$J$-groups of lens spaces

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§1. Introduction

The standard lens space mod $m$ is the orbit manifold

$$L^n(m) = S^{2n+1}/\mathbb{Z}_m \quad (z \in S^1: z^m = 1)$$

of the $(2n+1)$-sphere $S^{2n+1}(\mathbb{C}^{n+1})$ by the diagonal action

$$z(z_0, \ldots, z_n) = (zz_0, \ldots, zz_n).$$

The $J$-groups of lens spaces were studied by several authors
(e.g. [2],[5],[6],[8],[10] and [11]).

Let $\eta_m$ be the canonical complex line bundle over $L^n(m)$. Then we have the following theorem by making use of the results due to J.F. Adams [1] and D. Quillen [12].

Theorem 1.1. Let $p$ be a prime and let $r(\eta_{pr}^1-1) \in \tilde{K}_0(L^n(p^r))$ ($p^r \geq 3$) be the real restriction of the stable class of the $1$-fold tensor product of $\eta_{pr}$. Then the order of the $J$-image

$$JR(\eta_{pr}^1 - 1) \in \tilde{J}(L^n(p^r))$$

is equal to $p_{r}^{f_p(n,r,v)}$,

$$f_p(n,r,v) = \max\{s - v + \lfloor n/p^s(p-1) \rfloor p^{s-v} : vs < r \text{ and } p^s(p-1)sn\},$$
where $v = v_p(1)$ is the exponent of $p$ in the prime power decomposition of $1$ and $\max \phi = 0$.

In this lecture, we prove the above theorem only the case $p=2$, since we can prove the above theorem for odd prime $p$ in the similar way (see [10]).

Remark 1.2. By Theorem 1.1 and Proposition 1.3 below, we can determine the order of $JR(\eta_m^{1} - 1)$ in $\tilde{J}(L_m^n(m))$ for any $m$.

Let $L_0^n(m)$ be the $2m$-skeleton of $L^n(m)$ and $m = \prod_p r(p)$ be the prime power decomposition of $m$. Then we have

**Proposition 1.3.** (1) The sequence

$$0 \rightarrow \tilde{J}(S^{2n+1}) \rightarrow \tilde{J}(L^n(m)) \xrightarrow{i^*} \tilde{J}(L^n_0(m)) \rightarrow 0$$

is a split extension for odd $m$.

(ii) There exists a natural isomorphism

$$f = \bigoplus_p \tilde{J}(L^n_0(m)) \rightarrow \bigoplus_p \tilde{J}((p^{r(p)})) \quad (m: \text{odd}),$$

$$f = \bigoplus_{p \text{ odd prime}} \tilde{J}(L^n_0(m)) \rightarrow \bigoplus_{p \text{ odd prime}} \tilde{J}(L^n_0(p^{r(p)})) \otimes \tilde{J}(L^n(2^{r(2)}))(m: \text{even}),$$

which satisfies

$$f(JR(\eta_m^{1} - 1)) = \Sigma_{p} JR(\eta_m^{1} - 1),$$

where $\pi_q : L^n(q) \rightarrow L^n(m)$ and $i_p : L^n_0(p^{r(p)}) \rightarrow L^n(p^{r(p)})$ are the natural projection and inclusion, respectively.

**Proof.** (i) is immediate from Puppe exact sequence in $KO$-theory and the fact that $\tilde{J}(L^n_0(m))$ is of odd order if $m$ is odd.

(ii) We can show the similar result for $KO$ instead of $\tilde{J}$ by noticing that $f$ is surjective and the both sides groups have the same order (cf. [3, Lemma 2.3 (ii)] and [13. Th.(0.1)]). The
last equality follows from the definitions of $f$, $\eta_m$ and $\eta_p r(p)$. q.e.d.

§2. The structure of $\tilde{J}(L^n(2^r))$

Let $\eta$ be the canonical complex line bundle over $L^n(2^r)$ and $\rho$ be the non-trivial real line bundle over $L^n(2^r)$ and put

$$\sigma(s) = \eta^{2^s-1}, \quad \sigma(0) = \sigma \in \tilde{K}(L^n(2^r)),$$

$$\kappa = \rho - 1 \in \tilde{KO}(L^n(2^r)).$$

Then $\tilde{KO}(L^n(2^r))$ is generated additively by the elements

$$\kappa$$

and $r(\sigma^d \sigma(s))$ $(0 \leq s \leq r-2, 0 \leq d < 2^s)$,

and its explicit additive and multiplicative structures are known ([9, Th.1.9]).

The calculation of Adams operations $\psi^k$ on $\tilde{K}(L^n(2^r))$ and the property $r \psi^k_c = \psi^k_{cr}$ of Adams operations on $\tilde{K}$ and $\tilde{KO}$ imply the following

**Lemma 2.1.** Let $J : \tilde{KO}(L^n(2^r)) \to \tilde{J}(L^n(2^r))$ be the $J$-homomorphism. Then Ker $J$ is generated additively by the elements

$$r(\sigma^d(1+\sigma)\sigma(s))$$

$(0 \leq s \leq r-1, 0 \leq d < 2^s-1)$.

From now on, we use the following notation

$$\alpha_s = J\sigma(s) \in \tilde{J}(L^n(2^r)).$$

Here, we notice that $\alpha_s = 0$ if $s\neq r$ and $\alpha_{r-1} = \kappa 2^r$, since $\eta^{2^s} = 1$ if $s\neq r$ and $\eta^{2^{r-1}} = 2\rho$. 

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From the above lemma, we see easily the following

**Proposition 2.2.** $\tilde{J}(L^n(2^r))$ is generated by

$$J_k \text{ and } a_s \text{ (0} \leq s \leq r-2).$$

Combining the relations of $\tilde{\mathcal{KO}}(L^n(2^r))$ given in [9, Th.1.9] and the relations arisen from $\text{Ker} J$ in Lemma 2.1, we have the following theorem on the group structure of the reduced $J$-group $\tilde{J}(L^n(2^r))$ ($r \geq 2$), where

$$a_s = [n/2^s], \quad b_s = n-2^s a_s \quad (0 \leq s < r),$$

$$X(d,v) = \sum_{j \in \mathbb{Z}} (-1)^{j(2^r+1)} (2d)^{d+2v_j},$$

$$Y(d,v) = \sum_{j \in \mathbb{Z}} (2d-1)^{d+2v_{2j+1}}.$$

**Theorem 2.3.** (1) [5, Th.4.5] $J: \tilde{\mathcal{KO}}(L^n(4)) \cong \tilde{J}(L^n(4))$.

(ii) The relations of $\tilde{J}(L^n(2^r))$ for $r \geq 3$ are given as follows:

(a) The case $n \equiv 1 \mod 4$:

(2.3.1) $2^{1+r-a_0-1} - l_k = 0$, $2^{r-l+2a_0} = 0$, $2^{r-1-s+a_0} = 0 \quad (1 \leq s \leq r-2)$.

(2.3.2) $2^{r-1} \alpha_0 - \sum_{v=0}^{r-2} 2^{r-1-v(l+a_0-1-2)} \alpha_v = 0$ if $a_0 \geq 2^{r-2}$.

(2.3.3) $2^{r-s+2a_0} + \sum_{v=0}^{r-2} 2^{r-1-s+2^s-v(l+a_0)} \alpha_v = 0 \quad (1 \leq s \leq r-2, 2^s \leq a_0)$.

(2.3.4) $\sum_{v=0}^{r-s} (-1)^{2^s-v} 2^{r-s-4+2^s+1-v(s+1)} X(d,v) \alpha_v = 0 \quad (1 \leq s \leq r-2, 1 \leq d < 2^s, 2^s \leq d \alpha_1)$,

where $\delta = 1$ if $2d \leq 2^{s+1}$, $= 0$ otherwise.

(2.3.5) $2^{2s-2} \alpha_0 - \sum_{v=1}^{r-t} Y(1,v) \alpha_v = 0$ where $2^{t-1} < 2^{t+1} \leq (a_1 < 2^{r-1})$.

(b) The case $n \equiv 1 \mod 4$: The relations in (a), excluded the
one in (2.3.4) for \( s=r-2, \ 2d=1+b_{r-1} \) and the one in (2.3.5) for 
\( i=a_1+1 \), and in addition,

\[
(2.3.6) \ 2^{2i-2}a_0 - \sum_{v=1}^{t} Y(i, v) a_v = 0 \text{ where } 2^t s a_1 + 1 < 2^{t+1} \text{ if } a_1 < 2^{r-2}.
\]

For the special case that \( n=2^{r-1} a \), we can reduce the relations of \( \tilde{J}(L^n(2^r)) \) in (ii) of the above theorem to more simple ones, and \( \tilde{J}(L^n(2^r)) \) is given by the following explicit form, where \( z_h(x) \) denotes the cyclic group of order \( h \) generated by the element \( x \).

**Theorem 2.4.** If \( n=2^{r-1} a \ (r \geq 3, \ a \geq 2) \), then \( \tilde{J}(L^n(2^r)) \) is the direct sum

\[
 Z_{2^r-1-n} \langle a_0 \rangle \oplus Z_{2^{r-1}} a_1 a_{r-1} \langle a_0 a_{r-1} a_{r-2} \rangle.
\]

By using the above theorem, the known fact about the kernel of \( i^*: \tilde{K}(L^n(2^r)) \to \tilde{K}(L^{n-1}(2^r)) \) ([9, Prop. 4.4]) and the calculation of Adams operation \( \Psi^3 \) on \( \tilde{K}(L^n(2^r)) \), we can determine the kernel of \( (2.5) \ i^*: \tilde{J}(L^n(2^r)) \to \tilde{J}(L^{n-1}(2^r)) \)

as follows:

**Proposition 2.6.** \( i^* \) in (2.5) is isomorphic if \( n=3 \mod 4 \), epimorphic otherwise, and

\[
 \text{Ker } i^* = \begin{cases} 
 Z_4 \langle 2J\sigma^{2m+1} \rangle & \text{if } n = 4m+2 \\
 Z_2 \langle J\sigma^{2m+1} \rangle & \text{if } n = 4m+1 \\
 Z_u \langle J\sigma^{2m} \rangle & \text{if } n = 4m>0,
\end{cases}
\]

where \( \sigma = (n-1) \in \tilde{K}(L^n(2^r)) \) and

\[
u = 2 \min\{r+1, \ell+2\} \text{ for } n=4m=2^\ell q \text{ with } (2, q) = 1.
\]
§3. Proof of Theorem 1.1

To prove Theorem 1.1 for \( p=2 \), we prepare some lemmas.

**Lemma 3.1.** The following equality holds in \( \tilde{J}(L^n(2^r)) \) (\( r \geq 2 \)):

\[
J_r(n^i - 1) = J_r(c)(v) = a_v \quad \text{for} \quad i \equiv 1,
\]

where \( v = v_2(i) \) is the exponent of 2 in the prime power decomposition of \( i \).

**Proof.** By Lemma 2.1, we notice that the kernel of \( J \):

\[
\widetilde{K}_0(L^n(2^r)) \to \tilde{J}(L^n(2^r))
\]

is generated additively by

\[
r(n^i \sigma(s)) \quad (0 \leq s < r, \ 1 \leq j < 2^s).
\]

If \( 2^s i < 2^{s+1} \), then \( n^i - 1 = n^i \sigma(s) + n^j - 1 \) when \( j = 1 - 2^s \). If \( j > 0 \) in addition, then \( J_r(n^i - 1) = J_r(n^j - 1) \) by the above notice and \( \sigma(s) = 0 \) (s2r). By continuing this process, we have the desired equality.

q.e.d.

Now, let \( f_2(n,r;v) \) be the non-negative integer such that

\[
\#J_r(c)(v) = \#a_v = 2^{f_2(n,r;v)} \quad \text{in} \quad \tilde{J}(L^n(2^r)) \quad (n \geq 0, \ r \geq 2),
\]

where \( \#a \) denotes the order of \( a \). Then by the definition of \( a_v \), we see that

(3.2) \( f_2(n,r;v) = 0 \) if \( n = 0 \) or \( v \geq r \).

**Lemma 3.3.** If \( n = 2^r - 1 \) and \( r \geq 3 \), then

\[
f_2(n,r;v) = r - 1 - v + 2^{r-1} - v \quad \text{for} \quad n > 0, \ 0 \leq v < r.
\]

**Proof.** The lemma for \( r \geq 2 \) is easily seen from Theorem 2.4 and \( \alpha_{r-1} = 2J_k \).
Consider the case $n=2^{r-1}$. Then, by Proposition 2.6,
$$\#\J\sigma^{2m} = 2^{r+1} \text{ in } \tilde{\J}(L^{2^{r-1}}(2^r)) \quad (4m=2^{r-1}).$$

On the other hand, $2^{r-2m} = 2^{r+4m-2} \sigma$ in $\tilde{\K}(L^{2^{r-1}}(2^r))$ by [7, Lemma 2.3]. Thus, we obtain
$$\#\alpha_0 = \#\J\sigma = 2^{r-1+2r-1}.$$  
(3.4)

Furthermore, we have the following relations in $\tilde{\J}(L^{2^{r-1}}(2^r))$ by Theorem 2.3:
$$2^v\alpha_v = 2^{a_v+1} \alpha_{v-1} \quad (1 \leq v < r-3),$$  
(3.5)
$$2^2\alpha_{r-2} + 2^5\alpha_{r-3} = 0 = 2\kappa + 2^5\alpha_{r-2}.$$  

The relations (3.4) and (3.5) imply immediately
$$\#\alpha_v = r-1-v+2^{r-1-v} \quad (0 \leq v < r),$$
which is the equality for $a=1$. q.e.d.

Consider the commutative diagram (r≥3)
$$\begin{array}{ccc}
\text{Ker } i^* \subset \tilde{\J}(L^n(2^r)) & \xrightarrow{i^*} & \tilde{\J}(L^{n-1}(2^r)) \\
\pi^* \downarrow \quad & & \downarrow \pi'^* \\
\text{Ker } i'^* \subset \tilde{\J}(L^n(2^{r-1})) & \xrightarrow{i'^*} & \tilde{\J}(L^{n-1}(2^{r-1}))
\end{array}$$  
(3.6)

of the induced homomorphisms, where $i$ and $i'$ are the inclusions and $\pi$ and $\pi'$ are the natural projections. Then we have the following

Lemma 3.7. If $n \not\equiv 0 \mod 2^{r-1}$ (r≥3), then
$$\pi^*|\text{Ker } i^* : \text{Ker } i^* \rightarrow \text{Ker } i'^*$$
is isomorphic.

Proof. If $n = 4m = 2^g q$ (q: odd), then the assumption $n \not\equiv 0 \mod 2^{r-1}$
implies \( r-l \geq \ell \) and so \( \min\{r+1, \ell+2\} = \ell+2 = \min\{r, \ell+2\} \). Thus, we see immediately the lemma by Proposition 2.6, by noticing that 
\( \pi^* \mathfrak{m} = r \mathfrak{m} \mathfrak{n} = r \mathfrak{n} \) and hence \( \pi^* \mathfrak{j} \mathfrak{d}^{-1} = \mathfrak{j} \mathfrak{d}^{-1} \).

q.e.d.

Lemma 3.8. If \( n \neq 0 \mod 2^{r-1} (r \geq 3) \), then

\[
f_2(n, r; \nu) = \max\{f_2(n-1, r; \nu), f_2(n, r-1; \nu)\}.
\]

Proof. Consider the diagram (3.6). Then the definition of 
\( f_2(n, r; \nu) \) implies that

\[
f_2(n, r; \nu) \geq \max\{f_2(n-1, r; \nu), f_2(n, r-1; \nu)\},
\]

since \( \nu \mathfrak{a}_\nu = \mathfrak{a}_\nu \) and \( \pi^* \mathfrak{a}_\nu = \mathfrak{a}_\nu \). Moreover, if

\[
f_2(n, r; \nu) > \max\{f_2(n-1, r; \nu), f_2(n, r-1; \nu)\},
\]

then the non-zero element 
\( f_2(n, r; \nu) - 1 \)

\( \mathfrak{a}_\nu \) in \( \mathfrak{j}(L^{n}(2^{r})) \) is mapped to 0 by \( \nu^* \) and \( \pi^* \). This contradicts Lemma 3.7. Thus we have the lemma.

q.e.d.

Proof of Theorem 1.2. By (3.2), it is sufficient to show that

\[
f_2(n, r; \nu) = \max\{s-\nu+[n/2^s]2^{s-\nu} : v \leq s < r \text{ and } 2^s \mathfrak{n} \} \quad (0 \leq \nu < r).
\]

(3.9) for \( r = 2 \) is easy consequence of Theorem 2.3 (1) and [4, Th. B]. By Lemma 3.3, (3.9) holds if \( r \geq 3 \) and \( n \equiv 0 \mod 2^{r-1} \).

For the case \( r \geq 3 \) and \( 2^{r-1} a < n < 2^{r-1}(a+1) \), assume inductively that (3.9) holds for \( (n-1, r; \nu) \) and \( (n, r-1; \nu) \) instead of \( (n, r; \nu) \). Then, we see easily that the right hand side of the equality in Lemma 3.8 is equal to

\[
\begin{cases}
  f_2(n, r-1; \nu) & \text{if } a = 0, \\
  \max\{f_2(n, r-1; \nu), r-1-\nu+[(n-1)/2^{r-1}]2^{r-1-\nu}\} & \text{if } a > 0,
\end{cases}
\]
and hence to the right hand side of (3.9). Thus Lemma 3.8 implies (3.9) by the induction on \( n \) and \( r \).

These complete the proof of Theorem 1.2. q.e.d.

References


(1978), 469-489.


