数理解析研究所講究録
419巻 1981年 28-37

J-groups of lens spaces

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§1. Introduction

The standard lens space mod m is the orbit manifold

$$L^n(m) = S^{2n+1}/Z_m \quad (Z_m=\{z\in S^1 : z^m=1\})$$

of the (2n+1)-sphere $S^{2n+1}(\subset C^{n+1})$ by the diagonal action

$$z(z_0,\cdots,z_n) = (zz_0,\cdots,zz_n).$$

The J-groups of lens spaces were studied by several authors (e.g. [2],[5],[6],[8],[10] and [11]).

Let η_m be the canonical complex line bundle over $L^n(m)$. Then we have the following theorem by making use of the results due to J.F.Adams [1] and D.Quillen [12].

Theorem 1.1. Let p be a prime and let $r(\eta^i_{p^r}-1)\in\widetilde{KO}(L^n(p^r))$ $(p^r\geq 3)$ be the real restriction of the stable class of the i-fold tensor product of η_{p^r}. Then the order of the J-image

$$Jr(\eta^i_{p^r}-1) \in \tilde{J}(L^n(p^r))$$

is equal to $p^{f_p(n,r;\nu)}$,

$$f_p(n,r;\nu) = \max\{s-\nu+[n/p^s(p-1)]p^{s-\nu} : \nu\leq s<r \text{ and } p^s(p-1)\leq n\},$$

where $\nu=\nu_p(i)$ is the exponent of p in the prime power decomposition of i and max $\phi=0$.

In this lecture, we prove the above theorem only the case p=2, since we can prove the above theorem for odd prime p in the similar way (see [10]).

Remark 1.2. By Theorem 1.1 and Proposition 1.3 below, we can determine the order of $Jr(\eta_m^i-1)$ in $\tilde{J}(L^n(m))$ for any m.

Let $L_0^n(m)$ be the 2n-skeleton of $L^n(m)$ and $m=\Pi p^{r(p)}$ be the prime power decomposition of m. Then we have

Proposition 1.3. (i) The sequence

$$0 \longrightarrow \tilde{J}(S^{2n+1}) \longrightarrow \tilde{J}(L^n(m)) \xrightarrow{i^*} \tilde{J}(L_0^n(m)) \longrightarrow 0$$

is a split extension for odd m.

(ii) There exists a natural isomorphism

$$f = \oplus \pi_p^* : \tilde{J}(L_0^n(m)) \longrightarrow \bigoplus_p \tilde{J}(L_0^n(p^{r(p)})) \quad (m:\text{odd}),$$

$$f=\bigoplus_p (i_p \circ \pi_p)^* \oplus \pi_2^* : \tilde{J}(L^n(m)) \to \bigoplus_{p:\text{odd prime}} \tilde{J}(L_0^n(p^{r(p)})) \oplus \tilde{J}(L^n(2^{r(2)})) \, (m:\text{even}),$$

which satisfies

$$f(Jr(\eta_m^i-1)) = \sum_p Jr(\eta_{p^{r(p)}}^i-1),$$

where $\pi_q : L^n(q) \to L^n(m)$ and $i_p : L_0^n(p^{r(p)}) \to L^n(p^{r(p)})$ are the natural projection and inclusion, respectively.

Proof. (i) is immediate from Puppe exact sequence in KO-theory and the fact that $\tilde{J}(L_0^n(m))$ is of odd order if m is odd. (ii) We can show the similar result for $\widetilde{KO}$ instead of $\tilde{J}$ by noticing that f is surjective and the both sides groups have the same order (cf. [3, Lemma 2.3 (ii)] and [13. Th.(0.1)]). The last equality follows from the definitions of f, η_m and $\eta_{p^{r(p)}}$.

q.e.d.

§2. The structure of $\tilde{J}(L^n(2^r))$

Let η be the canonical complex line bundle over $L^n(2^r)$ and ρ be the non-trivial real line bundle over $L^n(2^r)$ and put

$$\sigma(s) = \eta^{2^s} - 1, \quad \sigma(0) = \sigma \quad \in \tilde{K}(L^n(2^r)),$$

$$\kappa = \rho - 1 \quad \in \widetilde{KO}(L^n(2^r)).$$

Then $\widetilde{KO}(L^n(2^r))$ is generated additively by the elements

$$\kappa \quad \text{and} \quad r(\sigma^d\sigma(s)) \quad (0 \leq s \leq r-2,\ 0 \leq d < 2^s),$$

and its explicit additive and multiplicative structures are known ([9, Th.1.9]).

The calculation of Adams operations Ψ^k on $\tilde{K}(L^n(2^r))$ and the property $r \circ \Psi^k_C = \Psi^k_R \circ r$ of Adams operations on $\tilde{K}$ and $\widetilde{KO}$ imply the following

<u>Lemma</u> 2.1. Let $J : \widetilde{KO}(L^n(2^r)) \to \tilde{J}(L^n(2^r))$ be the J-homomorphism. Then Ker J is generated additively by the elements

$$r(\sigma^d(1+\sigma)\sigma(s)) \quad (0 \leq s \leq r-1,\ 0 \leq d < 2^s - 1).$$

From now on, we use the following notation

$$\alpha_s = Jr\sigma(s) \in \tilde{J}(L^n(2^r)).$$

Here, we notice that $\alpha_s = 0$ if $s \geq r$ and $\alpha_{r-1} = 2J\kappa$, since $\eta^{2^s} = 1$ if $s \geq r$ and $\eta^{2^{r-1}} = 2\rho$.

From the above lemma, we see easily the following

Proposition 2.2. $\tilde{J}(L^n(2^r))$ is generated by

$$J\kappa \quad \text{and} \quad \alpha_s \quad (0 \leq s \leq r-2).$$

Combining the relations of $\widetilde{KO}(L^n(2^r))$ given in [9, Th.1.9] and the relations arisen from Ker J in Lemma 2.1, we have the following theorem on the group structure of the reduced J-group $\tilde{J}(L^n(2^r))$ $(r \geq 2)$, where

$$a_s = [n/2^s], \quad b_s = n - 2^s a_s \quad (0 \leq s < r),$$

$$X(d,v) = \Sigma_{j \in Z}(-1)^{j(2^v+1)}\binom{2d}{d+2^v j},$$

$$Y(d,v) = \Sigma_{j \in Z}\binom{2d-1}{d+2^v(2j+1)}.$$

Theorem 2.3. (i) [5, Th.4.5] $J:\widetilde{KO}(L^n(4)) \cong \tilde{J}(L^n(4))$.

(ii) The relations of $\tilde{J}(L^n(2^r))$ for $r \geq 3$ are given as follows:

(a) The case $n \not\equiv 1 \bmod 4$:

(2.3.1) $2^{1+a_{r-1}}J\kappa = 0$, $2^{r-1+2a_1}\alpha_0 = 0$, $2^{r-1-s+a_s}\alpha_s = 0$ $(1 \leq s \leq r-2)$.

(2.3.2) $2^{a_{r-1}}J\kappa + \Sigma_{v=0}^{r-2} 2^{r-1-v(1+a_{r-1})-2}\alpha_v = 0$ if $a_1 \geq 2^{r-2}$.

(2.3.3) $2^{r-s-2+a_s}\alpha_s + \Sigma_{v=0}^{s-1} 2^{r-s-3+2^{s-v}(1+a_s)}\alpha_v = 0$ $(1 \leq s \leq r-2,\ 2^s \leq a_1)$.

(2.3.4) $\Sigma_{v=0}^{s}(-1)^{2^{s-v}} 2^{r-s-4+2^{s+1-v}(a_{s+1}+\delta)} X(d,v)\alpha_v = 0$

$(1 \leq s \leq r-2,\ 1 \leq d < 2^s,\ 2^s + d \leq a_1)$,

where $\delta = 1$ if $2d \leq b_{s+1}$, $= 0$ otherwise.

(2.3.5) $2^{2i-2}\alpha_0 - \Sigma_{v=1}^{t} Y(i,v)\alpha_v = 0$ where $2^t \leq i < 2^{t+1}$ $(a_1 < i < 2^{r-1})$.

(b) The case $n \equiv 1 \bmod 4$: The relations in (a), excluded the one in (2.3.4) for $s=r-2$, $2d=1+b_{r-1}$ and the one in (2.3.5) for $i=a_1+1$, and in addition,

(2.3.6) $2^{2i-2}\alpha_0 - \Sigma_{v=1}^{t} Y(i,v)\alpha_v = 0$ where $2^t \leqq a_1+1 < 2^{t+1}$ if $a_1 < 2^{r-2}$.

For the special case that $n=2^{r-1}a$, we can reduce the relations of $\tilde{J}(L^n(2^r))$ in (ii) of the above theorem to more simple ones, and $\tilde{J}(L^n(2^r))$ is given by the following explicit form, where $Z_h(x)$ denotes the cyclic group of order h generated by the element x.

Theorem 2.4. If $n=2^{r-1}a$ $(r \geqq 3,\ a \geqq 2)$, then $\tilde{J}(L^n(2^r))$ is the direct sum

$$Z_{2^{r-1}-n}\langle\alpha_0\rangle \oplus \oplus_{s=1}^{r-2} Z_{2^{a_s-1}}\langle\alpha_s - 2^{a_{s-1}-a_s+1}\alpha_{s-1}\rangle$$

$$\oplus Z_{2^{a_{r-1}}}\langle J\kappa + 2^{a_{r-2}-a_{r-1}}\alpha_{r-2}\rangle .$$

By using the above theorem, the known fact about the kernel of $i^*:\widetilde{KO}(L^n(2^r)) \to \widetilde{KO}(L^{n-1}(2^r))$ ([9, Prop.4.4]) and the calculation of Adams operation Ψ^3 on $\widetilde{KO}(L^n(2^r))$, we can determine the kernel of

(2.5) $\quad i^* : \tilde{J}(L^n(2^r)) \longrightarrow \tilde{J}(L^{n-1}(2^r))$

as follows :

Proposition 2.6. i^* in (2.5) is isomorphic if $n \equiv 3 \bmod 4$, epimorphic otherwise, and

$$\operatorname{Ker} i^* = \begin{cases} Z_4\langle 2J\bar{\sigma}^{2m+1}\rangle & \text{if } n = 4m+2 \\ Z_2\langle J\bar{\sigma}^{2m+1}\rangle & \text{if } n = 4m+1 \\ Z_u\langle J\bar{\sigma}^{2m}\rangle & \text{if } n = 4m>0, \end{cases}$$

where $\bar{\sigma}=r(\eta-1) \in \widetilde{KO}(L^n(2^r))$ and

$$u = 2^{\min\{r+1,\ell+2\}} \quad \text{for} \quad n=4m=2^{\ell}q \quad \text{with } (2,q)=1.$$

§3. Proof of Theorem 1.1

To prove Theorem 1.1 for $p=2$, we prepare some lemmas.

Lemma 3.1. The following equality holds in $\tilde{J}(L^n(2^r))$ $(r\geqq 2)$:

$$Jr(\eta^i-1) = Jr\sigma(\nu) = \alpha_\nu \quad \text{for} \quad i\geqq 1,$$

where $\nu=\nu_2(i)$ is the exponent of 2 in the prime power decomposition of i.

Proof. By Lemma 2.1, we notice that the kernel of J: $\widetilde{KO}(L^n(2^r)) \to \tilde{J}(L^n(2^r))$ is generated additively by

$$r(\eta^j\sigma(s)) \quad (0\leqq s<r,\ 1\leqq j<2^s).$$

If $2^s\leqq i<2^{s+1}$, then $\eta^i-1 = \eta^i\sigma(s)+\eta^j-1$ wher $j = i-2^s$. If $j>0$ in addition, then $Jr(\eta^i-1) = Jr(\eta^j-1)$ by the above notice and $\sigma(s)=0$ $(s\geqq r)$. By continuing this process, we have the desired equality.

q.e.d.

Now, let $f_2(n,r;\nu)$ be the non-negative integer such that

$$\#Jr\sigma(\nu) = \#\alpha_\nu = 2^{f_2(n,r;\nu)} \quad \text{in } \tilde{J}(L^n(2^r)) \quad (n\geqq 0,\ r\geqq 2),$$

where $\#\alpha$ denotes the order of α. Then by the definition of α_ν, we see that

(3.2) $f_2(n,r;\nu) = 0$ if $n = 0$ or $\nu \geqq r$.

Lemma 3.3. If $n=2^{r-1}a$ and $r\geqq 3$, then

$$f_2(n,r;\nu) = r-1-\nu+2^{r-1-\nu}a \quad \text{for} \quad n>0,\ 0\leqq\nu<r.$$

Proof. The lemma for $a\geqq 2$ is easily seen from Theorem 2.4 and $\alpha_{r-1}=2J\kappa$.

Consider the case $n=2^{r-1}$. Then, by Proposition 2.6,

$$\#J\bar{\sigma}^{2m} = 2^{r+1} \quad \text{in} \quad \tilde{J}(L^{2^{r-1}}(2^r)) \quad (4m=2^{r-1}).$$

On the other hand, $2^r\bar{\sigma}^{2m} = 2^{r+4m-2}\bar{\sigma}$ in $\widetilde{KO}(L^{2^{r-1}}(2^r))$ by [7, Lemma 2.3]. Thus, we obtain

$$(3.4) \qquad \#\alpha_0 = \#J\bar{\sigma} = 2^{r-1+2^{r-1}}.$$

Furthermore, we have the following relations in $\tilde{J}(L^{2^{r-1}}(2^r))$ by Theorem 2.3 :

$$(3.5) \qquad \begin{aligned} &2^{a_v}\alpha_v = 2^{a_{v-1}+1}\alpha_{v-1} \quad (1 \leq v \leq r-3), \\ &2^2\alpha_{r-2} + 2^5\alpha_{r-3} = 0 = 2J\kappa + 2^2\alpha_{r-2}. \end{aligned}$$

The relations (3.4) and (3.5) imply immediately

$$\#\alpha_\nu = r-1-\nu+2^{r-1-\nu} \quad (0 \leq \nu < r),$$

which is the equality for $a=1$. q.e.d.

Consider the commutative diagram ($r \geq 3$)

$$(3.6) \qquad \begin{tikzcd} \mathrm{Ker}\, i^* \subset \tilde{J}(L^n(2^r)) \arrow[r, "i^*"] \arrow[d, "\pi^*"'] & \tilde{J}(L^{n-1}(2^r)) \arrow[d, "\pi'^*"] \\ \mathrm{Ker}\, i'^* \subset \tilde{J}(L^n(2^{r-1})) \arrow[r, "i'^*"] & \tilde{J}(L^{n-1}(2^{r-1})) \end{tikzcd}$$

of the induced homomorphisms, where i and i' are the inclusions and π and π' are the natural projections. Then we have the following

Lemma 3.7. If $n \not\equiv 0 \bmod 2^{r-1}$ ($r \geq 3$), then

$$\pi^* | \mathrm{Ker}\, i^* : \mathrm{Ker}\, i^* \longrightarrow \mathrm{Ker}\, i'^*$$

is isomorphic.

Proof. If $n=4m=2^\ell q$ (q:odd), then the assumption $n \not\equiv 0 \bmod 2^{r-1}$

implies $r-1>\ell$ and so $\min\{r+1, \ell+2\} = \ell+2 = \min\{r, \ell+2\}$. Thus, we see immediately the lemma by Proposition 2.6, by noticing that $\pi^*r\eta = r\pi^*\eta = r\eta$ and hence $\pi^*J\bar{\sigma}^{i} = J\bar{\sigma}^{i}$. q.e.d.

Lemma 3.8. If $n\not\equiv 0 \bmod 2^{r-1}$ ($r\geq 3$), then

$$f_2(n,r;\nu) = \max\{f_2(n-1,r;\nu),\ f_2(n,r-1;\nu)\}.$$

Proof. Consider the diagram (3.6). Then the definition of $f_2(n,r;\nu)$ implies that

$$f_2(n,r;\nu) \geq \max\{f_2(n-1,r;\nu),\ f_2(n,r-1;\nu)\},$$

since $i^*\alpha_\nu = \alpha_\nu$ and $\pi^*\alpha_\nu = \alpha_\nu$. Moreover, if $f_2(n,r;\nu) > \max\{f_2(n-1,r;\nu), f_2(n,r-1;\nu)\}$, then the non-zero element $2^{f_2(n,r;\nu)-1}\alpha_\nu$ in $\tilde{J}(L^n(2^r))$ is mapped to 0 by i^* and π^*. This contradicts Lemma 3.7. Thus we have the lemma. q.e.d.

Proof of Theorem 1.2. By (3.2), it is sufficient to show that

$$f_2(n,r;\nu) = \max\{s-\nu+[n/2^s]2^{s-\nu} : \nu\leq s<r \text{ and } 2^s\leq n\} \quad (0\leq\nu<r). \tag{3.9}$$

(3.9) for $r=2$ is easy consequence of Theorem 2.3 (i) and [4, Th.B]. By Lemma 3.3, (3.9) holds if $r\geq 3$ and $n\equiv 0 \bmod 2^{r-1}$.

For the case $r\geq 3$ and $2^{r-1}a<n<2^{r-1}(a+1)$, assume inductively that (3.9) holds for $(n-1,r;\nu)$ and $(n,r-1;\nu)$ instead of $(n,r;\nu)$. Then, we see easily that the right hand side of the equality in Lemma 3.8 is equal to

$$\begin{cases} f_2(n,r-1;\nu) & \text{if } a=0, \\ \max\{f_2(n,r-1;\nu),\ r-1-\nu+[(n-1)/2^{r-1}]2^{r-1-\nu}\} & \text{if } a>0, \end{cases}$$

and hence to the right hand side of (3.9). Thus Lemma 3.8 implies (3.9) by the induction on n and r.

These complete the proof of Theorem 1.2. q.e.d.

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