On the order of certain elements of $J(X)$

and

the Adams conjecture

by

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§1. Introduction

The Adams conjecture [2] was proved by several mathematicians in different methods (cf. [7],[8],[9],[10],[14],[15] and [19]).

But in their methods, the localization plays an important role and so we can not estimate the order of an element

$$J^*(\psi^k - 1)(x).$$

Let $n_n$ be the canonical (complex) line bundle over $\mathbb{C}P^n$ and $k$ an integer. Let $m(n,k)$ be the minimal positive integer such that
\[ \kappa^m(n,k)j_\ast(\psi^k - 1)(\eta_n) = 0, \]

which exists by the Adams conjecture for complex line bundles [2].

We put

\[ e(n,k) = m([\frac{n}{2}], k). \]

Then the purpose of this paper is to show

**Theorem 1.** If \( X \) is an \( n \)-dimensional CW complex, then

\[ \kappa^e(n,k)j_\ast(\psi^k - 1)(x) = 0 \]

for any \( x \in K(X) \).

On the other hand let

\[ e(n,k) \quad \text{if} \quad k \quad \text{is odd}, \]

\[ e'(n,k) = e(n,k) + 1 \quad \text{if} \quad k \quad \text{is even}. \]

Then by a quite similar method, we have

**Theorem 2.** If \( X \) is an \( n \)-dimensional CW complex, then

\[ \kappa^{e'}(n,k)j_\ast(\psi^k - 1)(x) = 0 \]
for any element $x \in KO(X)$.

To prove the above theorems, we do not use the Adams conjecture for general vector bundles. So as a corollary of Theorem 2, the Adams conjecture is proved. The proof of the above theorems is similar to the proof of the Adams conjecture of Nishida [14] and Hashimoto [10]. But we use relations between the induction homomorphisms and the Adams operations in [12] instead of the localization. We also use the cellular approximation of the Becker-Gottlieb transfer used by Sigrist and Suter in [18] instead of the usual Becker-Gottlieb transfer [8].

The paper is organized as follows:

In §2 some properties of the Becker-Gottlieb transfer are reviewed. Theorem 1 and Theorem 2 are proved in §3 and §4.
respectively. A property of the real induction homomorphism used in this paper is proved in Appendix.

By a quite similar method to the proof of Theorem 1, we can prove Theorem 1 of Sigrist and Suter [18].
§2. Properties of the Becker-Gottlieb transfer

In this section $X$ is an $n$-dimensional finite cell complex, $G$ is a compact Lie group and $H$ is a closed subgroup of $G$. Let $E$ be the total space of a principal $G$-bundle over $X$. Then $p : E/H \times X$ is a fibre bundle whose fibre is a compact smooth manifold $G/H$ and whose structure group is a compact Lie group $G$ acting smoothly on $G/H$. Let $t(p) : (E/H)_+ \times X_+$ be the $s$-map defined by Becker and Gottlieb in [8]. Since $X$ and $(E/H)_+$ are finite complexes, $t(p)$ is represented by a map

$$t : \Sigma^k \wedge X_+ \to \Sigma^k \wedge (E/H)_+$$

for some $k$. Let $(E/H)^{(n)}$ be the $n$-skelton of $E/H$ (for some cellular decomposition) and $j : (E/H)^{(n)} \subset E/H$ be the inclusion. Then by the cellular approximation theorem, there is a map

$$t' : \Sigma^k \wedge X_+ \to \Sigma^k \wedge ((E/H)^{(n)})_+$$
such that

\[
\Sigma^2 \wedge X_* \xrightarrow{t} \Sigma^2 \wedge (E/H)_+ \\
\xrightarrow{j} \Sigma^2 \wedge ((E/H)(n))_+
\]

commutes. Define \( p_1^* \) by the commutative diagram:

\[
\begin{array}{ccc}
K((E/H)(n)) & \xrightarrow{=} & \tilde{K}^0((E/H)(n))_+ \\
\downarrow p_1^* & & \downarrow t^* \\
K(X) & \xrightarrow{=} & \tilde{K}^0(X_+) \\
\end{array}
\]

where \( \sigma \) is the suspension isomorphism defined by the Bott periodicity theorem ([4]). The Becker-Gottlieb transfer \( p_! \) \( K(E) \to K(X) \) is defined by a similar way. Then by definitions the following diagram is commutative:

\[
\begin{array}{ccc}
K((E/H)(n)) & \xrightarrow{j^*} & K(E/H) \\
\xrightarrow{p_1^*} \downarrow & & \downarrow \ xto{p_!} \\
K(X) & \end{array}
\]
Let \( V \) be a complex \( H \)-module and \( \alpha : R(H) \to K(E/H) \) be a homomorphism defined by \( V \to (E \times_H V \to E/H) \). Define \( \alpha' : R(H) \to K((E/H)^{(n)}) \) by \( \alpha' = j^* \circ \alpha \). Then we have

**Lemma 2.1.** The following diagram is commutative:

\[
\begin{array}{ccc}
R(H) & \xrightarrow{\alpha'} & K((E/H)^{(n)}) \\
\downarrow{\text{Ind}}^G_H & & \downarrow{\text{p}^!} \\
R(G) & \xrightarrow{\alpha} & K(X),
\end{array}
\]

where \( \text{Ind}^G_H \) is the induction homomorphism defined by Segal [16] (see also [10]).

**Proof.** This is an easy consequence of the commutative diagram:

\[
\begin{array}{ccc}
R(H) & \xrightarrow{\alpha} & K(E/H) \\
\downarrow{\text{Ind}}^G_H & & \downarrow{\text{p}^!} \\
R(G) & \xrightarrow{\alpha} & K(X)
\end{array}
\]
which is Proposition 5.4 of Nishida [14].

Let \( \widetilde{\text{Sph}}^*() \) be the generalized cohomology theory defined by the stable spherical fibrations and \( \text{Sph}(X) = \widetilde{\text{Sph}}^0(X_+) \). Define

\[
p^*_\# : K((E/H)^{(n)}) \to K(X)
\]
or

\[
p^*_\# : \text{Sph}((E/H)^{(n)}) \to \text{Sph}(X)
\]

by a similar way to \( p^*_J \) using the suspension isomorphisms defined by the infinite loop space structures defined by the \( \Gamma \)-structures (cf. Segal [17]). Since \( J \) is an infinite loop map with respect to these infinite loop space structures, we have (cf. Nishida [14])

**Lemma 2.2.** The following diagram is commutative:

\[
\begin{array}{ccc}
K((E/H)^{(n)}) & \to & \text{Sph}((E/H)^{(n)}) \\
\uparrow p^*_J & & \uparrow p^*_J \\
K(X) & \to & \text{Sph}(X).
\end{array}
\]
By May [13], the infinite loop space structure of \( BU \times \mathbb{Z} \) defined by the \( \Gamma \)-structure is equivalent to that defined by the Bott periodicity theorem. Then \( p_1^* = p_\mathbb{Z}^* \) and so we have

**Theorem 2.3.** The diagram

\[
\begin{array}{ccc}
R(H) & \overset{\alpha^I}{\rightarrow} & K((E/H)(n)) \overset{J}{\rightarrow} \text{Sph}((E/H)(n)) \\
\uparrow \text{Ind}_H^G & & \uparrow p_\mathbb{Z}^I \\
R(G) & \overset{\alpha}{\rightarrow} & K(X) \overset{J}{\rightarrow} \text{Sph}(X) \\
\end{array}
\]

is commutative.

Quite similarly we have (cf. Hashimoto [10])

**Theorem 2.4.** The diagram

\[
\begin{array}{ccc}
RO(H) & \overset{\alpha^I}{\rightarrow} & KO((E/H)(n)) \overset{J}{\rightarrow} \text{Sph}((E/H)(n)) \\
\uparrow \text{Ind}_H^G & & \uparrow p_\mathbb{Z}^I \\
RO(G) & \overset{\alpha}{\rightarrow} & KO(X) \overset{J}{\rightarrow} \text{Sph}(X) \\
\end{array}
\]
is commutative where $\text{Ind}_H^G$ is the induction homomorphism of real representation rings defined by Hashimoto [10].
§3. **Proof of Theorem 1**

First recall the following lemmas.

**Lemma 3.1.** Let \( f : Y \to Y' \) be a (continuous) map and \( y \in K(Y') \). If \( k^eJ^o(\psi^k - 1)(y) = 0 \), then \( k^eJ^o(\psi^k - 1)(f^*(y)) = 0 \).

**Proof.** This is an easy consequence of the following commutative diagram:

\[
\begin{array}{ccc}
  K(Y') & \xrightarrow{f^*} & K(Y) \\
  \updownarrow J & & \updownarrow J \\
  \text{Sph}(Y') & \xrightarrow{f^*} & \text{Sph}(Y).
\end{array}
\]

**Lemma 3.2.** For any complex line bundle \( x \) over an \( n \)-dimensional CW complex \( X \),

\[ k^e(n,k)J^o(\psi^k - 1)(x) = 0. \]

**Proof.** Since \( x = f^*(\eta_{[\frac{n}{2}]}) \) for some \( f : X \to \mathbb{C}P^{[\frac{n}{2}]} \), this lemma follows immediately from Lemma 3.1.

To prove Theorem 1, we may assume that \( X \) is a finite cell
complex by Lemma 3.1, since \( BU \times \mathbb{Z} \) is skeleton finite (under a suitable cellular decomposition). So from now on \( X \) is an \( n \)-dimensional finite cell complex.

For any \( x \in K(X) \) we may assume that \( x \) is an \( m \)-dimensional complex vector bundle for some \( m \). Let \( E \) be the total space of the associated principal \( U(m) \)-bundle. Let

\[
\beta_m : U(1) \times U(m-1) \to U(1)
\]

be the first projection and

\[
i_m : U(m) \to U(m)
\]

be the identity map. Put \( G = U(m) \) and \( H = U(1) \times U(m-1) \subset U(m) \).

The following is due to [11] (see also Appendix):

**Lemma 3.3.** \( \text{Ind}_{H}^{G}(\beta_m) = i_m \).

Note that \( \alpha(\beta_m) = x \). Since \( G \) is connected we have
Lemma 3.4. For any integer $k$, $\psi^k \circ \text{Ind}_H^G = \text{Ind}_H^G \circ \psi^k$.

A proof is given in [12].

Now we can prove Theorem 1. Note that $\alpha \circ \psi^k = \psi^k \circ \alpha$ or $\alpha' \circ \psi^k = \psi^k \circ \alpha'$ by definitions and

$$J \circ (\psi^k - 1)(x) = J \circ (\psi^k - 1)(\alpha(i_m))$$

$$= J \circ (\psi^k - 1)(\alpha(\text{Ind}_H^G(\beta_m))) \quad \text{(by Lemma 3.3)}$$

$$= J \circ \alpha \circ \text{Ind}_H^G(\psi^k - 1)(\beta_m) \quad \text{(by Lemma 3.4)}$$

$$= p' \circ J \circ \alpha' \circ (\psi^k - 1)(\beta_m) \quad \text{(by Theorem 2.3)}$$

$$= p' \circ J \circ (\psi^k - 1) \circ \alpha'(\beta_m).$$

Since $\alpha'(\beta_m)$ is a complex line bundle over an $n$-dimensional finite cell complex $(E/H)^{(n)}$, 
\[ k^e(n,k) J_\psi(\psi^k - 1)(\alpha'(\beta_m)) = 0 \]

by Lemma 3.2. So

\[ k^e(n,k) J_\psi(\psi^k - 1)(x) = k^e(n,k) J_\psi(\psi^k - 1)(\alpha'(\beta_m)) = 0. \]

This completes the proof.
§4. Proof of Theorem 2

Let \( r : K(X) \to KO(X) \) be the realization homomorphism defined by forgetting complex structures. Then the following lemmas are well known (see [4]):

**Lemma 4.1.** \( 2KO(X) \subseteq \text{Im } r \).

**Lemma 4.2.** The diagram

\[
\begin{array}{ccc}
K(X) & \xrightarrow{r} & KO(X) \\
\downarrow J & & \downarrow J \\
\text{Sph}(X) & & \\
\end{array}
\]

is commutative.

If \( k \) is even, then \( kx \in \text{Im } r \) for any \( x \in KO(X) \). So

\[
k^{e'}(n,k)J^*\left(\psi^k - 1\right)(x) = k^{e}(n,k)J^*\left(\psi^k - 1\right)(kx) = 0
\]

by Theorem 1.

From now on \( k \) is an odd integer. First we prove
Lemma 4.3. If $X$ is an $n$-dimensional CW complex and $x \in KO(X)$ is a linear combination of one or two dimensional real vector bundles, then

$$k^e(n,k)J^*(\psi^k - 1)(x) = 0.$$ 

Proof. By Theorem 1, Lemma 4.1 and Lemma 4.2,

$$2k^e(n,k)J^*(\psi^k - 1)(x) = k^e(n,k)J^*(\psi^k - 1)(2x) = 0.$$ 

But by the Adams conjecture for one or two dimensional real vector bundles [2], $J^*(\psi^k - 1)(x)$ is an odd torsion. This completes the proof. Q.E.D.

Lemma 4.4. Let $G$ be a compact Lie group and $H$ be its closed subgroup. If $(|G/G^0|, k) = 1$ ($G^0$ denotes the connected component of the identity), then

$$\psi^k \cdot \text{Ind}_H^G = \text{Ind}_H^G \psi^k : \text{RO}(H) \rightarrow \text{RO}(G).$$

A proof is given in Appendix.
In particular we have

**Corollary 4.5.** If $G = O(2n+1)$ and $H = O(2) \times O(2n-1)$ 
c $O(2n+1)$, then $\psi^k \text{Ind}_H^G = \text{Ind}_H^G \psi^k$ for any odd integer $k$.

Let $\iota$ be the identity of $G$, $\nu : H \to O(2)$ be the first 
projection and $\mu : G \to O(1)$ be the determinant (cf. Hashimoto
[10]). Then the following is Proposition 5 of [10] :

**Lemma 4.6.** $\iota = \text{Ind}_H^G (\nu) + \mu$.

Now using Lemma 4.3, Lemma 4.6 and Theorem 2.4 instead of
Lemma 3.2, Lemma 3.4 and Theorem 2.3 respectively, we can prove
Theorem 2 by a similar way.

**Remark 4.7.** We can prove Theorem 1 of Sigrist and Suter
[18] by making use of Theorem 2.4 and Lemma 4.6. In the proof
of [18], the fact that $s$-map induces a homomorphism of $J''$ ([2]) is not clear, since $s$-map does not commute with the Adams operations. Moreover the Atiyah transfer does not commute with the Adams operations. The fact that the Atiyah transfer coincides with the Becker-Gottlieb transfer, which is an easy consequence of the Atiyah-Singer index theorem for elliptic families ([6]), seems to be necessary.
Appendix

Let $G$ be a compact Real Lie group and $\text{RR}(G)$ be the Real representation ring. By forgetting involutions, a homomorphism $r : \text{RR}(G) \rightarrow \text{R}(G)$ is defined. As is well known $r$ is a monomorphism (cf. Atiyah-Segal [5]). Moreover we know the diagram

$$\begin{array}{ccc}
\text{RR}(G) & \xrightarrow{r} & \text{R}(G) \\
\downarrow \psi^k & & \downarrow \psi^k \\
\text{RR}(G) & \xrightarrow{r} & \text{R}(G)
\end{array}$$

is commutative. Let $H$ be a Real subgroup of $G$ and $\text{Ind}_H^G$ be the induction homomorphism defined by Hashimoto [10]. Then the diagram
\[
\begin{align*}
    \text{RR}(H) & \xrightarrow{r} \text{R}(H) \\
    \xrightarrow{\text{Ind}_H^G} & \xrightarrow{\text{Ind}_H^G} \\
    \text{RR}(G) & \xrightarrow{r} \text{R}(G)
\end{align*}
\]

is commutative (cf. [10]). Now applying Theorem 1 of [12], we have

**Lemma A.1.** If \(|G/G^0|, k\) = 1, then

\[
\psi^k \circ \text{Ind}_H^G = \text{Ind}_H^G \circ \psi^k : \text{RR}(H) \rightarrow \text{RR}(G).
\]

If the involution of \(G\) is trivial, then \(\text{RR}(G) = \text{RO}(G)\) and \(\psi^k\) and \(\text{Ind}_H^G\) on \(\text{RO}(\ )\) coincide with those on \(\text{RR}(\ )\). So Lemma 4.4 is proved.
References


