Unstable Cohomology Operations

W. Stephen Wilson

Let $E^*(-)$ be a multiplicative generalized cohomology theory. This is represented by a spectrum $E$ which can be represented as an $\Omega$-spectrum

$$E_* = \{ E_k \}_k, \quad \Omega E_{k+1} \simeq E_k.$$ 

Then we have

$$E^k_X \simeq [X, E_k],$$

or

$$E^* X = [X, E_*].$$

We are interested in the unstable $E^*(-)$ cohomology operations, or the natural transformations

$$E^k_X \longrightarrow E^n_X.$$ 

We have that

$$E^k_X \xrightarrow{n.t.} E^n_X$$

and so the natural transformations are given by

$$[E_k, E_n] \simeq E^n E_k.$$ 

Consequently, $E^* E_*$ is of interest. However, we will restrict our attention to additive operations, i.e. those $r$ where

$$r(x+y) = r(x) + r(y).$$
To do this we will assume that

$$E^*(E_k^* \times E_k^*) \simeq E_k^* \otimes E_k^* \otimes E_k^*$$

Then the additive operations are just the primitives:

$$r \in PE^* E_* \text{ if } r \equiv r \otimes 1 + 1 \otimes r.$$ We can rigorously make $PE^* E_*$ into a ring such that for any space $X$, $E_*^X$ is an "unstable $E^* E$ module" over the ring $PE^* E_*$. The details will appear elsewhere but the concept is fairly clear. In the case of $E^* X$ we have a map

$$PE^n E_k \otimes E^k X \to E^n X$$

with a number of obvious compatibility conditions; among them the commuting of the diagram:

$$
\begin{array}{c}
PE^i E_n \otimes PE^n E_k \otimes E^k X \to PE^i E_n \otimes E^k X \\
\downarrow \\
PE^i E_k \otimes E^k X \to E^i X,
\end{array}
$$

where the "ring" structure on $PE^* E_*$ is clearly going to be given by composition of maps:

$$
PE^i E_n \otimes PE^n E_k \to PE^i E_k
$$

$$
\begin{array}{c}
[E_n, E_i] \otimes [E_k, E_n] \to [E_k, E_i].
\end{array}
$$

There is a map, cohomology suspension;

$$E^{k-n} E \to PE^k E_n$$

from the stable operations to the unstable operations. This is just given by restricting a stable operation to classes of degree $n$. An example of the potential usefulness is the non-desuspension problem.
If $X$ has a desuspension $\Sigma^{-1}X$, then by the suspension isomorphism and the fact that stable operations commute with suspension, we have a stable $E^*E$ module structure on $\tilde{E}^*(\Sigma^{-1}X)$. However, if $\Sigma^{-1}X$ exists, it must also have an unstable module structure compatible with the stable structure, i.e. we must be able to complete the diagram:

$$
\begin{align*}
E^{k-n} \otimes \tilde{E}^{n} \Sigma^{-1}X & \rightarrow \tilde{E}^{k} \Sigma^{-1}X \\
PE^{k} \otimes \tilde{E}^{n} \Sigma^{-1}X & \rightarrow
\end{align*}
$$

If this cannot be done, then $\Sigma^{-1}X$ does not exist.

We have specific examples for $E$ in mind. In particular we want $E$ to give complex cobordism or Brown-Peterson cohomology. The definition above, however, works for standard mod (p) cohomology as well.

In particularly nice cases,

$$E^*E_K \simeq \text{hom}_{E_*}(E_*E_K, E_*)$$

and

$$PE^*E_K \simeq \text{hom}_{E_*}(QE_*E_K, E_*).$$

Both BP and MU satisfy this property. Much more can be said.

Hence forth,

let $E = MU$ or BP.

In these cases

$$E_*(E_K \times E_K) \simeq E_*E_K \otimes_{E_*} E_*E_K$$

and the diagonal map

$$E_K \rightarrow E_K \times E_K$$

turns $E_*E_K$ into a coalgebra.
Because $E_k$ is a homotopy commutative H-space, $E_*E_k$ is a commutative Hopf algebra, with conjugation, over $E_*$; or, in other words, an abelian group object in the category of coalgebras over $E_*$. Even more structure exists; since $E_*$ is a ring spectrum we have maps

$$E_k \wedge E_n \to E_{k+n}$$

giving us a product

$$\circ : E_k E_k \otimes E_n E_n \to E_k E_{k+n},$$

and turning $E_*E_* = \{E_k E_k\}_k$ into a graded ring object over the category of coalgebras over $E_*$. This goes as: $E_*X$ is a graded ring, so $E_*E_*$ is a graded ring object in the homotopy category, so $E_*E_*$ is a graded ring object in the category of $E_*$-coalgebras.

The distributivity in this "ring", known as a "Hopf ring", uses the coproduct: let

$$x \to \Sigma x' \times x'',$$

then

$$x \circ (y \ast z) = \Sigma (x' \circ y) \ast (x'' \circ z)$$

where $\ast$ is the Hopf algebra product, or "addition" in our "ring".

$$E^*CP^\infty \cong E^*[ [x] ] \text{ for } x \in E^2CP^\infty.$$  

Dual to $x^i$ we have $\beta_i \in E_2CP^\infty$.

We obtain a formal group law over $E_*$ by applying $E^*(-)$ to the usual map

$$CP^\infty \times CP^\infty \to CP^\infty.$$  

Then

$$x \to \Sigma a_{ij} x'_i \otimes x''_j = F(x_1, x_2).$$
Define
\[ x \ast y = p(x, y) = \sum_{i,j} a_{ij} x^i y^j. \]

We define a few elements in \( E_\ast E_\ast \).

Using
\[ x \in E^2 \text{CP}^\infty = [\text{CP}^\infty, E_2] \]
we define
\[ b_i \equiv x_\ast(b_i) \in E_{2i}E_2. \]

Also for
\[ a \in E^k = [\text{pt}., E_k] \]
we have
\[ [a] \equiv a_\ast(1) \in E_0E_k. \]

we define
\[ x + [F] y = \sum_{i,j} a_{ij} x^i \circ y^j. \]

In "The Hopf ring for complex cobordism", Journal of Pure and Applied Algebra, 1977, Ravenel and Wilson prove the following about MU and BP. Let \( b(s) = \sum_{i \geq 0} b_i s^i. \)

**Theorem.** In \( E_\ast E_\ast[[s, t]] \), \( E = MU \) or \( BP \),

\[ b(s +_p t) = b(s) + [F] b(t). \]

The Hopf ring \( E_\ast E_2 \) is generated over \( E_\ast \) by the \( b_i \)'s and \([E^\ast] \), and the only relations come from above. To obtain \( E_\ast E_\ast \) just add \( e_1 \in E_1E_1 \) and \( e_1 \circ e_1 = b_1. \)

These formulas, by duality, give all information about unstable MU and BP operations. However, there is another way to look at these unstable operations. For \( n > 0 \) we have the rational isomorphisms
\[ E^*E_Q \simeq PE^*_E nQ. \]

Since there is no torsion anywhere we have

\[ E^{*+n}E \hookrightarrow E^{*+n}E_Q \simeq \text{hom}_{E_*}(E_{*+n}E, E_{*Q}) \]

and we can represent an unstable operation by a rational stable operation, however, we have the following surprising result:

**Theorem.** For \( E = MU \) or \( BP \), the coker in

\[ 0 \rightarrow E^{*-n}E \rightarrow PE^*_E nQ \rightarrow \text{coker} \rightarrow 0 \]

has no torsion. \( \square \)

This may seem like a contradiction, but because of completion problems it is not. We have that

\[ E^*E \simeq E^* \otimes S \]

where \( S \) has only nonnegative degrees. \( E^* \) has only non positive degrees. When we say "rationally" we mean

\[ E^*E_Q \simeq E^*_Q \otimes S, \]

not tensor product with \( Q \). In this completed tensor product, an element which is non trivial in the coker is an infinite sum

\[ \sum a_i \otimes s_i, \ a_i \in E^*_Q, \ s_i \in S, \]

with the denominators of the \( a_i \) going to infinity as \( i \) does.
A candidate for an unstable operation can be checked now. If we are given an element of $E^*E_Q$, we can evaluate it in

$$\text{hom}_{E^*}(E^*_n, E^*_Q)$$

and if we find that all of our values are really in

$$E^*_s \subset E^*_Q,$$

then we have a legitimate element of

$$PE^*_E^n.$$

It is at this stage that the detailed knowledge of $E^*_s$ developed in "The Hopf ring for complex cobordism" is useful.

An example of an unstable operation found in this way is the Adams operation $\psi^k$. These have been studied by several authors rationally, however we can obtain the following by use of the above technique.

**Theorem.** For $E = MU$ or $BP$, the rational operations $k^i\psi^k$ actually lie in

$$PE^{2i}E_{2i} \text{ and } PE^{2i+1}E_{2i+1}, \text{ all } i.$$  \( \square \)

In order to prove this type of result, techniques for evaluating

$$E^*_s(r) : E^*_sE_k + E^*_sE_n \text{ for } r : E_k + E_n$$

are necessary.

The details of these techniques, the last two theorems, and the rigorous definition of general unstable operations will appear elsewhere.
This paper represents a portion of the lectures I gave at a conference at the Research Institute for Mathematical Sciences at Kyoto University in October 1980. I would like to thank the participants and organizers for a most enjoyable conference.