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Kyoto University
On the $\beta$-family in stable homotopy
of spheres at the prime 3

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The stable homotopy of spheres $\pi^S_*(p)$ localized at an odd prime
$p$ has the Adams filtration associated to BP, the Brown-Peterson
spectrum at $p$:

$$\pi^S_*(p) \supset F^1 \supset F^2 \supset \ldots,$$

and $F^1/F^2$ is a direct summand isomorphic to the image of $J$. In case
$p \geq 5$, there is an infinite family $\{\beta_t\}$, called the $\beta$-family, in
$F^2 - F^3$ (Smith, [10]), but in case $p = 3$ only a part of $\beta$'s exists,
namely, $\beta_1, \beta_2, \beta_3, \text{ exist (Toda, [12])}, \beta_4 \text{ does not and } \beta_5 \text{ does (Oka, [5])},$
$\beta_6 \text{ does (Nakamura, [4], Tangora[11])}, \beta_7 \text{ and } \beta_8 \text{ do not (Ravenel,}
unpublished), and $\beta_9 \text{ does exists (Ravenel, Knapp[2],[3]).}$ Karl Heinz
Knapp [2],[3] proved that for $p \geq 5$, $\beta_{p+1}$ is not in the image of the
bi-stable $J$-homomorphism $J' : \pi^S_*(SO)(p) \rightarrow \pi^S_*(p)$. This gives a
counterexample to the conjecture of G.W. Whitehead : $J'$ is onto.

Unfortunately his proof does not work for $p = 3$, because $\beta_4$ does not
exist. He told me the first candidate giving a counterexample at $p=3$
is $\beta_{10}$, and asked me whether or not $\beta_{10}$ exists. In 1977-1978, Doug
Ravenel wrote me that the $BP_*$-module $BP_*/(3, v_1^2, v_2^9)$ is realized by an
8-cell complex and it would follow that $\beta_t$ exists whenever
t $\not\equiv 4,7$ or $8 \mod 9$ (cf. [8], p.144). His proof of the realization is
based on his extensive calculation of BP-Adams spectral sequence up to
dim $\leq 144$, and I do not know his publication of the result. I have,
however, proved that the realization of $BP_*/(3, v_1^2, v_2^9)$ implies the
existence of $\beta_t$ for $t \equiv 0,1,2,5,6 \mod 9$, and I feel there is a gap
in proving for $t \equiv 3 \mod 9$. My proof on $\beta_{10}$ here is independent of
Ravenel's. I use the result on $\pi^S_*(3)$ up to dim 80, though dim $\beta_{10} = 154$.

Lemma 1. For $t = 1,5$, there is a map

$$b_t : S^{16t} \rightarrow V = S^0 u_3 e_1 u_1 e_5 u_3 e_6$$

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such that \((b_t)_* = v_2^t\) and \(\pi_0 b_t = b_t \in \pi_5^S\), where \(\pi_0 : V \longrightarrow S^6\) collapses the 5-skeleton of \(V\).

Proof. \(V = V(1)\) in [10],[15], and \(b_1\) is the attaching map of the top cell in \(V(l_{14}^1) = V \cup e_1\). By the results on \(\pi_5^S\), \(\dim_7 \leq 80\), \(b_5\) has a factorization \(S^{80} \longrightarrow V \longrightarrow S^6\). Then \(b_5\) has a property \((b_5)_* = v_2^5\) by the Geometric Boundary Theorem (=G.B. Th.) [1].

Put \(M = S^0 u_3 e_1\). Then \(V\) is a mapping cone of some map \(\alpha : \Sigma^4 M \longrightarrow M\), and we have the cofibrations.

\[
\begin{array}{ccc}
M & \overset{i_1}{\longrightarrow} & V \\
\downarrow \quad \quad \quad \gamma & \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \
Hence $q_\#$ is the inverse of the $BU_*BU$-comodule homomorphism $(i_0^1)_\#$, or the zero homomorphism. Therefore $q_\#$ is a comodule homomorphism, though it is not a induced homomorphism of $BU$-homology. Similarly, $\gamma_\#$ is also a comodule homomorphism.

Now $(b_5 \wedge b_5)_\#(1) \in BP_{160}(V \wedge V)$ and we have $q_\#(b_5 \wedge b_5)_\#(1) = v_2^5v_2^5 = v_2^{10}$ because $q$ gives the multiplication on $BP \wedge V$ such that $1 \wedge 1_0 : BP \to BP \wedge V$ is a map of ring spectra [16]. By the G.B.th., we have

**Theorem 1.** The composite $s_{160} \xrightarrow{b_5 \wedge b_5} V \wedge V \xrightarrow{u} \Sigma^8 M \xrightarrow{\pi} s^6$ projects to $\beta_{10} \in \operatorname{Ext}_{BP_*}^{2,*}(BP_*, BP_*)$. Thus $\beta_{10} \in \pi_{154}^6$ exists and its order is 3.

**Remark.** Let $D$ be the Spanier-Whitehead dual functor (contravariant). Here the duality map for a finite CW complex (spectrum) $X$ is taken to be the map $X \wedge D(X) \to s^0$. Then $D(V) = \Sigma^{-6}V$, $D(S^n) = S^{-n}$ so $D(b_5) : \Sigma^{-6}V \to S^{-80}$. Then the above $\beta_{10} = \pi(u(b_5 \wedge b_5))$ is the composite $D(b_5)b_5$. As in Lemma 1, $\beta_1$ has a similar property, so we have also $\beta_2 = \pi(u(b_1 \wedge b_1))$, $\beta_6 = \pi(u(b_1 \wedge b_5))$ in $\pi_*^S(3)$.

Let $V' = S^0 u_3 e_1 u_2 e_9 u_3 e_10$, $V'' = S^0 u_3 e_1 u_3 e_13 u_3 e_{14}$, then

$BP_*V' = BP_*/(3, v_1^2)$, $BP_*V'' = BP_*/(3, v_1^3)$. Let $\lambda : \Sigma^4 V \to V'$, $\lambda' : \Sigma^4 V' \to V''$ be the maps such that $\lambda_*$ and $\lambda'_*$ are the multiplication by $v_1$.

Let $\beta : \Sigma^{16} V \wedge B \to V \wedge B$ be the map in [7], and define $B_t : \Sigma^{16t-11} B \to V$ to be the composite

$\xymatrix{ \Sigma^{16t-11} B \ar[r]^{i_0^1} & \Sigma^{16t-11} V \wedge B \ar[r]^{\beta t} & \Sigma^{-11} V \wedge B \ar[r]^{1 \wedge k} & V.}$

Then $BP_*(B) = BP_* + \Sigma^{11} BP_*$, and $(B_t)_* = 0$ on the bottom cell generator and $(B_t)_* = v_2$ on the top cell generator.

**Lemma 2.** For $s = 0, 1, 2, 5, 6$, there is a map $c_s : S^{16s+4} \to V'$ such that $(c_s)_* = v_1^sv_2$. -3-
Proof. For \( s = 0 \), put \( c_0 = i_0 \), and for \( s = 1, 5 \), put \( c_s = b_s \). Let \( j : S^0 \rightarrow B \) be the inclusion. Then, for \( s = 2, 6 \), 
\[ B_s, j = i_1 \xi_s \text{ for some } \xi_s \in \pi_{16s-11}(M), \text{ and } \lambda B_s j = i_1 \alpha_c \xi_s = 0 \text{ because} \]
\[ \alpha \hat{\omega}_{16s-11}(M) \subset \pi_{16s-7}(M) = 0. \text{ Hence } \lambda B_s = c_s k \text{ and } (c_s)_* = v_1 v_2^s. \]
For \( s = 3 \), \( B_3 j = i_1 \iota c' \), so \( \lambda B_3 j = i_1 \alpha c' \neq 0 \) because
\[ \alpha_1 c' = \beta_1^4 \neq 0 [14]. \text{ From this, the lemma is not true for } s = 3. \]
We have \( \lambda' \lambda B_3 j = 0 \), and

**Lemma 2**. There is a map \( c'_3 : S^{56} \rightarrow V' \) with \( (c'_3)_* = v_1^2 v_2^3. \)
Now the map \( v_3^2 : BP_* \rightarrow BP_*/(3, v_1^2) \) is the element in
\[ H^0 BP_*/(3, v_1^2) = H^0 BP_*(V'), \text{ and } d_5 (v_2^3) \neq 0 \text{ in the BP-Adams spectral sequence converging to } \pi_*(V'). \text{ V' has a multiplication [6], so the spectral sequence is multiplicative. Although the multiplication}
\]
on \( V' \) is not associative (because, the sub ring spectrum \( M \) is not associative [13], [15]), we have \( d_5 (x^3) = 3x^2 d_5 (x) \) for \( x = v_2^3 \), so \( d_5 (v_2^9) = 0 \). The next differentials possibly killing \( v_2^9 \) are \( d_9, d_{13}, ... \). By calculating \( H^* BP_*/(3, v_1^2) \) up to \( \text{dim } \leq 144 \), Ravenel claimed that there are no such differentials, that is,

**Claim.** \( v_2^9 \) \( H^0, 144 \) \( BP_*(V') \) is a permanent cycle.

Then there is a map \( v : S^{144} \rightarrow V' \) with \( v_* = v_2^9. \)
The composite \( \bar{v} : \Sigma^{144} V' \xrightarrow{v \Lambda 1} V' \wedge V' \rightarrow V' \) also satisfies
\( (\bar{v})_* = v_2^9 \), and hence the mapping cone of \( \bar{v} \) clearly realizes \( BP_*/(3, v_1^2, v_2^9). \)

**Theorem 2.** Claim implies that \( \beta_t \) is a permanent cycle if \( t \equiv 0, 1, 2, 5, 6 \text{ mod } 9. \)

**Proof.** Put \( t = 9k + s \), \( 0 \leq s < 9 \). The composite
\[ c_t : S^{16t+4} = S^{144k+16s+4} \xrightarrow{c_s} \Sigma^{144k} V' \xrightarrow{v^{k}} V' \]
satisfies \( (c_t)_* = v_1^t v_2^t. \) Then, using the G.B.Th. twice, we see that \( \beta_t \in H^2 BP_* \) is a permanent cycle and converges to

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$$\pi_0^{C_t} : s^{16t+4} \to v' \to s^{10}.$$ Using Lemma 2' instead of 2, we have

**Theorem 2'**. If $v_2^9 \ H^0,144BP_*/(3,v_1^3) = H^0,144BP_*(v'')$ is a permanent cycle, then $\beta_{g+3}$ is a permanent cycle.

**Theorem 2''**. If claim holds and the corresponding homotopy element $v$ satisfies $\{v, 3, \beta_4^1\} = \{0\}$, then $\beta_{g+3}$ is a permanent cycle.

For, the additional assumption implies the existence of the map $c_{12}$ as in Lemma 2.

**References**


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