Title
J-Groups of Lens Spaces (Topics in Homotopy Theory and Cohomology Theory)

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J-groups of lens spaces

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§1. Introduction

The standard lens space mod m is the orbit manifold

\[ L^n(m) = S^{2n+1}/\mathbb{Z}_m \quad (\mathbb{Z}_m = \{ z \in S^1 : z^m = 1 \}) \]

of the (2n+1)-sphere \( S^{2n+1} \) by the diagonal action

\[ z(z_0, \cdots, z_n) = (zz_0, \cdots, zz_n). \]

The J-groups of lens spaces were studied by several authors (e.g. [2],[5],[6],[8],[10] and [11]).

Let \( \eta_m \) be the canonical complex line bundle over \( L^n(m) \).

Then we have the following theorem by making use of the results due to J.F.Adams [1] and D.Quillen [12].

Theorem 1.1. Let \( p \) be a prime and let \( r(\eta_{pr}^i - 1) \in \tilde{KO}(L^n(p^r)) \) \((p^r \geq 3)\) be the real restriction of the stable class of the 1-fold tensor product of \( \eta_{pr}^i \). Then the order of the J-image

\[ Jr(\eta_{pr}^i - 1) \in \tilde{J}(L^n(p^r)) \]

is equal to \( f_p(n, r; v) \),

\[ f_p(n, r; v) = \max \{ s - v + \lceil n/p^{s(p-1)} \rceil p^{s-v} : v \leq s < r \text{ and } p^{s(p-1)} \leq n \}, \]
where \( v = v_p(1) \) is the exponent of \( p \) in the prime power decomposition of \( 1 \) and \( \max \phi = 0 \).

In this lecture, we prove the above theorem only the case \( p = 2 \), since we can prove the above theorem for odd prime \( p \) in the similar way (see [10]).

**Remark 1.2.** By Theorem 1.1 and Proposition 1.3 below, we can determine the order of \( JR(\eta^{1-1}_m) \) in \( \tilde{J}(L^n(m)) \) for any \( m \).

Let \( L^n_0(m) \) be the \( 2n \)-skeleton of \( L^n(m) \) and \( m = \prod_p r(p) \) be the prime power decomposition of \( m \). Then we have

**Proposition 1.3.** (1) The sequence

\[
0 \rightarrow \tilde{J}(S^{2n+1}) \rightarrow \tilde{J}(L^n(m)) \xrightarrow{i^*} \tilde{J}(L^n_0(m)) \rightarrow 0
\]

is a split extension for odd \( m \).

(ii) There exists a natural isomorphism

\[
f = \bigoplus_p : \tilde{J}(L^n_0(m)) \rightarrow \bigoplus_p \tilde{J}(L^n_0(p^{r(p)})) \quad (m: \text{odd}),
\]

\[
f = \bigoplus_p (1 - \pi_p)^* \otimes 2 : \tilde{J}(L^n(m)) \rightarrow \bigoplus_{p: \text{odd prime}} \tilde{J}(L^n_0(p^{r(p)})) \otimes \tilde{J}(L^n(2^{r(2)}))(m: \text{even}),
\]

which satisfies

\[
f(\eta^{1-1}_m) = \Sigma_p \eta^{1-1}_p,
\]

where \( \pi_q : L^n(q) \rightarrow L^n(m) \) and \( i_p : L^n_0(p^{r(p)}) \rightarrow L^n(p^{r(p)}) \) are the natural projection and inclusion, respectively.

**Proof.** (i) is immediate from Puppe exact sequence in \( K \)-theory and the fact that \( \tilde{J}(L^n_0(m)) \) is of odd order if \( m \) is odd.

(ii) We can show the similar result for \( \tilde{K} \) instead of \( \tilde{J} \) by noticing that \( f \) is surjective and the both sides groups have the same order (cf. [3, Lemma 2.3 (ii)] and [13, Th.(0.1)]). The
last equality follows from the definitions of $f$, $\eta_m$ and $\eta_{pr(p)}$, q.e.d.

§2. The structure of $\tilde{J}(L^N(2^R))$

Let $\eta$ be the canonical complex line bundle over $L^N(2^R)$ and $\rho$ be the non-trivial real line bundle over $L^N(2^R)$ and put

$$\sigma(s) = \eta^{2^s} - 1, \quad \sigma(0) = \sigma \in \tilde{K}(L^N(2^R)),$$

$$\kappa = \rho - 1 \in \tilde{KO}(L^N(2^R)).$$

Then $\tilde{KO}(L^N(2^R))$ is generated additively by the elements $\kappa$ and $r(\sigma^d\sigma(s))$ ($0 \leq s \leq r-2$, $0 \leq d < 2^s$),

and its explicit additive and multiplicative structures are known ([9, Th.1.9]).

The calculation of Adams operations $\psi^k$ on $\tilde{K}(L^N(2^R))$ and the property $r \psi^k = \psi^k R r$ of Adams operations on $\tilde{K}$ and $\tilde{KO}$ imply the following

**Lemma 2.1.** Let $J : \tilde{KO}(L^N(2^R)) \to \tilde{J}(L^N(2^R))$ be the $J$-homomorphism. Then $\text{Ker } J$ is generated additively by the elements

$$r(\sigma^d(1+\sigma)\sigma(s))$$

($0 \leq s \leq r-1$, $0 \leq d < 2^s-1$).

From now on, we use the following notation

$$\alpha_s = J r \sigma(s) \in \tilde{J}(L^N(2^R)).$$

Here, we notice that $\alpha_s = 0$ if $s \geq r$ and $\alpha_{r-1} = \kappa 2J\kappa$, since $\eta^{2^s} = 1$ if $s \geq r$ and $\eta^{2^{r-1}} = 2\rho$.
From the above lemma, we see easily the following

**Proposition 2.2.** $\tilde{J}(L^n(2^r))$ is generated by

$$J^r \quad \text{and} \quad \alpha_s \quad (0 \leq s \leq r-2).$$

Combining the relations of $\tilde{\text{KO}}(L^n(2^r))$ given in [9, Th.1.9] and the relations arisen from $\text{Ker} J$ in Lemma 2.1, we have the following theorem on the group structure of the reduced $J$-group $\tilde{J}(L^n(2^r)) \quad (r \geq 2)$, where

$$a_s = [n/2^s], \quad b_s = n - 2^s a_s \quad (0 \leq s < r),$$

$$X(d,v) = \sum_{j \in \mathbb{Z}} (-1)^j (2^{r+1})^{2d} (d+2v)^j,$$

$$Y(d,v) = \sum_{j \in \mathbb{Z}} (d+2v)^j (2^{r+1}).$$

**Theorem 2.3.** (i) [5, Th.4.5] $J:\tilde{\text{KO}}(L^n(4)) \cong \tilde{J}(L^n(4))$.

(ii) The relations of $\tilde{J}(L^n(2^r))$ for $r \geq 3$ are given as follows:

(a) The case $n \equiv 1 \pmod{4}$:

(2.3.1) \[
2^{a_{r-1}} - 1 \equiv 0, \quad 2^{r-1} \equiv 2a_0 + 0, \quad 2^{r-1} \equiv 2^{a_0} \equiv 0 \quad (1 \leq s \leq r-2).
\]

(2.3.2) \[
2^{a_{r-1}} + \sum_{v=0}^{r-2} 2^{r-1-v} (1+a_{r-1})^{-2} \equiv 0 \quad \text{if} \quad a_1 \geq 2^r - 2.
\]

(2.3.3) \[
2^{r-s} + \sum_{v=0}^{s-1} 2^{r-s} \equiv 2^{s-1} \equiv 2^{a_0} \equiv 0 \quad (1 \leq s \leq r-2, \quad 2^s a_1).
\]

(2.3.4) \[
\sum_{v=0}^{s} (-1)^{2s-v} 2^{r-s} - 2^{s+1} - 2^{a_0+\delta} \equiv 0 \quad (1 \leq s \leq r-2, \quad 1 \leq d < 2^s, \quad 2^s + d a_1),
\]

where $\delta = 1$ if $2d \equiv 1 \pmod{4}$, $= 0$ otherwise.

(2.3.5) \[
2^{a_0} - 2^t \equiv Y(1,v) \equiv 0 \quad \text{where} \quad 2^t \leq 2^{t+1} (a_1 < 2^{r-1}).
\]

(b) The case $n \equiv 1 \pmod{4}$: The relations in (a), excluded the
one in (2.3.4) for $s=r-2$, $d=1+b_{r-1}$ and the one in (2.3.5) for $i=a_1+1$, and in addition,

$$(2.3.6) \ 2^{2i-2}a_0 - \sum_{v=1}^{t} Y(1,v)a_v = 0 \text{ where } 2^{2i}a_1+1 < 2^{t+1} \text{ if } a_1 < 2^{r-2}.$$

For the special case that $n=2^{r-1}a$, we can reduce the relations of $\tilde{J}(L^n(2^r))$ in (ii) of the above theorem to more simple ones, and $\tilde{J}(L^n(2^r))$ is given by the following explicit form, where $z_h(x)$ denotes the cyclic group of order $h$ generated by the element $x$.

**Theorem 2.4.** If $n=2^{r-1}a$ ($r \geq 3$, $a \geq 2$), then $\tilde{J}(L^n(2^r))$ is the direct sum

$$Z_{2^{r-1}-n} <a_0 \sigma_{s=1}^{r-2} z_{a_s-1} <z_{a_s-1} - a_s+1 \alpha_{s-1} > \sigma Z_{2^{a_r-1}} <Jk+2^{a_r-2}a_r-1 \alpha_{r-2} >.$$

By using the above theorem, the known fact about the kernel of $i^* : \widetilde{KO}(L^n(2^r)) \rightarrow \widetilde{KO}(L^{n-1}(2^r))$ ([9, Prop. 4.4]) and the calculation of Adams operation $\Psi^3$ on $\widetilde{KO}(L^n(2^r))$, we can determine the kernel of (2.5) $i^* : \tilde{J}(L^n(2^r)) \rightarrow \tilde{J}(L^{n-1}(2^r))$

as follows:

**Proposition 2.6.** $i^*$ in (2.5) is isomorphic if $n \equiv 3 \mod 4$, epimorphic otherwise, and

$$\text{Ker } i^* = \begin{cases} 
Z_4 <2J\sigma^{2m+1}> & \text{if } n = 4m+2 \\
Z_2 <J\sigma^{2m+1}> & \text{if } n = 4m+1 \\
Z_u <J\sigma^{2m}> & \text{if } n = 4m>0,
\end{cases}$$

where $\sigma = r(n-1) \in \widetilde{KO}(L^n(2^r))$ and

$$u = 2\min(r+1, \ell+2) \text{ for } n = 4m = 2^\ell q \text{ with } (2,q)=1.$$
§3. Proof of Theorem 1.1

To prove Theorem 1.1 for \( p=2 \), we prepare some lemmas.

**Lemma 3.1.** The following equality holds in \( \tilde{J}(L^n(2^r)) (r \geq 2) \):

\[
J_r(n^i-1) = J_r(\sigma(v)) = \alpha_v \quad \text{for } i \geq 1,
\]

where \( v = v_2(i) \) is the exponent of 2 in the prime power decomposition of \( i \).

**Proof.** By Lemma 2.1, we notice that the kernel of \( J : \tilde{\mathcal{O}}(L^n(2^r)) \rightarrow \tilde{J}(L^n(2^r)) \) is generated additively by

\[
r(n^j\sigma(s)) \quad (0 \leq s < r, 1 \leq j < 2^s).
\]

If \( 2^s \leq i < 2^{s+1} \), then \( n^i - 1 = n^j\sigma(s) + n^j - 1 \) where \( j = i - 2^s \). If \( j > 0 \) in addition, then \( J_r(n^i - 1) = J_r(n^j - 1) \) by the above notice and \( \sigma(s) = 0 \) (see r). By continuing this process, we have the desired equality.

q.e.d.

Now, let \( f_2(n,r;v) \) be the non-negative integer such that

\[
\#J_r(\sigma(v)) = \#\alpha_v = 2^{f_2(n,r;v)} \quad \text{in } \tilde{J}(L^n(2^r)) \quad (n \geq 0, r \geq 2),
\]

where \( \#\alpha \) denotes the order of \( \alpha \). Then by the definition of \( \alpha_v \), we see that

\[
(3.2) \quad f_2(n,r;v) = 0 \quad \text{if } n = 0 \text{ or } v \geq r.
\]

**Lemma 3.3.** If \( n = 2^{r-1}a \) and \( r \geq 3 \), then

\[
f_2(n,r;v) = r-1-v+2^{r-1-v}a \quad \text{for } n > 0, 0 \leq v < r.
\]

**Proof.** The lemma for \( a \geq 2 \) is easily seen from Theorem 2.4 and \( \alpha_{r-1} = 2J_r \).
Consider the case $n=2^{r-1}$. Then, by Proposition 2.6,

$$\#\tilde{J}(2^{r-1}) = 2^{r+1} \text{ in } \tilde{J}(\mathbb{L}^{2^{r-1}}(2^r)) \quad (4m=2^{r-1}).$$

On the other hand, $2^{r-2}m=2^{r+4}m-2$ in $\tilde{J}(\mathbb{L}^{2^{r-1}}(2^r))$ by [7, Lemma 2.3]. Thus, we obtain

$$\#\alpha = \#J(2^{r-1}) = 2^{r-1}+2^{r-1}.$$  

Furthermore, we have the following relations in $J(\mathbb{L}^{2^{r-1}}(2^r))$ by Theorem 2.3:

$$2^{av} \alpha_v = 2^{av-1} \alpha_{v-1} \quad (1 \leq v \leq r-3),$$

$$(3.5) 
2^2 \alpha_{r-2} + 2^5 \alpha_{r-3} = 0 = 2J \kappa + 2^2 \alpha_{r-2}.$$  

The relations (3.4) and (3.5) imply immediately

$$\#\alpha = r-1-v+2^{r-1} \quad (0 \leq v \leq r),$$

which is the equality for $a=1$. q.e.d.

Consider the commutative diagram (r≥3)

$$\begin{array}{ccc}
\text{Ker } i^* & \xrightarrow{i^*} & \tilde{J}(\mathbb{L}^n(2^r)) \\
\downarrow \pi^* & & \downarrow \pi'^* \\
\text{Ker } i'^* & \xrightarrow{i'^*} & \tilde{J}(\mathbb{L}^n(2^{r-1}))
\end{array}$$

(3.6)

of the induced homomorphisms, where $i$ and $i'$ are the inclusions and $\pi$ and $\pi'$ are the natural projections. Then we have the following

**Lemma 3.7.** If $n \not\equiv 0 \mod 2^{r-1}$ (r≥3), then

$$\pi^* \mid \text{Ker } i^* : \text{Ker } i^* \to \text{Ker } i'^*$$

is isomorphic.

**Proof.** If $n=4m=2^q \, q$ (q:odd), then the assumption $n \not\equiv 0 \mod 2^{r-1}$
implies \(r-1>\ell\) and so \(\min(r+1, \ell+2) = \ell+2 = \min(r, \ell+2)\). Thus, we see immediately the lemma by Proposition 2.6, by noticing that \(\pi^*\eta = r\pi^*\eta = r\eta\) and hence \(\pi^*\psi^{-1} = \psi^{-1}\).

\textbf{Lemma 3.8.} If \(n \neq 0 \mod 2^{r-1}\) \((r \geq 3)\), then

\[ f_2(n, r; v) = \max\{f_2(n-1, r; v), f_2(n, r-1; v)\}. \]

\textbf{Proof.} Consider the diagram (3.6). Then the definition of \(f_2(n, r; v)\) implies that

\[ f_2(n, r; v) \geq \max\{f_2(n-1, r; v), f_2(n, r-1; v)\}, \]

since \(i^*\alpha_v = \alpha_v\) and \(\pi^*\alpha_v = \alpha_v\). Moreover, if

\[ f_2(n, r; v) > \max\{f_2(n-1, r; v), f_2(n, r-1; v)\}, \]

then the non-zero element \(2^{f_2(n, r; v)-1}\alpha_v\) in \(\bar{J}(L^n(2^r))\) is mapped to 0 by \(i^*\) and \(\pi^*\). This contradicts Lemma 3.7. Thus we have the lemma. \(\text{q.e.d.}\)

\textbf{Proof of Theorem 1.2.} By (3.2), it is sufficient to show that

(3.9) \[ f_2(n, r; v) = \max\{s-v+[n/2^s]2^{s-v}: \forall s \leq r \text{ and } 2^s \leq n\} \quad (0 \leq v < r). \]

(3.9) for \(r=2\) is easy consequence of Theorem 2.3 (1) and [4, Th.B]. By Lemma 3.3, (3.9) holds if \(r \geq 3\) and \(n \neq 0 \mod 2^{r-1}\).

For the case \(r \geq 3\) and \(2^{r-1}a_n < 2^{r-1}(a+1)\), assume inductively that (3.9) holds for \((n-1, r; v)\) and \((n, r-1; v)\) instead of \((n, r; v)\). Then, we see easily that the right hand side of the equality in Lemma 3.8 is equal to

\[ \begin{cases} 
 f_2(n, r-1; v) & \text{if } a=0, \\
 \max\{f_2(n, r-1; v), r-1-v+[(n-1)/2^{r-1}]2^{r-1-v}\} & \text{if } a>0,
\end{cases} \]

\[ -8 - \]
and hence to the right hand side of (3.9). Thus Lemma 3.8 implies (3.9) by the induction on $n$ and $r$.

These complete the proof of Theorem 1.2. q.e.d.

References


