

We put $\hat{H}^n = \Sigma^{-1} \hat{H}^n$, $\tilde{q} = \Sigma^{-1} \tilde{q}$ and $\tilde{t} = \Sigma^{-1} \tilde{t}$.

Then we have following stable maps:

\[
\begin{array}{ccc}
CP_+ & \xrightarrow{\tilde{q}} & HP_+ \\
\& & \xrightarrow{t} & CP_+
\end{array}
\]

\[
\begin{array}{ccc}
CP_+ & \xrightarrow{\tilde{q}} & HP_+ \\
\& & \xrightarrow{\tilde{t}} & CP_+
\end{array}
\]

\[
\begin{array}{ccc}
\Sigma^2 \hat{H}^n & \xrightarrow{r} & HP^n
\end{array}
\]

We can easily calculate the homomorphisms induced by these maps on the ordinaly
homology theory.

Let \( y^{\text{MSp}} \) be the euler class of \( \text{MSp} \), \( y \) the class of ordinary homology. Then in \( H \) \( \text{MSp}-\text{theory} \), we have
\[
y^{\text{MSp}} = h(y) = \sum_{i \geq 0} h_i y^{i+1}.
\]

Let \( x \) be the complex euler class of the ordinary homology \( H \).

Now we can define the symplectic formal system.

Let \( R \) be a commutative ring with unit and \( R[[X, \bar{X}, Y, \bar{Y}]] \) formal power series ring with four variables \( X, \bar{X}, Y \) and \( \bar{Y} \).

**Definition 4.1.** A symplectic formal system is a set of formal power series \( E(X), F_k(X, \bar{X}, Y, \bar{Y}) \) and \( G_k(X, \bar{X}, Y, \bar{Y}) \) (for \( k \geq 1 \)) such that satisfy

(i) \[
E(X) = \sum_{i \geq 1} a_i X^i,
\]

\[
F_k(X, \bar{X}, Y, \bar{Y}) = \sum_{i, j \geq 0} b_{i,j}^{(k)} X^i Y^j + \sum_{i, j \geq 1} c_{i,j}^{(k)} \bar{X}^i \bar{Y}^{j-1} Y^j + \sum_{i, j \geq 0} d_{i,j}^{(k)} \bar{X}^{i-1} \bar{Y}^j + Y^i \bar{Y}^{j-1} Y^j,
\]

\[
G_k(X, \bar{X}, Y, \bar{Y}) = \sum_{i, j \geq 0} e_{i,j}^{(k)} (X^i \bar{X}^{j-1} Y^j + \bar{Y}^i Y^{j-1} X^j)
\]

and under \( \bar{X}^2 = E(X), \bar{Y}^2 = E(Y) \), satisfy also

(ii) (unitary relation) \( b_{1,0}^{(1)} = d_{1,0}^{(1)} = 1, b_{n,0}^{(1)} = d_{n,0}^{(1)} = 0 \) for \( n \neq 1 \),

(iii) (associative relation)
\[
D(F_1(X, \bar{X}, Y, \bar{Y}), G_1(X, \bar{X}, Y, \bar{Y}), Z, \bar{Z}) = D(X, \bar{X}, F_1(Y, \bar{Y}, Z, \bar{Z}), G_1(Y, \bar{Y}, Z, \bar{Z}))
\]

for \( D = F_1 \) or \( G_1 \),

(iv) (commutative relation) \( b_{i,j}^{(1)} = b_{j,i}^{(1)}, c_{i,j}^{(1)} = c_{j,i}^{(1)} \),

(v) (differential relation) \( c_{1,1}^{(1)} = -2, c_{1,n}^{(1)} = e_{n,1}^{(1)} = 0 \) for \( n \neq 1 \),

(vi) (power relation) \( F_k(X, \bar{X}, Y, \bar{Y}) = (F_1(X, \bar{X}, Y, \bar{Y}))^k \),

\[
G_k(X, \bar{X}, Y, \bar{Y}) = G_1(X, \bar{X}, Y, \bar{Y}) \cdot F_{k-1}(X, \bar{X}, Y, \bar{Y}) \quad \text{and}
\]

(vii) (square relation) \( (G_1(X, \bar{X}, Y, \bar{Y}))^2 = E(F_1(X, \bar{X}, Y, \bar{Y})) \).
Definition 4.2.

Let \( \Gamma = \langle E, F_k, G_k \rangle \) be a symplectic formal system over \( R \).

An associated symplectic ring for \( \Gamma \), \( \Gamma_R \), is the subring of \( R \) which is generated by the elements \( 8a_{i,j}, 4b_{i,j}, 2b_{(2k-1)i,j}, c_{(k)i,j}, d_{(k)i,j} \) and \( 1 \).

Now we can define symplectic Lazard ring \( LMSp \) as follows.

Let \( S = \mathbb{Z}[a_i, b_i, c_i, d_i, c_{(k)i,j}, d_{(k)i,j}] \) where \( a_i, b_i, c_i, d_i, c_{(k)i,j}, d_{(k)i,j} \) are variables and \( I \) the ideal of relations that appear in (i) \( \sim \) (vii) of (4.1).

Then we get a universal symplectic formal system over \( S/I \).

We denote \( \Gamma_{univ} \) as this system over \( S/I \) and do \( LMSp \) as \( (S/I) \Gamma_{univ} \).

Next we want to construct a symplectic formal system over \( H_\#(MSP) \).

For simplicity, we denote \( f(x) \) and \( \overline{f}(x) \) as \( h(-x^2) \) and \( \frac{1}{2} \frac{d}{dx} h(-x^2) \)

\( H_\#(MSP)[[x]] \) where \( h(-x^2) \) is as previous.

We denote a symplectic formal system \( \Gamma_H \) by setting,

\[
E_H(f(x)) = (\overline{f}(x))^2,
\]

\[
F_k(H(f(x), \overline{f}(x), \overline{f}(y), \overline{f}(y))) = (f(x+y))^k \quad \text{and}
\]

\[
G_k(H(f(x), \overline{f}(x), \overline{f}(y), \overline{f}(y))) = \overline{f}(x+y) \cdot (f(x+y))^{k-1} \quad \text{for } k \geq 1.
\]

Then the relations (i) \( \sim \) (vii) except (v) are almost trivial.

Proposition 4.4. In \( \Gamma_H \), differential relation holds.

We have a ring homomorphism \( \theta': LMSp \rightarrow H_\#(MSP) \), by the universality.

Theorem. \( \text{Im}(\theta') \subseteq \text{Im}(\text{hurewicz homomorphism} : MSP \rightarrow H_\#(MSP)) \).