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On relations between the Brown-Peterson cohomology theory
and the ordinary mod $p$ cohomology theory

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§1. Introduction. We recall the bordism theory with singularities
Let $S$ be a close (weakly complex) manifold. An $S$-manifold
(manifold with singularities of type $S$,) means
\[ \hat{M} = M \cup (\text{cone}S) \times N \]
where $M$ is an open manifold with
\[ \partial M \simeq S \times N \]
and $N$ is a close manifold.

\[ (\text{cone}S) \times N \]

\[ \hat{M} \to X \]

The bordism group $\text{MU}(S)_*(X)$ is a group of bordism classes of maps
$f : M \to X$.

Define a bordism operation $Q_S$ by
\[ Q_S[\hat{M}, f] = [N, f|N]. \]
We will show that this operation exhibits relationship between
$BP_*(X)$ and $H_*(X; \mathbb{Z}_p)$.

§2. $BP(S)_*(X)$. A main reference of this section is [11, 6].
Let $\text{MU}_* \cong \mathbb{Z}[x_1, \ldots]$ and let $x_{p^{i-1}}v_i$, (where we take $v_i$ as a Milnor
manifold, i.e., $c_{p^{i-1}}(v_i) = p \mod p^2$). For each sequence $S=(p_1, \ldots, p_i)$
these manifold, Sullivan [5] also defined $\text{MU}(S)_*(X)$. The most
important tool of this theory is the following Sullivan exact
sequence.
From this exact sequence, if $S$ is regular, then

$$\text{MU}(S) \overset{\gamma}{\to} \text{MU}(S,P) \overset{\delta}{\to} \text{MU}(S,P) \overset{\epsilon}{\to} \text{MU}(S,P).$$

Hence $\text{MU}(\ldots, x_i, \ldots, i=p^j-1) \otimes (X) \otimes \text{BP}_*(X).$

By Quillen's splitting theorem, we can prove

$$\text{MU}(p,v_1,\ldots) \otimes (X) \cong \text{MU}(p)^{\otimes} \otimes \text{BP}_*(p,v_1,\ldots) \otimes (X).$$

In particular,

$$\text{MU}(p,v_1,v_2,\ldots) \otimes (X) \cong \text{MU}(p)^{\otimes} \otimes \text{BP}_*(p,v_1,v_2,\ldots) \otimes (X)$$

$$\cong \text{MU}(p)^{\otimes} \text{BP}_*(p;X/Z_p) \text{ (since BP}(p,v_1,\ldots) = Z_p).$$

This fact shows that $H_*(X;Z_p)$ is essentially the bordism theory of type $(p,v_1,v_2,\ldots).$ (Of course, $H_*(X;Z_p)$ is the bordism theory of type $(p,x_1,x_2,\ldots)$ but the above fact shows singularities of $x_i, i=p^j-1$ do not appear.)

§3. Milnor operations. Taking Spanier-Whitehead dual, the bordism operation induces cohomology operation. (Quillen's geometric picture arguments [4], also induce cohomology operation.)

Recall the Milnor operation $Q_i$, namely, $Q_0 = \text{the Bockstein operation}$ and $Q_{i+1} = \delta^i Q_i - Q_i \otimes P^i$.

Theorem 1. In $H^*(X;Z_p)\otimes = \text{BP}(p,v_1,\ldots)^* (X), the bordism operation induced from a Milnor manifold is the Milnor operation, i.e., $Q_{v_1} = Q_i$. 

Proof. Since $Q_{v_1}$ is defined from taking the boundaries, $Q_{v_1}$ is a derivation. Recall that the product of Lens spaces $L^m_p \times \cdots \times L^m_p$ is a retract of the Eilenberg MacLane spectrum $KZ_p$. 


Hence we have only to prove $Q_{v_i} = Q_i$ in $H^*(L_p; Z_p)$.

It is well known

$$H^*(L_p; Z_p) \cong Z_p[x]/(x^m) \otimes \langle \alpha \rangle, \quad Q_1 \alpha = x^{p^i} \quad \text{and} \quad Q_i \alpha = 0.$$ 

By the Gysin sequence, it is also well known

$$BP^*(L_p) \cong BP^*[x]/(x^m)$$

where $[p] = c_1(\gamma \otimes \cdots \otimes \gamma) = px + a_1 x^2 + \cdots = px + v_1 x^{p^i} + \cdots$,

$a_{p-1-i} = v_i \mod (p, \ldots, v_{i-1}).$

In $BP^*(L_p)$, that $[p] = 0$ means there is a manifold $Y$ such that

$$\ominus Y = px \vee a_1 x^2 \vee \cdots \vee a_{p-1-i} x^{p^i} \cdots$$

$$= px \vee (v_1 x^{p^i} \vee v_1 x w_1) \vee (v_2 x^{p^i} \vee v_2 x w_2) \vee \cdots,$$

where $w_i \in BP_+ BP^*(L_p)$.

Attach the cones.

Then we have

$$Q_{v_i} \hat{Y} = x^{p^i} \vee w_i.$$ 

In $H^*(X; Z_p)$, $BP_+ BP^*(L_p) = 0$ and hence $Q_{v_i} \hat{Y} = x^{p^i}$. In particular, $Q_p$-points $\hat{Y} = x$. It is immediate that $Q_p$-points = the Bockstein $Q_0$.

Since there is only one $\alpha$ such that $Q_0 \alpha = x$, we have $\hat{Y} = \hat{\alpha}$. Hence

$$Q_{v_i} \hat{\alpha} = Q_{v_i} \hat{Y} = x^{p^i} = Q_i \alpha.$$ 

Since $x$ is a closed manifold and $x$ has no singularities, $Q_i x = 0$.

Hence we have the theorem. q.e.d.

§4. Relations $BP^*(X)$ to $H^*(X; Z_p)$.

Theorem 2. If $p b_0 + v_1 b_1 + \cdots + v_n b_n = 0$ in $BP^*(X)$, then there is $y \in H^*(X; Z_p)$ such that $Q_j(y) = i(b_j)$ where $i : BP \to KZ_p$ is the natural inclusion map (Thom map).
Proof. Because there is $Y$ such that

$$\exists Y = p\cdot b_0 \vee v_1 b_1 \vee \ldots \vee v_n b_n.$$ 

Construct $Y$ as in the proof of Theorem 1. Then

$$Q_{v_i} = b_i.$$ 

q.e.d.

Example 1. Let $X$ be a finite $H$-space. By [2], there is a system of even dimensional indecomposable elements in $H(X; \mathbb{Z}_p)$

$$(y_k, \ldots, y_1), \quad |y_i| = q^{k+1-1}/(p-1) - p^i,$$ 

(*)

such that there is $i \leq j$ with $y_i = q^{j-i} q'(1 \leq k \leq q' \leq k)$.

Conjecture 1. If there is a finite $H$-space of type (*), then there are $y'_k$ such that $i(y'_k) = v_k$ and

$$v_h y'_k + v_k y'_h = 0 \quad \text{in } BP^*(X).$$

This conjecture is true for Lie groups, for example, we have

1. $H^*(F_4; \mathbb{Z}_3) \cong \mathbb{Z}_3[x_8]/(x_8) \otimes \Lambda (x_{15}, \ldots), O_1 x_3 = 0.$

In $BP^*(F_4)$, $v_1 x'_8 = 0$.

2. $H^*(E_8; \mathbb{Z}_3) \cong \mathbb{Z}_3[x_8, x_{20}]/(x_8^3, x_{20}^3) \otimes \Lambda (x_3, x_7, x_{15}, \ldots).$

In $BP^*(E_8)$, $v_1 x'_8 + v_2 x'_2 = 0, \quad v_1 x'_{20} = 0$.

Example 2. Let $K$ be the Eilenberg MacLane space $K(\mathbb{Z}, 3)$.

The mod $p$ cohomology ring is

$$H^*(K; \mathbb{Z}_p) \cong \mathbb{Z}_p \langle \delta \delta \gamma, \ldots \rangle \otimes \Lambda \langle \delta, v, \delta^p \gamma, \ldots \rangle$$

For simplicity of notations, let denote $\delta \delta^p \gamma = c_n, \quad \delta c_n = b_n$. Then $Q_m c_n = (b_{m-n})^{n-1}$ for $n > m > 0, \quad Q_m c_n = (b_{m-n})^{n-1}$ for $m > n \geq 0, \quad Q_m c_n = 0$ and $Q_m b_n = 0$.

Conjecture 2. There are $b'_j$ in $BP^*(K)$ such that $i(b'_j) = b_j$ and

$$v_1 b'_1 + v_2 b'_2 + \ldots = 0.$$
Proposition 1. In $BP^*(K^{2p^4+2p^3};Z_p)$, there are $b_j'$ for $1 \leq j \leq 4$ such that $i(b_j')=b_j$ and mod $(p,v_1,\ldots)^2$

\[v_1b_1' + \ldots + v_4b_4' = 0\]
\[v_1b_3'^p + v_2b_2'^p + v_3b_1'^p = 0, \quad v_1b_2'^p + v_2b_1'^p = 0, \quad v_1b_1'^p = 0.\]

Proof. We will prove only the first relation. We notice that $|b_1'|=2(p^n+1)$ and $|c_1'|=2(p^n+1)-1$. The degrees of the differentials of Atiyah-Hirzebruch spectral sequence $P(1)^*E_r$ which convergents $P(1)^*(X)$ are $4m-1$. Hence we can prove $b_1,\ldots,b_4$ are permanent in $P(1)^*E_r$. Since $d_{2p-1}=0$, we have $v_1b_1'=0$ in $P(1)^*E_r$. This means $v_1b_1'\notin E_r^s$ (the associated filtration), i.e. $b_1'$. But $4m$-dimensional elements are generated, as an $P(1)^*$-module, by $b_1',b_2'^4,b_3'^4$.

Therefore there is a relation

\[v_1b_1' + \ldots = 0 \text{ in } P(1)^*(K^{2p^4+2p^3}).\]

The fact that there is only one $\xi$ with $\xi b_1'=b_1$ implies $i(b_j')=\xi b_j'.\quad \text{q.e.d.}$

§5. Relations between A-H spectral sequence and the Sullivan exact sequence.

Lemma. Let $wx=0$ in $P(1)^*(X)=BP^*(X;Z_p)=BP(p)^*(X)$ for $w\in P(1)^*$, and let $i(x)=\lambda w=0$ in $H^*(X;Z_p)$. From Sullivan exact sequence there is $y\in BP(p,w)^*(X)$. Then in A-H spectral sequence $P(1)^*E_r^{*,*}$,

\[d_r(y')=\lambda wx,\]

where $\lambda \neq 0$ in $Z_p$, $y'\in P(1)^*E_r^{*,*}$ corresponds to $y$.

Theorem 3. Let $(w_1,\ldots,w_s)=J_s$, $|w_i|>|w_{i+1}|$, regular sequence in $P(1)^*=BP^*/p$. Let $b_j\in P(1)^*$, $0\neq i(b_j)$ in $H^*(X;Z_p)$. Suppose $w_1b_1'+\ldots+w_sb_s=0$ in $P(1)^*(X)$. Then there is $y\in P(1)^*E_r^{*,*}$ such that

\[d_r(y)=\lambda t^{v_1i(b_t)} \text{ in } BP(p,J_{t-1})E_r^{*,*}, 0<t\leq s\]

where $\lambda_t \neq 0$ in $Z_p$. 
When we study relations in $BP^*(X)$ with decomposable elements of $BP^*$, Theorem 3 is useful. For example, there is a relation in $BP(p)^{(K^{2p^4+2p^3})}$
\[ v_1^pb_2^v_2b_1^p+v_3b_2^pv_4b_3^p=0 \mod (p,v_1,v_2,...)^2v_1^2 \].

References


