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On the $\beta$-family in stable homotopy of spheres at the prime 3

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The stable homotopy of spheres $\pi_*^S(p)$ localized at an odd prime $p$ has the Adams filtration associated to $BP$, the Brown-Peterson spectrum at $p$:

$$\pi_*^S(p) \supseteq F^1 \supseteq F^2 \supseteq \ldots,$$

and $F^1/F^2$ is a direct summand isomorphic to the image of $J$. In case $p \geq 5$, there is an infinite family $\{\beta_t\}$, called the $\beta$-family, in $F^2 - F^3$ (Smith, [10]), but in case $p = 3$ only a part of $\beta$'s exists, namely, $\beta_1, \beta_2, \beta_3, \beta_4$ exist (Toda, [12]), $\beta_4$ does not and $\beta_5$ does (Oka, [5]), $\beta_6$ does (Nakamura, [4], Tangora[11]), $\beta_7$ and $\beta_8$ do not (Ravenel, unpublished), and $\beta_9$ does exist (Ravenel, Knapp[2],[3]). Karl Heinz Knapp [2],[3] proved that for $p \geq 5$, $\beta_{p+1}$ is not in the image of the bi-stable $J$-homomorphism $J' : \pi_*^S(SO)(p) \to \pi_*^S(p)$. This gives a counterexample to the conjecture of G.W. Whitehead : $J'$ is onto.

Unfortunately his proof does not work for $p = 3$, because $\beta_4$ does not exist. He told me his first candidate giving a counterexample at $p=3$ is $\beta_{10}$, and asked me whether or not $\beta_{10}$ exists. In 1977-1978, Doug Ravenel wrote me that the $BP_*$-module $BP_*/(3, v_1^2, v_2^9)$ is realized by an 8-cell complex and it would follow that $\beta_t$ exists whenever $t \neq 4,7$ or 8 mod 9 (cf. [8], p.144). His proof of the realization is based on his extensive calculation of $BP$-Adams spectral sequence up to dim $\leq 144$, and I do not know his publication of the result. I have, however, proved that the realization of $BP_*/(3, v_1^2, v_2^9)$ implies the existence of $\beta_t$ for $t \equiv 0,1,2,5,6$ mod 9, and I feel there is a gap in proving for $t \equiv 3$ mod 9. My proof on $\beta_{10}$ here is independent of Ravenel's. I use the result on $\pi_*^S(3)$ up to dim 80, though dim $\beta_{10} = 154$.

Lemma 1. For $t = 1,5$, there is a map

$$b_t : S^{16t} \to v = S^0 u_3 e_1 u_1 e_5 u_3 e_6$$

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such that \((b_L)_\ast = v_2^\ast\) and \(\pi_0b_t = \beta_t \in \pi_t^S\), where \(\pi_0 : V \rightarrow S^6\) collapses the 5-skeleton of \(V\).

**Proof.** \(V = V(1)\) in [10],[15], and \(b_1\) is the attaching map of the top cell in \(V(1)) \in \Sigma V\). By the results on \(\pi_0^S\), \(\dim \leq 80\), \(\beta_5\) has a factorization \(S^80 \xrightarrow{b_5} V \rightarrow S^6\). Then \(b_5\) has a property \((b_5)_\ast = v_2^5\) by the Geometric Boundary Theorem (=G.B. Th.) [1].

Let \(M = S^0\), Then \(V\) is a mapping cone of some map
\[
\alpha : \Sigma^4M \rightarrow M, \text{and we have the cofibrations,}
\]
\[
\begin{array}{ccc}
M & \xrightarrow{\ i_1 \ } & V \\
\pi_1 & \xrightarrow{\ } & \Sigma^5M, \\
S^0 & \xrightarrow{\ i \ } & M \rightarrow \pi_1 S^1.
\end{array}
\]

Using a similar homotopy as in [9], p.374, we can construct a map \(u : \nu\nuV \rightarrow \Sigma^5M\) such that \(u(i_1\Lambda 1) = \mu_M^0(l\Lambda i_1)\), \(u(l\Lambda i_1) = \mu_M^0(l\Lambda 1)\), where \(\mu_M = M \wedge M \rightarrow M\) is the multiplication. \(\nu\) is unique, and \(\pi u : \nu\nuV \rightarrow S^6\) gives the self-duality of \(V\). If \(V\) has a multiplication, \(\nu\) is defined to be \(\nu\nuV \xrightarrow{\nu} V \rightarrow V/M = \Sigma^5M\), but \(\nu\) does not exist for \(p = 3\) (Toda). Let \(U\) be the fiber of \(u\), then there is a commutative diagram :

\[
\begin{array}{c}
\Sigma^4M \xrightarrow{\ i\Lambda 1 \ } U \xrightarrow{\ u \ } V \xrightarrow{\ \nu\nuV \ } \Sigma^5M
\end{array}
\]

Here \(\iota : S^0 \rightarrow BP\), \(i_0 = i_1 i : S^0 \rightarrow V\) are the inclusion and \(q\) is a unique map such that \(q(i_0\Lambda 1) = \nu\Lambda 1 = q(l\Lambda i_0)\) [5]. Then we have a comm. diagram

\[
\begin{array}{c}
0 \rightarrow BP_\ast(\Sigma^4M) \rightarrow BP_\ast(U) \rightarrow BP_\ast(V\Lambda V) \rightarrow 0
\end{array}
\]

\[
\begin{array}{c}
0 \rightarrow BP_\ast(\Sigma^4M) \rightarrow BP_\ast(U) \rightarrow BP_\ast(V\Lambda V) \rightarrow 0,
\end{array}
\]

where \(q_\#\) is induced by \(BP_\Lambda V \xrightarrow{\nu\Lambda q} BP_\Lambda BP_\Lambda V \xrightarrow{\nu\Lambda 1} BP_\Lambda V\) and \(\gamma_\#\) is similarly defined. The composite \(BP_\ast(V) \xrightarrow{\nu\Lambda 1} BP_\Lambda V\) and \(\nu\Lambda 1\) is isomorphic in \(\dim \equiv 0 \mod 4\).
Hence $q_\#$ is the inverse of the $BP_*BP$-comodule homomorphism $(i_0 \wedge 1)^*$ or the zero homomorphism. Therefore $q_\#$ is a comodule homomorphism, though it is not a induced homomorphism of $BP$-homology. Similarly, $\gamma_\#$ is also a comodule homomorphism.

Now $(b_5 \wedge b_5)_*(1) \in BP_{160}(V \wedge V)$ and we have $q_\#(b_5 \wedge b_5)_*(1) = v_2^5 v_2^5 = v_2^{10}$ because $q$ gives the multiplication on $BP \wedge V$ such that $1 \wedge i_0 : BP \to BP \wedge V$ is a map of ring spectra [16]. By the G.B.th., we have

**Theorem 1.** The composite $S^{160} \xrightarrow{b_5 \wedge b_5} V \wedge V \xrightarrow{u} \Sigma^{5M} \xrightarrow{\pi} S^6$ projects to $\beta_{10} \in \text{Ext}^2_*(BP_*, BP_*)$. Thus $\beta_{10} \in \pi_1^{s5}$ exists and its order is 3.

**Remark.** Let $D$ be the Spanier–Whitehead dual functor (contravariant). Here the duality map for a finite CW complex (spectrum) $X$ is taken to be the map $X \wedge D(X) \to S^0$. Then $D(V) = \Sigma^{-6} V$, $D(S^n) = S^{-n}$ so $D(b_5) : \Sigma^{-6} V \to S^{-80}$. Then the above $\beta_{10} = \pi u(b_5 \wedge b_5)$ is the composite $D(b_5)b_5$. As in Lemma 1, $\beta_1$ has a similar property, so we have also

$\beta_2 = \pi u(b_1 \wedge b_1)$, $\beta_6 = \pi u(b_1 \wedge b_5)$ in $\pi_1^{s5}(3)$.

Let $V' = S^0 u_3 e^9 u_3 e^{10}$, $V'' = S^0 u_3 e^9 u_3 e^{13} u_3 e^{14}$, then

$BP_*V' = BP_*/(3, v_2^2)$, $BP_*V'' = BP_*/(3, v_1^2)$. Let $\lambda : \Sigma^4 V \to V'$, $\lambda' : \Sigma^4 V' \to V''$ be the maps such that $\lambda_*$ and $\lambda'_*$ are the multiplication by $v_1$.

Let $\beta : \Sigma^{16} V \wedge B \to V \wedge B$ be the map in [7], and define

$B_t : \Sigma^{16t-11} B \to V$ to be the composite

$\Sigma^{16t-11} B \xrightarrow{i_0 \wedge 1} \Sigma^{16t-11} V \wedge B \xrightarrow{\beta^t} \Sigma^{-11} V \wedge B \xrightarrow{1 \wedge k} V$.

Then $BP_*(B) = BP_* + \Sigma^1 BP_*$, and $(B_t)^*_0 = 0$ on the bottom cell generator and $(B_t)^*_* = v_2$ on the top cell generator.

**Lemma 2.** For $s = 0, 1, 2, 5, 6$, there is a map $c_s : S^{16s+4} \to V'$ such that $(c_s)^*_* = v_1 v_2^s$. 

---
Proof. For \( s = 0 \), put \( c_0 = \lambda i_0 \), and for \( s = 1,5 \), put \( c_s = b_s^\lambda \). Let \( j : S^0 \rightarrow B \) be the inclusion. Then, for \( s = 2,6 \), \( b_s^j = i_1^1 \xi_s \) for some \( \xi_s \in \pi_{16s-11}(M) \), and \( \lambda b_s^j = i_1^1 \alpha_0 \xi_s = 0 \) because \( a_0 \pi_{16s-11}(M) \subset \pi_{16s-7}(M) = 0 \). Hence \( \lambda b_s^j = c_s^k \) and \( (c_s^\lambda)_* = v_1^2 v_2^s \).

For \( s = 3 \), \( B_3^j = i_1^1 i_1^1 \epsilon' \), so \( \lambda B_3^j = i_1^1 \alpha_1 \epsilon ' \neq 0 \) because \( \alpha_1 \epsilon ' = \beta_1^4 \neq 0 [14] \). From this, the lemma is not true for \( s = 3 \).

We have \( \lambda' \lambda B_3^j = 0 \), and

Lemma 2'. There is a map \( c_3' : S^6 \rightarrow V' \) with \( (c_3')_* = v_1^2 v_2^3 \).

Now the map \( v_2^3 : BP_* \rightarrow BP_*/(3, v_1^2) \) is the element in \( H^0 BP_*/(3, v_1^2) = H^0 BP_*(V') \), and \( d_5(v_2^3) \neq 0 \) in the BP-Adams spectral sequence converging to \( \pi_*(V') \). \( V' \) has a multiplication [6], so the spectral sequence is multiplicative. Although the multiplication on \( V' \) is not associative (because, the sub ring spectrum \( M \) is not associative [13],[15]), we have \( d_5(x^3) = 3x^2 d_5(x) \) for \( x = v_2^3 \), so \( d_5(v_2^9) = 0 \). The next differentials possibly killing \( v_2^9 \) are \( d_9 \), \( d_{13} \), ..., . By calculating \( H^* BP_*/(3, v_1^2) \) up to \( \text{dim} \leq 144 \), Ravenel claimed that there are no such differentials, that is,

Claim. \( v_2^9 \in H^0,144 BP_*(V') \) is a permanent cycle.

Then there is a map \( v : S^{144} \rightarrow V' \) with \( v_* = v_2^9 \).

The composite \( \bar{v} : \Sigma^{144}V' \xrightarrow{v} V' \xrightarrow{\text{val}} V' \) also satisfies \( (\bar{v})_* = v_2^9 \), and hence the mapping cone of \( \bar{v} \) clearly realizes \( BP_*/(3, v_1^2, v_2^9) \).

Theorem 2. Claim implies that \( \beta_t \) is a permanent cycle if \( t = 0, 1, 2, 5, 6 \) mod 9.

Proof. Put \( t = 9k + s \), \( 0 \leq s < 9 \). The composite

\[
c_t : S^{16t+4} = S^{144k+16s+4} \xrightarrow{c_s} \Sigma^{144k}V' \xrightarrow{\bar{v}} V'
\]
satisfies \( (c_t)_* = v_1^t v_2^s \). Then, using the G.B.Th. twice, we see that \( \beta_t \in H^2 BP_* \) is a permanent cycle and converges to

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\[ \pi'_0 C_t : s^{16t+4} \longrightarrow v' \longrightarrow s^{10}. \]

Using Lemma 2' instead of 2, we have

**Theorem 2'.** If \[ v^9 \in H^0,^{144}_{BP_*/(3,v^3_1)} = H^0,^{144}_{BP_*}(v'') \] is a permanent cycle, then \( \beta_{9k+3} \) is a permanent cycle.

**Theorem 2".** If claim holds and the corresponding homotopy element \( v \) satisfies \( \{v, 3, \beta^4_1\} = \{0\} \), then \( \beta_{9k+3} \) is a permanent cycle.

For, the additional assumption implies the existence of the map \( c_{12} \) as in Lemma 2.

References


