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On the order of certain elements of $J(X)$ and the Adams conjecture

by

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§1. Introduction

The Adams conjecture [2] was proved by several mathematicians in different methods (cf. [7],[8],[9],[10],[14],[15] and [19]). But in their methods, the localization plays an important role and so we can not estimate the order of an element

$$J\circ(\psi^k - 1)(x).$$

Let η_n be the canonical (complex) line bundle over CP^n and k an integer. Let $m(n,k)$ be the minimal positive integer such that

$$k^{m(n,k)} J\circ(\psi^k - 1)(\eta_n) = 0,$$

which exists by the Adams conjecture for complex line bundles [2].

We put

$$e(n,k) = m([\tfrac{n}{2}],k).$$

Then the purpose of this paper is to show

Theorem 1. If X is an n-dimensional CW complex, then

$$k^{e(n,k)} J\circ(\psi^k - 1)(x) = 0$$

for any $x \in K(X)$.

On the other hand let

$$e'(n,k) = \begin{cases} e(n,k) & \text{if } k \text{ is odd}, \\ e(n,k) + 1 & \text{if } k \text{ is even}. \end{cases}$$

Then by a quite similar method, we have

Theorem 2. If X is an n-dimensional CW complex, then

$$k^{e'(n,k)} J\circ(\psi^k - 1)(x) = 0$$

for any element $x \in KO(X)$.

To prove the above theorems, we do not use the Adams conjecture for general vector bundles. So as a corollary of Theorem 2, the Adams conjecture is proved. The proof of the above theorems is similar to the proof of the Adams conjecture of Nishida [14] and Hashimoto [10]. But we use relations between the induction homomorphisms and the Adams operations in [12] instead of the localization. We also use the cellular approximation of the Becker-Gottlieb transfer used by Sigrist and Suter in [18] instead of the usual Becker-Gottlieb transfer [8].

The paper is organized as follows :

In §2 some properties of the Becker-Gottlieb transfer are reviewed. Theorem 1 and Theorem 2 are proved in §3 and §4 respectively. A property of the real induction homomorphism used in this paper is proved in Appendix.

By a quite similar method to the proof of Theorem 1, we can prove Theorem 1 of Sigrist and Suter [18].

§2. Properties of the Becker-Gottlieb transfer

In this section X is an n-dimensional finite cell complex, G is a compact Lie group and H is a closed subgroup of G. Let E be the total space of a principal G-bundle over X. Then $p : E/H \to X$ is a fibre bundle whose fibre is a compact smooth manifold G/H and whose structure group is a compact Lie group G acting smoothly on G/H. Let $t(p) : (E/H)_+ \to X_+$ be the s-map defined by Becker and Gottlieb in [8]. Since X and $(E/H)_+$ are finite complexes, $t(p)$ is represented by a map

$$t : \Sigma^{\ell} \wedge X_+ \to \Sigma^{\ell} \wedge (E/H)_+$$

for some ℓ. Let $(E/H)^{(n)}$ be the n-skelton of E/H (for some cellular decomposition) and $j : (E/H)^{(n)} \subset E/H$ be the inclusion. Then by the cellular approximation theorem, there is a map

$$t' : \Sigma^{\ell} \wedge X_+ \to \Sigma^{\ell} \wedge ((E/H)^{(n)})_+$$

such that

$$\begin{array}{ccc} \Sigma^\ell \wedge X_+ & \xrightarrow{t} & \Sigma^\ell \wedge (E/H)_+ \\ & t' \searrow \quad \nearrow \Sigma^\ell \wedge j & \\ & \Sigma^\ell \wedge ((E/H)^{(n)})_+ & \end{array}$$

commutes. Define $p_!^!$ by the commutative diagram :

$$\begin{array}{ccccc} K((E/H)^{(n)}) & \overset{=}{\rightarrow} & \tilde{K}^0(((E/H)^{(n)})_+) & \overset{\sigma}{\rightarrow} & \tilde{K}^\ell(\Sigma^\ell \wedge ((E/H)^{(n)})_+) \\ \downarrow p_!^! & & & & \downarrow t'^* \\ K(X) & \overset{=}{\rightarrow} & \tilde{K}^0(X_+) & \overset{\sigma}{\rightarrow} & \tilde{K}^\ell(\Sigma^\ell \wedge X_+) \end{array}$$

where σ is the suspension isomorphism defined by the Bott periodicity theorem ([4]). The Becker-Gottlieb transfer $p_!$ $K(E) \rightarrow K(X)$ is defined by a similar way. Then by definitions the following diagram is commutative :

$$\begin{array}{ccc} K((E/H)^{(n)}) & \overset{j^*}{\leftarrow} & K(E/H) \\ & p_!^! \searrow \quad \swarrow p_! & \\ & K(X) . & \end{array}$$

Let V be a complex H-module and $\alpha : R(H) \rightarrow K(E/H)$ be a homomorphism defined by $V \rightarrow (E \times_H V \rightarrow E/H)$. Define

$$\alpha' : R(H) \rightarrow K((E/H)^{(n)})$$

by $\alpha' = j^* \circ \alpha$. Then we have

<u>Lemma 2.1.</u> The following diagram is commutative :

$$\begin{array}{ccc} R(H) & \xrightarrow{\alpha'} & K((E/H)^{(n)}) \\ \downarrow \mathrm{Ind}_H^G & & \downarrow p_! \\ R(G) & \xrightarrow{\alpha} & K(X), \end{array}$$

where Ind_H^G is the induction homomorphism defined by Segal [16] (see also [10]).

Proof. This is an easy consequence of the commutative diagram

$$\begin{array}{ccc} R(H) & \xrightarrow{\alpha} & K(E/H) \\ \downarrow \mathrm{Ind}_H^G & & \downarrow p_! \\ R(G) & \xrightarrow{\alpha} & K(X) \end{array}$$

which is Proposition 5.4 of Nishida [14].

Let $\widetilde{Sph}^*(\ )$ be the generalized cohomology theory defined by the stable spherical fibrations and $Sph(X) = \widetilde{Sph}^0(X_+)$. Define

$$p_*^! : K((E/H)^{(n)}) \to K(X)$$

or

$$p_*^! : Sph((E/H)^{(n)}) \to Sph(X)$$

by a similar way to $p_!$ using the suspension isomorphisms defined by the infinite loop space structures defined by the Γ-structures (cf. Segal [17]). Since J is an infinite loop map with respect to these infinite loop space structures, we have (cf. Nishida [14])

<u>Lemma 2.2.</u> The following diagram is commutative :

$$\begin{array}{ccc} K((E/H)^{(n)}) & \xrightarrow{J} & Sph((E/H)^{(n)}) \\ \downarrow p_*^! & & \downarrow p_*^! \\ K(X) & \xrightarrow{J} & Sph(X). \end{array}$$

By May [13], the infinite loop space structure of $BU \times Z$ defined by the Γ-structure is equivalent to that defined by the Bott periodicity theorem. Then $p_! = p_*^!$ and so we have

Theorem 2.3. The diagram

$$\begin{array}{ccccc} R(H) & \xrightarrow{\alpha'} & K((E/H)^{(n)}) & \xrightarrow{J} & Sph((E/H)^{(n)}) \\ \downarrow Ind_H^G & & \downarrow p_*^! & & \downarrow p_*^! \\ R(G) & \xrightarrow{\alpha} & K(X) & \xrightarrow{J} & Sph(X) \end{array}$$

is commutative.

Quite similarly we have (cf. Hashimoto [10])

Theorem 2.4. The diagram

$$\begin{array}{ccccc} RO(H) & \xrightarrow{\alpha'} & KO((E/H)^{(n)}) & \xrightarrow{J} & Sph((E/H)^{(n)}) \\ \downarrow Ind_H^G & & \downarrow p_*^! & & \downarrow p_*^! \\ RO(G) & \xrightarrow{\alpha} & KO(X) & \xrightarrow{J} & Sph(X) \end{array}$$

is commutative where Ind_H^G is the induction homomorphism of real representation rings defined by Hashimoto [10].

§3. Proof of Theorem 1

First recall the following lemmas.

Lemma 3.1. Let $f : Y \to Y'$ be a (continuous) map and $y \in K(Y')$. If $k^e J\circ(\psi^k - 1)(y) = 0$, then $k^e J\circ(\psi^k - 1)(f^*(y)) = 0$.

Proof. This is an easy consequence of the following commutative diagram :

$$\begin{array}{ccc} K(Y') & \xrightarrow{f^*} & K(Y) \\ \downarrow J & & \downarrow J \\ Sph(Y') & \xrightarrow{f^*} & Sph(Y). \end{array}$$

Lemma 3.2. For any complex line bundle x over an n-dimensional CW complex X,

$$k^{e(n,k)} J\circ(\psi^k - 1)(x) = 0.$$

Proof. Since $x = f^*(\eta_{[\frac{n}{2}]})$ for some $f : X \to CP^{[\frac{n}{2}]}$, this lemma follows immidiately from Lemma 3.1.

To prove Theorem 1, we may assume that X is a finite cell complex by Lemma 3.1, since $BU \times Z$ is skeleton finite (under a suitable cellular decomposition). So from now on X is an n-dimensional finite cell complex.

For any $x \in K(X)$ we may assume that x is an m-dimensional complex vector bundle for some m. Let E be the total space of the associated principal $U(m)$-bundle. Let

$$\beta_m : U(1) \times U(m-1) \to U(1)$$

be the first projection and

$$\iota_m : U(m) \to U(m)$$

be the identity map. Put $G = U(m)$ and $H = U(1) \times U(m-1) \subset U(m)$. The following is due to [11] (see also Appendix) :

Lemma 3.3. $\mathrm{Ind}_H^G(\beta_m) = \iota_m$.

Note that $\alpha(\beta_m) = x$. Since G is connected we have

<u>Lemma 3.4.</u> For any integer k, $\psi^k \circ \mathrm{Ind}_H^G = \mathrm{Ind}_H^G \circ \psi^k$.

A proof is given in [12].

Now we can prove Theorem 1. Note that $\alpha \circ \psi^k = \psi^k \circ \alpha$ or $\alpha' \circ \psi^k = \psi^k \circ \alpha'$ by definitions and

$$
\begin{aligned}
J \circ (\psi^k - 1)(x) &= J \circ (\psi^k - 1)(\alpha(\iota_m)) \\
&= J \circ (\psi^k - 1)(\alpha(\mathrm{Ind}_H^G(\beta_m))) && \text{(by Lemma 3.3)} \\
&= J \circ \alpha \circ \mathrm{Ind}_H^G \circ (\psi^k - 1)(\beta_m) && \text{(by Lemma 3.4)} \\
&= p_*^! \circ J \circ \alpha' \circ (\psi^k - 1)(\beta_m) && \text{(by Theorem 2.3)} \\
&= p_*^! \circ J \circ (\psi^k - 1) \circ \alpha'(\beta_m).
\end{aligned}
$$

Since $\alpha'(\beta_m)$ is a complex line bundle over an n-dimensional finite cell complex $(E/H)^{(n)}$,

$$k^{e(n,k)} J \circ (\psi^k - 1)(\alpha'(\beta_m)) = 0$$

by Lemma 3.2. So

$$k^{e(n,k)} J \circ (\psi^k - 1)(x) = k^{e(n,k)} J \circ (\psi^k - 1)(\alpha'(\beta_m)) = 0.$$

This completes the proof.

§4. Proof of Theorem 2

Let $r : K(X) \to KO(X)$ be the realization homomorphism defined by forgetting complex structures. Then the following lemmas are well known (see [4]) :

<u>Lemma 4.1.</u> $2KO(X) \subset \operatorname{Im} r$.

<u>Lemma 4.2.</u> The diagram

$$\begin{array}{ccc} K(X) & \xrightarrow{r} & KO(X) \\ & J\searrow \quad \swarrow J & \\ & Sph(X) & \end{array}$$

is commutative.

If k is even, then $kx \in \operatorname{Im} r$ for any $x \in KO(X)$. So

$$k^{e'(n,k)} J \circ (\psi^k - 1)(x) = k^{e(n,k)} J \circ (\psi^k - 1)(kx) = 0$$

by Theorem 1.

From now on k is an odd integer. First we prove

Lemma 4.3. If X is an n-dimensional CW complex and $x \in KO(X)$ is a linear combination of one or two dimensional real vector bundles, then

$$k^{e(n,k)} J \circ (\psi^k - 1)(x) = 0.$$

Proof. By Theorem 1, Lemma 4.1 and Lemma 4.2,

$$2k^{e(n,k)} J \circ (\psi^k - 1)(x) = k^{e(n,k)} J \circ (\psi^k - 1)(2x) = 0.$$

But by the Adams conjecture for one or two dimensional real vector bundles [2], $J \circ (\psi^k - 1)(x)$ is an odd torsion. This completes the proof. Q.E.D.

Lemma 4.4. Let G be a compact Lie group and H be its closed subgroup. If $(|G/G^0|, k) = 1$ (G^0 denotes the connected component of the identity), then

$$\psi^k \circ \mathrm{Ind}_H^G = \mathrm{Ind}_H^G \circ \psi^k : RO(H) \to RO(G).$$

A proof is given in Appendix.

In particular we have

<u>Corollary 4.5.</u> If $G = O(2n+1)$ and $H = O(2) \times O(2n-1) \subset O(2n+1)$, then $\psi^k \circ \mathrm{Ind}_H^G = \mathrm{Ind}_H^G \circ \psi^k$ for any odd integer k.

Let ι be the identity of G, $\nu : H \to O(2)$ be the first projection and $\mu : G \to O(1)$ be the determinant (cf. Hashimoto [10]). Then the following is Proposition 5 of [10] :

<u>Lemma 4.6.</u> $\iota = \mathrm{Ind}_H^G(\nu) + \mu$.

Now using Lemma 4.3, Lemma 4.6 and Theorem 2.4 instead of Lemma 3.2, Lemma 3.4 and Theorem 2.3 respectively, we can prove Theorem 2 by a similar way.

<u>Remark 4.7.</u> We can prove Theorem 1 of Sigrist and Suter [18] by making use of Theorem 2.4 and Lemma 4.6. In the proof

of [18], the fact that s-map induces a homomorphism of J" ([2]) is not clear, since s-map does not commute with the Adams operations. Moreover the Atiyah transfer does not commute with the Adams operations. The fact that the Atiyah transfer coincides with the Becker-Gottlieb transfer, which is an easy consequence of the Atiyah-Singer index theorem for elliptic families ([6]), seems to be necessary.

Appendix

Let G be a compact Real Lie group and $RR(G)$ be the Real representation ring. By forgetting involutions, a homomorphism $r : RR(G) \to R(G)$ is defined. As is well known r is a monomorphism (cf. Atiyah-Segal [5]). Moreover we know the diagram

$$\begin{array}{ccc} RR(G) & \xrightarrow{r} & R(G) \\ \downarrow \psi^k & & \downarrow \psi^k \\ RR(G) & \xrightarrow{r} & R(G) \end{array}$$

is commutative. Let H be a Real subgroup of G and Ind_H^G be the induction homomorphism defined by Hashimoto [10]. Then the diagram

$$\begin{array}{ccc} RR(H) & \xrightarrow{r} & R(H) \\ \downarrow \mathrm{Ind}_H^G & & \downarrow \mathrm{Ind}_H^G \\ RR(G) & \xrightarrow{r} & R(G) \end{array}$$

is commutative (cf. [10]). Now applying Theorem 1 of [12], we have

<u>Lemma A.1.</u> If $(|G/G^0|,k) = 1$, then

$$\psi^k \circ \mathrm{Ind}_H^G = \mathrm{Ind}_H^G \circ \psi^k : RR(H) \to RR(G).$$

If the involution of G is trivial, then $RR(G) = RO(G)$ and ψ^k and Ind_H^G on $RO(\ )$ coincide with those on $RR(\ )$. So Lemma 4.4 is proved.

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