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$J$-groups of lens spaces

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§1. Introduction

The standard lens space mod $m$ is the orbit manifold

$$L^n_m = S^{2n+1}/\mathbb{Z}_m \quad (\mathbb{Z}_m = \langle z \in S^1 : z^m = 1 \rangle)$$

of the $(2n+1)$-sphere $S^{2n+1}(c^{n+1})$ by the diagonal action

$$z(z_0, \ldots, z_n) = (zz_0, \ldots, zz_n).$$

The $J$-groups of lens spaces were studied by several authors (e.g. [2],[5],[6],[8],[10] and [11]).

Let $\eta_m$ be the canonical complex line bundle over $L^n_m$. Then we have the following theorem by making use of the results due to J.F. Adams [1] and D. Quillen [12].

Theorem 1.1. Let $p$ be a prime and let $r(\eta^i_{pr-1}) \in KO(L^n(p^r))$ ($p^r \geq 3$) be the real restriction of the stable class of the $i$-fold tensor product of $\eta_{pr}$. Then the order of the $J$-image

$$Jr(\eta^i_{pr-1}) \in J(L^n(p^r))$$

is equal to $p f(p,n,r;\nu)$,

$$f(p,n,r;\nu) = \max \{ s - \nu + \lfloor n/p^s(p-1) \rfloor p^{s-\nu} : \nu s < r \text{ and } p^s(p-1) s n \},$$
where $v_p(1)$ is the exponent of $p$ in the prime power decomposition of $1$ and $\max \phi = 0$.

In this lecture, we prove the above theorem only the case $p=2$, since we can prove the above theorem for odd prime $p$ in a similar way (see [10]).

Remark 1.2. By Theorem 1.1 and Proposition 1.3 below, we can determine the order of $J_r(\eta_m^1-1)$ in $\tilde{J}(\mathbb{L}_m^n(m))$ for any $m$.

Let $\mathbb{L}_m^n(m)$ be the $2n$-skeleton of $\mathbb{L}_m^n(m)$ and $m=\prod_p r(p)$ be the prime power decomposition of $m$. Then we have

**Proposition 1.3.** (1) The sequence

$$0 \rightarrow \tilde{J}(S^{2n+1}) \rightarrow \tilde{J}(\mathbb{L}_m^n(m)) \rightarrow \tilde{J}(\mathbb{L}_m^n(m)) \rightarrow 0$$

is a split extension for odd $m$.

(ii) There exists a natural isomorphism

$$f = \bigoplus_p \tilde{J}(\mathbb{L}_m^n(p)) \rightarrow \bigoplus_p \tilde{J}(\mathbb{L}_m^n(p^r(p))) \quad (m: \text{odd}),$$

$$f = \bigoplus_p (1 \circ p \circ \pi_p) \tilde{J}(\mathbb{L}_m^n(p)) \rightarrow \bigoplus_{p: \text{odd prime}} \tilde{J}(\mathbb{L}_m^n(p^r(p))) \bigoplus \tilde{J}(\mathbb{L}_m^n(2^r(2))) \quad (m: \text{even}),$$

which satisfies

$$f(J_r(\eta_m^1-1)) = \Sigma r(p),$$

where $\pi_q : \mathbb{L}_m^n(q) \rightarrow \mathbb{L}_m^n(m)$ and $i_p : \mathbb{L}_m^n(p^r(p)) \rightarrow \mathbb{L}_m^n(p^r(p))$ are the natural projection and inclusion, respectively.

**Proof.** (i) is immediate from Puppe exact sequence in $KO$-theory and the fact that $\tilde{J}(\mathbb{L}_m^n(m))$ is of odd order if $m$ is odd.

(ii) We can show the similar result for $\tilde{K}$ instead of $\tilde{J}$ by noticing that $f$ is surjective and the both sides groups have the same order (cf. [3, Lemma 2.3 (ii)] and [13, Th.(0.1)]).
last equality follows from the definitions of $f$, $\eta_m$ and $\eta_p\nu(p)$.

q.e.d.

§2. The structure of $\tilde{J}(L^n(2^r))$

Let $\eta$ be the canonical complex line bundle over $L^n(2^r)$ and $\rho$ be the non-trivial real line bundle over $L^n(2^r)$ and put

$$\sigma(s) = \eta^{2^s-1}, \sigma(0) = \sigma \in \tilde{K}(L^n(2^r)),$$

$$\kappa = \rho - 1 \in \tilde{K}O(L^n(2^r)).$$

Then $\tilde{K}O(L^n(2^r))$ is generated additively by the elements $\kappa$ and $r(\sigma^d\sigma(s))$ ($0 \leq s \leq r - 2$, $0 \leq d < 2^s$),

and its explicit additive and multiplicative structures are known ([9, Th.1.9]).

The calculation of Adams operations $\psi^k$ on $\tilde{K}(L^n(2^r))$ and the property $r \psi^k_c = \psi^k_{\rho} r$ of Adams operations on $\tilde{K}$ and $\tilde{K}O$ imply the following

**Lemma 2.1.** Let $J : \tilde{K}O(L^n(2^r)) \to \tilde{J}(L^n(2^r))$ be the $J$-homomorphism. Then $\text{Ker } J$ is generated additively by the elements

$$r(\sigma^d(1+\sigma)\sigma(s)) \ (0 \leq s \leq r - 1, 0 \leq d < 2^s - 1).$$

From now on, we use the following notation

$$\alpha_s = J(\sigma(s)) \in \tilde{J}(L^n(2^r)).$$

Here, we notice that $\alpha_s = 0$ if $s \geq r$ and $\alpha_{r-1} = \kappa J\kappa$, since $\eta^{2^s} = 1$ if $s \geq r$ and $\eta^{2^{r-1}} = 2\rho$. 

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From the above lemma, we see easily the following

**Proposition 2.2.** \( \overline{J}(L^n(2^r)) \) is generated by

\[ J^K \text{ and } a_s \ (0 \leq s \leq r-2). \]

Combining the relations of \( \widetilde{K}O(L^n(2^r)) \) given in [9, Th.1.9] and the relations arisen from \( \text{Ker} J \) in Lemma 2.1, we have the following theorem on the group structure of the reduced \( J \)-group \( \overline{J}(L^n(2^r)) \) \((r \geq 2)\), where

\[ a_s = \lfloor n/2^s \rfloor, \quad b_s = n - 2^s a_s \quad (0 \leq s < r), \]

\[ X(d,v) = \sum_{j \in \mathbb{Z}} (-1)^j (2^{2^j} + 1)^{2d} \]

\[ Y(d,v) = \sum_{j \in \mathbb{Z}} (d+2^{2^j}(2j+1)). \]

**Theorem 2.3.** (i) [5, Th.4.5] \( J: \widetilde{K}O(L^n(4)) \cong \overline{J}(L^n(4)). \)

(ii) The relations of \( \overline{J}(L^n(2^r)) \) for \( r \geq 3 \) are given as follows:

(a) The case \( n \equiv 1 \mod 4 \):

(2.3.1) \( 2^{1+a_2-1} a_0 = 0, \quad 2^{r-1+2a_1} a_0 = 0, \quad 2^{r-1-s+a_2} a_s = 0 \quad (1 \leq s \leq r-2). \)

(2.3.2) \( 2^{a_2-1} + \sum_{r-1-v(1+a_2-1)} a_v = 0 \quad \text{if } a_2 \geq 2^{r-2}. \)

(2.3.3) \( 2^{a_2} + \sum_{r-s-3+2^s-v(1+a_2)} a_v = 0 \quad (1 \leq s \leq r-2, \ 2^s \leq a_2). \)

(2.3.4) \( \sum_{v=0}^{s} (-1)^{2^{s-v}} 2^{r-s-4+2^s-v(a_2+1)} X(d,v) a_v = 0 \)

\[ (1 \leq s \leq r, \ 1 \leq d < 2^s, \ 2^s \leq a_2), \]

where \( \delta = 1 \) if \( 2d \leq b_{s+1} \), = 0 otherwise.

(2.3.5) \( 2^{2^{s+1} a_0} - \sum_{v=1}^{t} Y(1,v) a_v = 0 \) where \( 2^t \leq 2^{t+1} (a_2 < 2^{r-1}). \)

(b) The case \( n \equiv 1 \mod 4 \): The relations in (a), excluded the
one in (2.3.4) for \( s=r-2, 2d=1+b_{r-1} \) and the one in (2.3.5) for \( i=a_1+1 \), and in addition,

\[
(2.3.6) \quad 2^{2d-2}a_0 - \sum_{v=1}^{t} Y(i,v)a_v = 0 \quad \text{where} \quad 2^{t}a_1 + 1 < 2^{t+1} \quad \text{if} \quad a_1 < 2^{r-2}.
\]

For the special case that \( n=2^{r-1}a \), we can reduce the relations of \( \tilde{J}(L^n(2^r)) \) in (ii) of the above theorem to more simple ones, and \( \tilde{J}(L^n(2^r)) \) is given by the following explicit form, where \( z_h(x) \) denotes the cyclic group of order \( h \) generated by the element \( x \).

**Theorem 2.4.** If \( n=2^{r-1}a \) (\( r \geq 3, a \geq 2 \)), then \( \tilde{J}(L^n(2^r)) \) is the direct sum

\[
Z_{2^{r-1}-n} \langle a_0 \rangle \bigoplus \bigoplus_{s=1}^{r-2} Z_{2a_s-1} \langle a_s-2a_s-1 \rangle \bigoplus Z_{2a_{r-1}} \langle Jk+2a_r-2a_r-1 \rangle.
\]

By using the above theorem, the known fact about the kernel of \( i^*:\widetilde{KO}(L^n(2^r)) \to \widetilde{KO}(L^{n-1}(2^r)) \) ([9, Prop.4.4]) and the calculation of Adams operation \( \psi^3 \) on \( \widetilde{KO}(L^n(2^r)) \), we can determine the kernel of

\[
i^*:\tilde{J}(L^n(2^r)) \to \tilde{J}(L^{n-1}(2^r))
\]
as follows:

**Proposition 2.6.** \( i^* \) in (2.5) is isomorphic if \( n \equiv 3 \mod 4 \), epimorphic otherwise, and

\[
\text{Ker } i^* = \begin{cases} 
Z_4 \langle 2J_4^{2m+1} \rangle & \text{if } n = 4m+2 \\
Z_2 \langle J_2^{2m+1} \rangle & \text{if } n = 4m+1 \\
Z_4 \langle J_4^{2m} \rangle & \text{if } n = 4m+0,
\end{cases}
\]

where \( \sigma=r(n-1) \in KO(L^n(2^r)) \) and

\[
u = 2^{\min(r+1,k+2)} \text{ for } n = 4m + 2^k q \text{ with } (2,q)=1.
\]
§3. Proof of Theorem 1.1

To prove Theorem 1.1 for $p=2$, we prepare some lemmas.

**Lemma 3.1.** The following equality holds in $\tilde{J}(L^n(2^r))$ ($r \geq 2$):

$$J_r(n^{i}-1) = J_r(\sigma) = a_\nu$$

for $i \geq 1$,

where $\nu = \nu_2(i)$ is the exponent of $2$ in the prime power decomposition of $i$.

**Proof.** By Lemma 2.1, we notice that the kernel of $J$:

$$\tilde{K}(L^n(2^r)) \to \tilde{J}(L^n(2^r))$$

is generated additively by

$$r(n^j\sigma(s)) \quad (0 \leq s < r, 1 \leq j < 2^8).$$

If $2^s i < 2^s + 1$, then $n^i - 1 = n^i \sigma(s) + n^i_1 - 1$ where $j = 1 - 2^s$. If $j > 0$ in addition, then $J_r(n^i - 1) = J_r(n^i_1 - 1)$ by the above notice and $\sigma(s) = 0$ $(s < r)$. By continuing this process, we have the desired equality.

q.e.d.

Now, let $f_2(n, r; \nu)$ be the non-negative integer such that

$$\#J_r(\sigma) = \#a_\nu = 2^{f_2(n, r; \nu)}$$

in $\tilde{J}(L^n(2^r))$ ($n \geq 0$, $r \geq 2$),

where $\#a$ denotes the order of $a$. Then by the definition of $a_\nu$, we see that

(3.2) $f_2(n, r; \nu) = 0$ if $n = 0$ or $\nu = r$.

**Lemma 3.3.** If $n = 2^{r-1}a$ and $r \geq 3$, then

$$f_2(n, r; \nu) = r - 1 - \nu + 2^{r-1} - a$$

for $n > 0$, $0 \leq \nu < r$.

**Proof.** The lemma for $a \geq 2$ is easily seen from Theorem 2.4 and $a_{r-1} = 2J_k$. 

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Consider the case \( n = 2^{r-1} \). Then, by Proposition 2.6,
\[
\#J_{\sigma}^{2m} = 2^{r+1} \text{ in } \tilde{J}(L^{2^{r-1}}(2^{r})) \quad (4m = 2^{r-1}).
\]
On the other hand, \( 2^{r-2m} = 2^{r+4m-2- \sigma} \) in \( \tilde{K}(L^{2^{r-1}}(2^{r})) \) by [7, Lemma 2.3]. Thus, we obtain
\[
(3.4) \quad \# \alpha_0 = \#J_{\sigma} = 2^{r-1+2r-1}.
\]
Furthermore, we have the following relations in \( \tilde{J}(L^{2^{r-1}}(2^{r})) \) by Theorem 2.3:
\[
2^{\alpha v_0} = 2^{\alpha v_0+1} \alpha_{v-1} \quad (1 \leq v \leq r-3),
\]
\[
(3.5) \quad 2^{2\alpha_{r-2} + 2^5 \alpha_{r-3}} = 0 = 2\kappa + 2^2 \alpha_{r-2}.
\]
The relations (3.4) and (3.5) imply immediately
\[
\# \alpha_v = r-1+v+2^{r-1-v} \quad (0 \leq v \leq r),
\]
which is the equality for \( a=1 \). q.e.d.

Consider the commutative diagram (\( r \geq 3 \))
\[
\begin{array}{ccc}
\ker i^* \subset \tilde{J}(L^n(2^r)) & \xrightarrow{i^*} & \tilde{J}(L^{n-1}(2^r)) \\
\pi^* \downarrow & & \downarrow \pi'^*
\ker i'^* \subset \tilde{J}(L^n(2^{r-1})) & \xrightarrow{i'^*} & \tilde{J}(L^{n-1}(2^{r-1}))
\end{array}
\]
of the induced homomorphisms, where \( i \) and \( i' \) are the inclusions and \( \pi \) and \( \pi' \) are the natural projections. Then we have the following

**Lemma 3.7.** If \( n \not\equiv 0 \text{ mod } 2^{r-1} \) (\( r \geq 3 \)), then
\[
\pi^* | \ker i^* : \ker i^* \to \ker i'^*
\]
is isomorphic.

**Proof.** If \( n=4m=2^q \) (\( q \) odd), then the assumption \( n \not\equiv 0 \text{ mod } 2^{r-1} \)
implies \( r-l \geq 1 \) and so \( \min\{r+1, l+2\} = l+2 = \min\{r, l+2\} \). Thus, we see immediately the lemma by Proposition 2.6, by noticing that 
\[
\pi^*\alpha_n = \pi\pi^*\alpha_n = \pi\alpha_n \quad \text{and hence} \quad \pi^*\mathcal{J}^{-1} = \mathcal{J}^{-1}.
\]
q.e.d.

**Lemma 3.8.** If \( n \equiv 0 \mod 2^{r-1} \) (\( r \geq 3 \)), then

\[
f_2(n, r; \nu) = \max\{f_2(n-1, r; \nu), f_2(n, r-1; \nu)\}.
\]

**Proof.** Consider the diagram (3.6). Then the definition of 
\( f_2(n, r; \nu) \) implies that

\[
f_2(n, r; \nu) = \max\{f_2(n-1, r; \nu), f_2(n, r-1; \nu)\},
\]

since \( i^*\alpha_\nu = \alpha_\nu \) and \( \pi^*\alpha_\nu = \alpha_\nu \). Moreover, if

\[
f_2(n, r; \nu) > \max\{f_2(n-1, r; \nu), f_2(n, r-1; \nu)\},
\]

then the non-zero element

\[
2^{f_2(n, r; \nu)-1}\alpha_\nu \quad \text{in} \quad \mathcal{J}(L^n(2^r))
\]

is mapped to 0 by \( i^* \) and \( \pi^* \). This contradicts Lemma 3.7. Thus we have the lemma.

q.e.d.

**Proof of Theorem 1.2.** By (3.2), it is sufficient to show that

\[
(3.9) \quad f_2(n, r; \nu) = \max\{s-\nu+[n/2^s]2^{s-\nu}: s < r \text{ and } 2^s \geq n\} \quad (0 \leq \nu < r).
\]

(3.9) for \( r=2 \) is easy consequence of Theorem 2.3 (1) and [4, Th.B]. By Lemma 3.3, (3.9) holds if \( r \geq 3 \) and \( n \equiv 0 \mod 2^{r-1} \).

For the case \( r \geq 3 \) and \( 2^{r-1}a < n < 2^{r-1}(a+1) \), assume inductively that (3.9) holds for \( (n-1, r; \nu) \) and \( (n, r-1; \nu) \) instead of \( (n, r; \nu) \). Then, we see easily that the right hand side of the equality in Lemma 3.8 is equal to

\[
\begin{cases}
  f_2(n, r-1; \nu) & \text{if } a=0, \\
  \max\{f_2(n, r-1; \nu), r-1-\nu+[n/2^{r-1}]2^{r-1-\nu}\} & \text{if } a>0,
\end{cases}
\]
and hence to the right hand side of (3.9). Thus Lemma 3.8 implies (3.9) by the induction on n and r.

These complete the proof of Theorem 1.2. q.e.d.

References

(1978), 469-489.


