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<td>STEPHEN WILSON, W.</td>
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Kyoto University
Unstable Cohomology Operations

W. Stephen Wilson

Let $E^*(-)$ be a multiplicative generalized cohomology theory. This is represented by a spectrum $E$ which can be represented as an $\Omega$-spectrum

$$E_* = \{E_k\}, \quad \Omega E_{k+1} \simeq E_k.$$ 

Then we have

$$E^k X \simeq [X, E_k],$$

or

$$E^* X = [X, E_*].$$

We are interested in the unstable $E^*(-)$ cohomology operations, or the natural transformations

$$E^k X \longrightarrow E^n X.$$ 

We have that

$$E^k X \xrightarrow{n.t.} E^n X$$

$$\simeq$$

$$[X, E_k] \xrightarrow{n.t.} [X, E_n],$$

and so the natural transformations are given by

$$[E_k, E_n] \simeq E^n E_k.$$ 

Consequently, $E^* E_*$ is of interest. However, we will restrict our attention to additive operations, i.e. those $r$ where

$$r(x+y) = r(x) + r(y).$$
To do this we will assume that
\[ E^*(E^* \times E^*_k) \cong E^*E_k \otimes E^*E_k. \]
Then the additive operations are just the primitives:
\[ r \in PE^*E_* \quad \text{if} \quad r = r \otimes 1 + 1 \otimes r. \]
We can rigorously make \( PE^*E_* \) into a ring such that for any space \( X \), \( E^*X \) is an "unstable \( E^*E \) module" over the ring \( PE^*E_* \). The details will appear elsewhere but the concept is fairly clear. In the case of \( E^*X \) we have a map
\[ PE^*E_k \otimes E^*X \longrightarrow E^*X \]
with a number of obvious compatibility conditions; among them the commuting of the diagram:
\[
\begin{array}{ccc}
PE^iE_n \otimes PE^*E_k \otimes E^*X & \longrightarrow & PE^iE_n \otimes E^*X \\
\downarrow & & \downarrow \\
PE^iE_k \otimes E^*X & \longrightarrow & E^*X,
\end{array}
\]
where the "ring" structure on \( PE^*E_* \) is clearly going to be given by composition of maps:
\[
PE^iE_n \otimes PE^*E_k \longrightarrow PE^iE_k
\]
\[
\bigotimes \bigotimes
\]
\[
[E_n, E_i] \otimes [E_k, E_n] \longrightarrow [E_k, E_i].
\]

There is a map, cohomology suspension;
\[ E^{k-n}E \longrightarrow PE^*E_n \]
from the stable operations to the unstable operations. This is just given by restricting a stable operation to classes of degree \( n \).
An example of the potential usefulness is the nondesuspension problem.
If $X$ has a desuspension $\Sigma^{-1}X$, then by the suspension isomorphism and the fact that stable operations commute with suspension, we have a stable $E^*E$ module structure on $\tilde{E}^*(\Sigma^{-1}X)$. However, if $\Sigma^{-1}X$ exists, it must also have an unstable module structure compatible with the stable structure, i.e. we must be able to complete the diagram:

$$E^{k-n} \otimes \tilde{E}^{n-1} \xrightarrow{} \tilde{E}^{k-n-1}$$

If this cannot be done, then $\Sigma^{-1}X$ does not exist.

We have specific examples for $E$ in mind. In particular we want $E$ to give complex cobordism or Brown-Peterson cohomology. The definition above, however, works for standard mod $(p)$ cohomology as well.

In particularly nice cases,

$$E^*E_k \cong \hom_{E^*}(E^*_k, E^*_k)$$

and

$$PE^*E_k \cong \hom_{E^*}(QE^*_k, E^*_k).$$

Both $BP$ and $MU$ satisfy this property. Much more can be said. Hence forth,

let $E = MU$ or $BP$.

In these cases

$$E^* (E_k \times E_k) \cong E^* E_k \otimes_{E^*} E^*_k E^*_k$$

and the diagonal map

$$E_k \xrightarrow{} E_k \times E_k$$

turns $E^*_k$ into a coalgebra.
Because $E_k$ is a homotopy commutative $H$-space, $E_*E_k$ is a commutative Hopf algebra, with conjugation, over $E_*$; or, in other words, an abelian group object in the category of coalgebras over $E_*$. Even more structure exists; since $E_*$ is a ring spectrum we have maps

$$E_k \wedge E_n \rightarrow E_{k+n}$$

giving us a product

$$\cdot: E_*E_k \wedge E_*E_n \rightarrow E_*E_{k+n},$$

and turning $E_*E_* = \{E_*E_k\}_k$ into a graded ring object over the category of coalgebras over $E_*$. This goes as: $E^*X$ is a graded ring, so $E_*$ is a graded ring object in the homotopy category, so $E_*E_*$ is a graded ring object in the category of $E_*$-coalgebras.

The distributivity in this "ring", known as a "Hopf ring", uses the coproduct: let

$$x \rightarrow \Sigma x' \times x'',$$

then

$$x \circ (y \star z) = \Sigma \pm (x' \circ y) \star (x'' \circ z)$$

where $\star$ is the Hopf algebra product, or "addition" in our "ring".

$$E^*CP^\infty \simeq E^*[\langle x \rangle] \text{ for } x \in E^*CP^\infty.$$

Dual to $x^i$ we have $\beta_i \in E_2CP^\infty$.

We obtain a formal group law over $E_*$ by applying $E^*(-)$ to the usual map

$$CP^\infty \times CP^\infty \rightarrow CP^\infty.$$

Then

$$x \rightarrow \Sigma_{i,j} a_{ij} x_1^{i} \otimes x_2^{j} = F(x_1, x_2).$$
Define
\[ x +_F y = F(x, y) = \sum_{i,j} a_{ij} x^i y^j. \]

We define a few elements in \( E_* E_* \).

Using
\[ x \in E^2 \mathbb{CP}^\infty = [\mathbb{CP}^\infty, E_2] \]
we define
\[ b_i \equiv x_*(\beta_i) \in E_{2i} E_2. \]

Also for
\[ a \in E^k = [\text{pt.}, E_k] \]
we have
\[ [a] \equiv a_*(1) \in E_0 E_k. \]

we define
\[ x +_F [F] y = \sum_{i,j} a_{ij} [F] \cdot x^i \cdot y^j. \]

In "The Hopf ring for complex cobordism", Journal of Pure and Applied Algebra, 1977, Ravenel and Wilson prove the following about \( \text{MU} \) and \( \text{BP} \). Let \( b(s) = \sum_{i \geq 0} b_i s^i \).

**Theorem.** In \( E_* E_* [[s,t]] \), \( E = \text{MU} \) or \( \text{BP} \),
\[ b(s +_F t) = b(s) +_F b(t). \]

The Hopf ring \( E_* E_{2*} \) is generated over \( E_* \) by the \( b_i \)'s and \([E_*]\), and the only relations come from above. To obtain \( E_* E_* \) just add \( e_1 \in E_1 E_1 \) and \( e_1 \circ e_1 = b_1 \).

These formulas, by duality, give all information about unstable \( \text{MU} \) and \( \text{BP} \) operations. However, there is another way to look at these unstable operations. For \( n > 0 \) we have the rational isomorphisms
\[ E^{*} Q \simeq PE^{*} E_{n} Q. \]

Since there is no torsion anywhere we have

\[
\begin{array}{c}
E^{*+n} E \subset E^{*+n} E_{Q} \simeq \hom_{E_{*}} (E^{*+n} E, E_{*} E_{Q}) \\
\downarrow \quad \sim \\
PE^{*} E_{n} \subset PE^{*} E_{n} Q \simeq \hom_{E_{*}} (OE_{*} E_{n}, E_{*} E_{Q})
\end{array}
\]

and we can represent an unstable operation by a rational stable operation,
However, we have the following surprising result:

**Theorem.** For \( E = MU \) or \( BP \), the coker in

\[ 0 \to E^{*-n} E \to PE^{*} E_{n} \to \text{coker} \to 0 \]

has no torsion. \( \square \)

This may seem like a contradiction, but because of completion problems it is not. We have that

\[ E^{*} E \simeq E^{*} \widehat{\otimes} S \]

where \( S \) has only nonnegative degrees. \( E^{*} \) has only non positive degrees.

When we say "rationally" we mean

\[ E^{*} E_{Q} \simeq E^{*} \widehat{\otimes} Q, \]

not tensor product with \( Q \). In this completed tensor product, an element which is non trivial in the coker is an infinite sum

\[ \sum a_{i} \widehat{\otimes} s_{i}, \quad a_{i} \in E^{*}_{Q}, \quad s_{i} \in S, \]

with the denominators of the \( a_{i} \) going to infinity as \( i \) does.
A candidate for an unstable operation can be checked now. If we are given an element of $\text{E}_*\text{E}^2_0$ we can evaluate it in
\[ \text{hom}_{\text{E}_*}(\text{E}_*\text{E}_n, \text{E}_*\text{Q}) \]
and if we find that all of our values are really in
\[ \text{E}_* \subseteq \text{E}_*\text{Q}, \]
then we have a legitimate element of
\[ \text{PE}_*\text{E}_n. \]
It is at this stage that the detailed knowledge of $\text{E}_*\text{E}_*$ developed in "The Hopf ring for complex cobordism" is useful.

An example of an unstable operation found in this way is the Adams operation $\psi^k$. These have been studied by several authors rationally, however we can obtain the following by use of the above technique.

**Theorem.** For $E = MU$ or $BP$, the rational operations $k^i\psi^k$ actually lie in
\[ \text{PE}^{2i}\text{E}_{2i} \text{ and PE}^{2i+1}\text{E}_{2i+1}, \text{ all i.} \]

In order to prove this type of result, techniques for evaluating
\[ E_*(r): E_*E_k \rightarrow E_*E_n \text{ for } r: E_k \rightarrow E_n \]
are necessary.

The details of these techniques, the last two theorems, and the rigorous definition of general unstable operations will appear elsewhere.
This paper represents a portion of the lectures I gave at a conference at the Research Institute for Mathematical Sciences at Kyoto University in October 1980. I would like to thank the participants and organizers for a most enjoyable conference.