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Unstable Cohomology Operations

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Let $E^*(-)$ be a multiplicative generalized cohomology theory. This is represented by a spectrum $E$ which can be represented as an $\Omega$-spectrum

$$E_* = \{E_k\}_k, \quad \Omega E_{k+1} \simeq E_k.$$ 

Then we have

$$E^kX \simeq [X, E_k],$$

or

$$E^*X = [X, E_*].$$

We are interested in the unstable $E^*(-)$ cohomology operations, or the natural transformations

$$E^kX \longrightarrow E^nX.$$

We have that

$$E^kX \overset{\text{n.t.}}{\longrightarrow} [X, E_k] \overset{\text{n.t.}}{\longrightarrow} E^nX.$$ 

and so the natural transformations are given by

$$[E_k, E_n] \simeq E^nE_k.$$ 

Consequently, $E^*E_*$ is of interest. However, we will restrict our attention to additive operations, i.e. those $r$ where

$$r(x+y) = r(x) + r(y).$$
To do this we will assume that
\[ E^*(E\times E) \cong E^*E \otimes E^*E. \]
Then the additive operations are just the primitives:
\[ r \in PE^*E_* \text{ if } r \to r \otimes 1 + 1 \otimes r. \]
We can rigorously make \( PE^*E_* \) into a ring such that for any space \( X \), \( E^*X \) is an "unstable \( E^*E \) module" over the ring \( PE^*E_* \). The details will appear elsewhere but the concept is fairly clear. In the case of \( E^*X \) we have a map
\[ PE^nE_k \otimes E^kX \to E^nX \]
with a number of obvious compatibility conditions; among them the commuting of the diagram:
\[ PE^iE_n \otimes PE^nE_k \otimes E^kX \to PE^iE_n \otimes E^nX \]
\[ \downarrow \quad \downarrow \]
\[ PE^iE_k \otimes E^kX \to E^iX, \]
where the "ring" structure on \( PE^*E_* \) is clearly going to be given by composition of maps:
\[ PE^iE_n \otimes PE^nE_k \to PE^iE_k \]
\[ \wedge \quad \wedge \]
\[ [E_n, E_i] \otimes [E_k, E_n] \to [E_k, E_i]. \]
There is a map, cohomology suspension;
\[ E^{k-n}E \to PE^kE_n \]
from the stable operations to the unstable operations. This is just given by restricting a stable operation to classes of degree \( n \).
An example of the potential usefulness is the non-desuspension problem.
If $X$ has a desuspension $\Sigma^{-1}X$, then by the suspension isomorphism and the fact that stable operations commute with suspension, we have a stable $E^*E$ module structure on $\tilde{E}^*(\Sigma^{-1}X)$. However, if $\Sigma^{-1}X$ exists, it must also have an unstable module structure compatible with the stable structure, i.e. we must be able to complete the diagram:

$$
\begin{array}{cc}
E^k & \otimes \sim^{nE}_{\Sigma^{-1}X} \\
\downarrow & \\
PE^k & \otimes \sim^{nE}_{\Sigma^{-1}X}
\end{array}
$$

If this cannot be done, then $\Sigma^{-1}X$ does not exist.

We have specific examples for $E$ in mind. In particular we want $E$ to give complex cobordism or Brown-Peterson cohomology. The definition above, however, works for standard mod (p) cohomology as well.

In particularly nice cases,

$$E^*E_k \simeq \text{hom}_{E^*}(E_*E_k, E_*)$$

and

$$PE^*E_k \simeq \text{hom}_{E^*}(QE_*E_k, E_*)$$

Both BP and MU satisfy this property. Much more can be said. Hence forth,

let $E = MU$ or $BP$.

In these cases

$$E^*(E_k \times E_k) \simeq E^*E_k \oplus E^*E_k$$

and the diagonal map

$$E_k \to E_k \times E_k$$

turns $E_*E_k$ into a coalgebra.
Because $E_k$ is a homotopy commutative $H$-space, $E_*E_k$ is a commutative Hopf algebra, with conjugation, over $E_*$; or, in other words, an abelian group object in the category of coalgebras over $E_*$. Even more structure exists; since $E_*$ is a ring spectrum we have maps

$$E_k \land E_n \to E_{k+n}$$

giving us a product

$$\cdot : E_kE_k \otimes E_nE_n \to E_{k+n},$$

and turning $E_kE_*$ into a graded ring object over the category of coalgebras over $E_*$. This goes as: $E^*X$ is a graded ring, so $E_*$ is a graded ring object in the homotopy category, so $E_*E_*$ is a graded ring object in the category of $E_*$-coalgebras.

The distributivity in this "ring", known as a "Hopf ring", uses the coproduct: let

$$x \to \Sigma x' \times x'',$$

then

$$x \circ (y \circ z) = \Sigma \pm (x' \circ y) \times (x'' \circ z)$$

where $\circ$ is the Hopf algebra product, or "addition" in our "ring".

$$E^*CP^\infty \simeq E^*[[x]] \text{ for } x \in E^2CP^\infty.$$ Dual to $x^i$ we have $\beta_i \in E^2CP^\infty$.

We obtain a formal group law over $E_*$ by applying $E^*(-)$ to the usual map

$$CP^\infty \times CP^\infty \to CP^\infty.$$ Then

$$x \to \Sigma a_{i,j}x_1^i \otimes x_2^j = F(x_1, x_2).$$
Define
\[ x +_F y = F(x, y) = \sum_{i,j} a_{ij} x^i y^j. \]

We define a few elements in \( E_* E_* \).

Using
\[ x \in E^2 \mathbb{CP}^\infty = [\mathbb{CP}^\infty, E_2] \]
we define
\[ b_i = x_* (\beta_i) \in E_{2i} E_2. \]

Also for
\[ a \in E^k = [\text{pt.}, E_k] \]
we have
\[ [a] = a_* (1) \in E_0 E_k. \]

we define
\[ x + [F] y = \sum_{i,j} a_{ij}^* x^i \circ y^j. \]

In "The Hopf ring for complex cobordism", Journal of Pure and Applied Algebra, 1977, Ravenel and Wilson prove the following about \( MU \) and \( BP \). Let \( b(s) = \sum_{i \geq 0} b_i s^i \).

**Theorem.** In \( E_* E_* [[s, t]] \), \( E = MU \) or \( BP \),

\[ b(s +_F t) = b(s) + [F] b(t). \]

The Hopf ring \( E_* E_* \) is generated over \( E_* \) by the \( b_i \)'s and \([E_*]\), and the only relations come from above. To obtain \( E_* E_* \) just add \( e_1 \in E_1 E_1 \) and \( e_1 \circ e_1 = b_1 \).

These formulas, by duality, give all information about unstable \( MU \) and \( BP \) operations. However, there is another way to look at these unstable operations. For \( n > 0 \) we have the rational isomorphisms
\[ E^* E \simeq PE^* E_{nQ}. \]

Since there is no torsion anywhere we have

\[
\begin{align*}
E^{*+n} E \subset E^{*+n} E \simeq \text{hom}_{E^*}(E^{*+n} E, E_{nQ}) \\
\downarrow \quad \sim \quad \downarrow \sim \\
PE^* E_n \subset PE^* E_{nQ} \simeq \text{hom}_{E^*}(QE^* E_n, E_{nQ})
\end{align*}
\]

and we can represent an unstable operation by a rational stable operation,

However, we have the following surprising result:

**Theorem.** For \( E = MU \) or \( BP \), the coker in

\[ 0 \rightarrow E^{*-n} E \rightarrow PE^* E_n \rightarrow \text{coker} \rightarrow 0 \]

has no torsion. \( \square \)

This may seem like a contradiction, but because of completion problems it is not. We have that

\[ E^* E \simeq E^* \mathcal{S} \]

where \( S \) has only nonnegative degrees. \( E^* \) has only non positive degrees.

When we say "rationally" we mean

\[ E^* E_{Q} \simeq E^* \mathcal{S}_{Q}, \]

not tensor product with \( Q \). In this completed tensor product, an element which is non trivial in the coker is an infinite sum

\[ \sum_{i} a_i \otimes s_i, \quad a_i \in E^*_{Q}, s_i \in S, \]

with the denominators of the \( a_i \) going to infinity as \( i \) does.
A candidate for an unstable operation can be checked now. If we are given an element of \( E^*E_\mathbb{Q} \) we can evaluate it in
\[
\text{hom}_{E_*}(E_*E_n, E_*E_\mathbb{Q})
\]
and if we find that all of our values are really in
\[
E_* \subseteq E_*E_\mathbb{Q},
\]
then we have a legitimate element of
\[
PE^*E_n.
\]
It is at this stage that the detailed knowledge of \( E_*E_* \) developed in "The Hopf ring for complex cobordism" is useful.

An example of an unstable operation found in this way is the Adams operation \( \psi^k \). These have been studied by several authors rationally, however we can obtain the following by use of the above technique.

**Theorem.** For \( E = MU \) or \( BP \), the rational operations \( k^i\psi^k \) actually lie in
\[
PE^{2i}E_{2i} \text{ and } PE^{2i+1}E_{2i+1}, \text{ all } i.
\]

In order to prove this type of result, techniques for evaluating
\[
E_*(r) : E_*E_k \to E_*E_n \text{ for } r : E_k \to E_n
\]
are necessary.

The details of these techniques, the last two theorems, and the rigorous definition of general unstable operations will appear elsewhere.
This paper represents a portion of the lectures I gave at a conference at the Research Institute for Mathematical Sciences at Kyoto University in October 1980. I would like to thank the participants and organizers for a most enjoyable conference.