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On The Symplectic Lazard Ring

by Kazumoto KOJIMA

0. Introduction

In 'Elementary proofs of some results of cobordism theory using Steenrod oparations' (Advances in Math.,7 (1971), 29-56.), D.Quillen determined complex cobordism ring MU_* using the formal group theory. This method is not applicable directly for the symplectic case.

However there are some works along in this line.
Espetialy, Buhstaber-Novikov stadied two-valued formal groups and gave some applications to symplectic cobordism ring MSp_* .

We will define symplectic formal system using formal power series like as (two-valued) formal group, and construct a geometrical example of symplectic formal system. To construct this geometrical example, We need some stable maps between the complex (or symplectic) projective and quasiprojective space.

Moreover, we can construct a ring assocated with symplectic formal system. We denote the symplectic Lazard ring LMSp as the associated ring for the universal symplectic formal system.

Then, we can construct a homomorphism θ : LMSp $\longrightarrow$ MSp_*/Torsion.
By some calculations and the result of R.$\bar{O}$kita ('On the MSp Hattori-Stong problem', Osaka J. math. 13 (1976), 547-566.), we can conclude that if we apply the rational indecomposable functor Q( ), then Q(θ) is an isomorphism.

1. Stable maps

There is a symplectification map $q : CP^{\infty} \longrightarrow HP^{\infty}$.

Since q is a fibre bundle whose fibre is S^2, there is a Becker-Gottlieb transfer $t : HP^{\infty}_{+} \xrightarrow[(s)]{} CP^{\infty}_{+}$.

Let F be C or H and S^n_F unit sphere in F^n.

Ler $G_n(C) = U(n)$ and $G_n(H) = Sp(n)$.

The quasiprojective space $Q_n(F)$ is defined to be the space of generalized reflections, that is, the image of

$$\phi : S^n_F \times S^1_F \longrightarrow G_n(F)$$

where $\phi(u,q)$ is the automorphism which leaves v fixed if $\langle u,v\rangle = 0$ and sends u to uq.

We may define $Q_n(F)$ as the space obtained $S^n_F \times S^1_F$ by imposing the equivalence relation $(u,q) \sim (ug, g^{-1}qg)$ $(g \in S^1_F)$, and collapsing $S^n_F \times 1$ to a point.

By the second definition, we can easily show that $Q_n(C) \approx \Sigma(CP^{n-1}_{+})$.

We put $\widetilde{CP}^n = Q_n(C)$ and $\widetilde{HP}^n = Q_n(H)$. Clearly we have a symplectification map $\tilde{q} : \widetilde{CP}^{\infty} \longrightarrow \widetilde{HP}^{\infty}$.

Now we construct a map from $\widetilde{HP}^n$ to $\widetilde{CP}^{2n}$.

Let $z \in H^n$ and $z = x + jy$ where $x,y \in C^n$.

We denote complexification map $c : H^n \longrightarrow C^{2n}$ by setting $c(z) = x \oplus y \in C^{2n}$.

Let $q = a + jb \in H$ where $a,b \in C$. Since S^1_C is a maximal torus of S^1_H, there is a $g \in S^1_H$ such that $g^{-1}qg \in S^1_C$. If $g^{-1}qg = e^{i\pi t}$, where $-1 < t < 0$, then $(gj)^{-1}qgj = e^{-i\pi t}$.

Thus there is a $g \in S^1_H$ such that $g^{-1}qg = e^{i\pi t}$ where $0 \leq t \leq 1$.

So a representative element of $\widetilde{HP}^n$ can be taken as $(x + jy, e^{i\pi t})$ where $x, y \in C^n$ and $0 \leqq t \leqq 1$.

We define $\tilde{t}_n : \widetilde{HP}^n \longrightarrow \widetilde{CP}^{2n}$ by the equation

$\tilde{t}_n[(x + jy, e^{i\pi t})] = [(x \oplus y, e^{2i\pi t})]$.

Then the following proposition holds.

Proposition. The diagram

$$\begin{tikzcd}
\widetilde{HP}^n \arrow[r, "\tilde{t}_n"] \arrow[d, "j"] & \widetilde{CP}^{2n} \arrow[d, "j"] \\
SP(n) \arrow[r, "c"] & U(2n)
\end{tikzcd}$$

commutes up to homotopy.

By the theorem of Becker-Segal,

$Q(HP^\infty) \simeq BSp \times F$ ~~as an infinite loop space~~ where $Q(\ )$ is a stabilize functor $\varinjlim_n \Omega^n S^n(\ )$.

So we have a map $r : \Sigma\widetilde{HP}^\infty \longrightarrow Q(HP^\infty)$ such that the diagram

$$\begin{tikzcd}
\Sigma\widetilde{HP}^\infty \arrow[r, "j"] \arrow[d, "r"] & \Sigma Sp \arrow[d, "\iota"] \\
Q(HP^\infty) \arrow[r, "j"] & BSp
\end{tikzcd}$$

commutes up to homotopy.

We may regard r as a stable map $r : \Sigma\widetilde{HP}^\infty \xrightarrow[(s)]{} HP^\infty$.

We put $\overline{HP}^\infty = \Sigma^{-1}\widetilde{HP}^\infty$, $\bar{q} = \Sigma^{-1}\tilde{q}$ and $\bar{t} = \Sigma^{-1}\tilde{t}$.

Then we have following stable maps :

$CP^\infty_+ \xrightarrow{q} HP^\infty_+ \xrightarrow{t} CP^\infty_+$,

$CP^\infty_+ \xrightarrow{\bar{q}} \overline{HP}^\infty_+ \xrightarrow{\bar{t}} CP^\infty_+$ and

$\Sigma^2\overline{HP}^\infty \xrightarrow{r} HP^\infty$.

We can easily calculate the homomorphisms induced by these maps on the ordinaly homology theory.

Let y^{MSp} be the euler class of MSp, y the class of ordinaly homology. Then in H MSp-theory, we have $y^{MSp} = h(y) = \sum_{i\geq 0} h_i y^{i+1}$.
Let x be the complex euler class of the ordinaly homology H.

Now we can define the symplectic formal system.

Let R be a commutative ring with unit and $R[[X,\bar{X},Y,\bar{Y}]]$ formal power series ring with four variables X, $\bar{X}$, Y and $\bar{Y}$.

Definition 4.1. A symplectic formal system is a set of formal power series $E(X)$, $F_k(X,\bar{X},Y,\bar{Y})$ and $G_k(X,\bar{X},Y,\bar{Y})$ (for $k \geq 1$) such that satisfy

(i) $E(X) = \sum_{i\geq 1} a_i X^i$,

$$F_k(X,\bar{X},Y,\bar{Y}) = \sum_{i,j\geq 0} b_{i,j}^{(k)} \cdot X^i \cdot Y^j + \sum_{i,j\geq 1} c_{i,j}^{(k)}\, \bar{X}\cdot X^{i-1}\, \bar{Y}\cdot Y^{j-1} ,$$

$$G_k(X,\bar{X},Y,\bar{Y}) = \sum_{i,j\geq 0} d_{i,j}^{(k)}\, (\bar{X}\cdot X^{i-1} \cdot Y^j + \bar{Y}\, Y^{i-1} \cdot X^j)$$

and under $\bar{X}^2 = E(X)$, $\bar{Y}^2 = E(Y)$, satisfy also

(ii) (unitary relation) $b_{1,0}^{(1)} = d_{1,0}^{(1)} = 1$, $b_{n,0}^{(1)} = d_{n,0}^{(1)} = 0$ for $n \neq 1$,

(iii) (associative relation)

$D(F_1(X,\bar{X},Y,\bar{Y}),G_1(X,\bar{X},Y,\bar{Y}),Z,\bar{Z})$

$= D(X,\bar{X},F_1(Y,\bar{Y},Z,\bar{Z}),G_1(Y,\bar{Y},Z,\bar{Z}))$ for $D = F_1$ or G_1 ,

(iv) (commutative relation) $b_{i,j}^{(1)} = b_{j,i}^{(1)}$, $c_{i,j}^{(1)} = c_{j,i}^{(1)}$,

(v) (differntial relation) $c_{1,1}^{(1)} = -2$, $c_{1,n}^{(1)} = c_{n,1}^{(1)} = 0$ for $n \neq 1$,

(vi) (power relation) $F_k(X,\bar{X},Y,\bar{Y}) = (F_1(X,\bar{X},Y,\bar{Y}))^k$,

$G_k(X,\bar{X},Y,\bar{Y}) = G_1(X,\bar{X},Y,\bar{Y})\, F_{k-1}(X,\bar{X},Y,\bar{Y})$ and

(vii) (squar relation) $(G_1(X,\bar{X},Y,\bar{Y}))^2 = E(F_1(X,\bar{X},Y,\bar{Y}))$.

Definition 4.2.

Let $\Gamma = \{E, F_k, G_k\}$ be a symplectic formal system over R. An associated symplectic ring for Γ, R_Γ, is the subring of R which is generated by the elements $8a_i$, $4b^{(2k-1)}_{i,j}$, $2b^{(2k)}_{i,j}$, $c^{(k)}_{i,j}$, $4d^{(k)}_{i,j}$ and 1.

Now we can define symplectic Lazard ring LMSp as follows. Let S be $Z[a_i, b^{(k)}_{i,j}, c^{(k)}_{i,j}, d^{(k)}_{i,j}]$ where a_i, $b^{(k)}_{i,j}$, $c^{(k)}_{i,j}$ and $d^{(k)}_{i,j}$ are variables and I the ideal of relations that appear in (i) $\sim$ (vii) of (4.1).

Then we get a universal symplectic formal system over S/I. We denote Γ_{univ} as this system over S/I and do LMSp as $(S/I)_{\Gamma_{univ}}$.

Next we want to construct a symplectic formal system over $H_*(MSp)$. For simplicity, we denote $f(x)$ and $\bar{f}(x)$ as $h(-x^2)$ and $\frac{1}{2}\frac{d}{dx}h(-x^2)$ $H_*(MSp)[[x]]$ where $h(-x^2)$ is as previous.

We denote a symplectic formal system Γ_H by setting,

$$E^H(f(x)) = (\bar{f}(x))^2,$$

$$F^H_k(f(x), \bar{f}(x), f(y), \bar{f}(y)) = (f(x+y))^k \quad \text{and}$$

$$G^H_k(f(x), \bar{f}(x), f(y), \bar{f}(y)) = \bar{f}(x+y)\cdot(f(x+y))^{k-1} \quad \text{for } k \geqq 1.$$

Then the relations (i) $\sim$ (vii) except (v) are almost trivial.

Proposition 4.4. In Γ_H, differential relation holds.

We have a ring homomorphism $\theta' : \text{LMSp} \longrightarrow H_*(MSp)_{\Gamma_H}$ by the universarity.

Theorem. $\mathrm{Im}(\theta') \subseteq \mathrm{Im}(\text{hurewicz homomorphism} : MSp_* \longrightarrow H_*(MSp))$.