Unstable Cohomology Operations

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Let $E^*(-)$ be a multiplicative generalized cohomology theory. This is represented by a spectrum $E$ which can be represented as an $\Omega$-spectrum

$$E_* = \{E_k\}_k, \quad \Omega E_{k+1} \simeq E_k.$$ 

Then we have

$$E^kX \simeq [X, E_k],$$

or

$$E^*X = [X, E_*].$$

We are interested in the unstable $E^*(-)$ cohomology operations, or the natural transformations

$$E^kX \longrightarrow E^nX.$$

We have that

$$E^kX \xrightarrow{n.t.} E^nX$$

$$\simeq$$

$$[X, E_k] \xrightarrow{n.t.} [X, E^n].$$

and so the natural transformations are given by

$$[E_k, E^n] \simeq E^nE_k.$$ 

Consequently, $E^*E_*$ is of interest. However, we will restrict our attention to additive operations, i.e. those $r$ where

$$r(x+y) = r(x) + r(y).$$
To do this we will assume that
\[ E^*(E_k \times E_k) \simeq E^*E_k \otimes E^*E_k. \]

Then the additive operations are just the primitives:
\[ r \in PE^*_E \text{ if } r \mapsto r \otimes 1 + 1 \otimes r. \]

We can rigorously make \( PE^*_E \) into a ring such that for any space \( X \), \( E^*X \) is an "unstable \( E^*E \) module" over the ring \( PE^*_E \). The details will appear elsewhere but the concept is fairly clear. In the case of \( E^*X \) we have a map
\[ PE^n_{E_k} \otimes E^k_X \longrightarrow E^n_X \]
with a number of obvious compatibility conditions; among them the commuting of the diagram:
\[
\begin{array}{ccc}
PE^i_{E_k} \otimes PE^n_{E_k} \otimes E^k_X & \longrightarrow & PE^i_{E_k} \otimes E^n_X \\
\downarrow & & \downarrow \\
PE^i_{E_k} \otimes E^k_X & \longrightarrow & E^i_X,
\end{array}
\]
where the "ring" structure on \( PE^*_E \) is clearly going to be given by composition of maps:
\[ PE^i_{E_k} \otimes PE^n_{E_k} \longrightarrow PE^i_{E_k} \]
\[ \bigwedge \bigwedge \]
\[ [E_n, E_i] \otimes [E_k, E_n] \longrightarrow [E_k, E_i]. \]

There is a map, cohomology suspension;
\[ E^{k-n}_E \longrightarrow PE^k_{E_n} \]
from the stable operations to the unstable operations. This is just given by restricting a stable operation to classes of degree \( n \). An example of the potential usefulness is the nondesuspension problem.
If $X$ has a desuspension $\Sigma^{-1}X$, then by the suspension isomorphism and the fact that stable operations commute with suspension, we have a stable $E^*E$ module structure on $\tilde{E}^*(E^{-1}X)$. However, if $E^{-1}X$ exists, it must also have an unstable module structure compatible with the stable structure, i.e. we must be able to complete the diagram:

$$
\begin{array}{ccc}
E^{k-n_E} \otimes \tilde{E}^{n_X} & \longrightarrow & \tilde{E}^{k-1}X \\
\downarrow & & \downarrow \\
\text{PE}^{k}E & \otimes \tilde{E}^{n_X} & \text{E}^{-1}X
\end{array}
$$

If this cannot be done, then $E^{-1}X$ does not exist.

We have specific examples for $E$ in mind. In particular we want $E$ to give complex cobordism or Brown-Peterson cohomology. The definition above, however, works for standard mod (p) cohomology as well.

In particularly nice cases,

$$E^*E_K \simeq \text{hom}_{E_*}(E_*E_K, E_*)$$

and

$$\text{PE}^*E_K \simeq \text{hom}_{E_*}(QE_*E_K, E_*).$$

Both $BP$ and $MU$ satisfy this property. Much more can be said. Henceforth,

let $E = MU$ or $BP$.

In these cases

$$E_*(E_K \times E_K) \cong E_*E_K \otimes_{E_*} E_*E_K$$

and the diagonal map

$$E_K \rightarrow E_K \times E_K$$

turns $E_*E_K$ into a coalgebra.
Because $E_k$ is a homotopy commutative H-space, $E_*E_k$ is a commutative Hopf algebra, with conjugation, over $E_*$; or, in other words, an abelian group object in the category of coalgebras over $E_*$. Even more structure exists; since $E_*$ is a ring spectrum we have maps

$$E_k \wedge E_n \to E_{k+n}$$

giving us a product

$$*: E_*E_k \otimes E_*E_n \to E_*E_{k+n},$$

and turning $E_*E_* = \{E_*E_k\}_k$ into a graded ring object over the category of coalgebras over $E_*$. This goes as: $E^*X$ is a graded ring, so $E_*$ is a graded ring object in the homotopy category, so $E_*E_*$ is a graded ring object in the category of $E_*$-coalgebras.

The distributivity in this "ring", known as a "Hopf ring", uses the coproduct: let

$$x \to \Sigma x' \times x'',$$

then

$$x * (y * z) = \Sigma \lambda (x' * y) \star (x'' * z)$$

where $\star$ is the Hopf algebra product, or "addition" in our "ring".

$$E^*CP^\infty \cong E^*[\{x\}] \text{ for } x \in E^2CP^\infty.$$ Dual to $x^i$ we have $\beta^i_j \in E^2CP^\infty$.

We obtain a formal group law over $E_*$ by applying $E^*(-)$ to the usual map

$$CP^\infty \times CP^\infty \to CP^\infty.$$ Then

$$x \to \Sigma_{i,j} a[ij] x_1^i \otimes x_2^j = F(x_1, x_2).$$
Define
\[ x \ast_p y = p(x, y) = \sum_{i,j} a_{i,j} x^i y^j. \]

We define a few elements in $E_\ast E_\ast$.

Using
\[ x \in E^n \text{CP}^\infty = [\text{CP}^\infty, E_\ast] \]
we define
\[ b_i = x_\ast (e_i) \in E_{2i} E_2. \]

Also for
\[ a \in E^k = [\text{pt.}, E_k] \]
we have
\[ [a] = a_\ast (1) \in E_0 E_k. \]

we define
\[ x \ast [F] y = \sum_{i,j} [a_{i,j}] \circ x^i \circ y^j. \]

In "The Hopf ring for complex cobordism", Journal of Pure and Applied Algebra, 1977, Ravenel and Wilson prove the following about $MU$ and $BP$. Let $b(s) = \sum s^i b_i$.

**Theorem.** In $E_\ast E_\ast[[s, t]]$, $E = MU$ or $BP$,
\[ b(s \ast_p t) = b(s) \ast [F] b(t). \]

The Hopf ring $E_\ast E_{2\ast}$ is generated over $E_\ast$ by the $b_i$'s and $[E_\ast]$ and the only relations come from above. To obtain $E_\ast E_\ast$ just add $e_1 \in E_1 E_1$ and $e_1 \circ e_1 = b_1$. □

These formulas, by duality, give all information about unstable $MU$ and $BP$ operations. However, there is another way to look at these unstable operations. For $n > 0$ we have the rational isomorphisms
\[ E^{*}E_{Q} \simeq PE^{*}E_{nQ}. \]

Since there is no torsion anywhere we have
\[
\begin{array}{c}
E^{*+n}E \subset E^{*+n}E_{Q} \simeq \text{hom}_{E^{*}}(E^{*+n}E, E_{Q}) \\
\downarrow \quad \downarrow \sim \\
PE^{*}E_{n} \subset PE^{*}E_{nQ} \simeq \text{hom}_{E^{*}}(QE^{*}E_{n}, E_{Q})
\end{array}
\]

and we can represent an unstable operation by a rational stable operation,

However, we have the following surprising result:

**Theorem.** For \( E = MU \) or \( BP \), the coker in
\[
0 \to E^{*-n}E \to PE^{*}E_{n} \to \text{coker} \to 0
\]

has no torsion. \( \square \)

This may seem like a contradiction, but because of completion problems it is not. We have that
\[ E^{*}E \simeq E^{*} \hat{\otimes} S \]

where \( S \) has only nonnegative degrees. \( E^{*} \) has only non positive degrees. When we say "rationally" we mean
\[ E^{*}E_{Q} \simeq E^{*} \hat{\otimes} Q_{Q} S, \]

not tensor product with \( Q \). In this completed tensor product, an element which is non trivial in the coker is an infinite sum
\[
\sum a_{i} \hat{\otimes} s_{i}, \quad a_{i} \in E^{*}_{Q}, \quad s_{i} \in S,
\]

with the denominators of the \( a_{i} \) going to infinity as \( i \) does.
A candidate for an unstable operation can be checked now. If we are given an element of $E^*_E_Q$ we can evaluate it in
\[ \text{hom}_{E_*}(E_*E_n, E_*Q) \]
and if we find that all of our values are really in
\[ E_* \subseteq E_*Q, \]
then we have a legitimate element of
\[ PE^*E_n. \]
It is at this stage that the detailed knowledge of $E_*E_*$ developed in "The Hopf ring for complex cobordism" is useful.

An example of an unstable operation found in this way is the Adams operation $\psi^k$. These have been studied by several authors rationally, however we can obtain the following by use of the above technique.

**Theorem.** For $E = MU$ or $BP$, the rational operations $k^i\psi^k$ actually lie in
\[ PE^{2i}E_{2i} \text{ and } PE^{2i+1}E_{2i+1}, \text{ all } i. \]

In order to prove this type of result, techniques for evaluating
\[ E_*(r) : E_*E_k + E_*E_n \text{ for } r : E_k \rightarrow E_n \]
are necessary.

The details of these techniques, the last two theorems, and the rigorous definition of general unstable operations will appear elsewhere.
This paper represents a portion of the lectures I gave at a conference at the Research Institute for Mathematical Sciences at Kyoto University in October 1980. I would like to thank the participants and organizers for a most enjoyable conference.