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Equivariant homotopy groups, Power operations
and the equivariant Kahn-Priddy Theorem

Osaka City Univ., Shôrô Araki

1. Introduction. Let \( G \) be a finite group and \( V \) a finite
dimensional real \( G \)-module with an invariant metric. \( S^V \) and \( B^V \)
will denote the unit sphere and ball in \( V \), and \( \Sigma^V = B^V/S^V \). Let
\( X \) be a finite \( G \)-complex with base point \((e)^X_G\). The stable
\( G \)-cohomotopy group \( \omega^G_d(X) \) is usually defined for \( d \in \text{RO}(G) \) by
the formula

\[
\omega^G_d(X) = \text{colim} \left[ \Sigma^U \oplus W_X, \Sigma^U \oplus V \right]_G,
\]

where \( d = V - W \) and \( U \) runs through finite dimensional
\( G \)-modules. But we are interested here in multiplicative
structure with respect to smath products, so we restrict
ourselves to \((\text{RO}(G)+\mathbb{Z})\)-graded theory, i.e.,
\( \omega^G_d(X) \) is defined only for \( d \in \text{RO}(G)+\mathbb{Z} \) by the same formula
as above restricting all \( G \)-modules \( U \), \( V \) and \( W \) to complex
ones up to oriented real trivial summands. Then the multipli-
cation is always commutative in graded sense.

In equivariant homotopy theory there are two types of
natural homomorphisms for \( H \triangleleft G \):

1.
\[ \psi_H = \psi_H^G : \omega_G^\partial(X) \to \omega_H^\partial(X). \]
called the \textit{forgetful (or restriction) morphism}, and
\[ \phi_H = \phi_H^G : \omega_G^\partial(X) \to \omega_W^\partial(X), \]
called the \textit{fixed-point morphism}, where, for \[ \partial = V-W \in R(G)+Z, \]
\[ \psi_H^\partial = \text{res}_H^G - \text{res}_H^G \in R(H)+Z, \]
\[ \phi_H^\partial \in \psi_H^\partial - \psi_H^W \in R(W(H))+Z, \]
\[ W(H) = N(H)/H. \]
These are multiplicative as is easily seen.

In case \( G = Z/2 \), there are exact sequences involving the forgetful and fixed-point morphisms (after Landweber), which played important roles in our previous work to compute \( \tau_{p,q}^S \), \( p+q \leq 13 \) (jointly with K. Iriye). We observed also that the combination of these two exact sequences and squaring operation gives the Kahn-Priddy theorem for \( Z/2 \).

Here we observe how these machines for \( G = Z/2 \) can be generalized for more general groups, which implies the equivariant Kahn-Priddy theorem at least for \( p = 2 \).

2. The forgetful and fixed-point exact sequences.

Let \( G \succ H \), and \( X \) be a finite \( H \)-complex. There holds the canonical isomorphism
\[ c : \omega_G^\partial(G_+ \wedge_H X) \simeq \omega_H^\partial(X). \]
When \( X \) is a \( G \)-complex, \( G_+ \wedge_H X \simeq G/(G/H)_+ \wedge X \) and the map \( G/H \to \text{pt} \) induces the forgetful morphism
\[ \psi_H^G : \omega_G^\partial(X) \to \omega_H^\partial(X). \]
Thus the G-cofibration \( (G/H)_+ \to C(G/H)_+ \to \Sigma(G/H) \) (where \( C \) and \( \Sigma \) denote unreduced cone and suspension) induces an exact sequence involving the forgetful morphism, which may be called the forgetful exact sequence (even though \( \omega^d_G(\Sigma(G/H) \wedge X) \) might be generally not simple to discuss).

As to the fixed-point morphism

\[
\phi_H : \omega^d_G(X) \to \omega^d_W(H)(X^H),
\]

first we remark that it can be decomposed as

\[
\phi_H = \phi^N_H \circ \gamma^G_N(H),
\]

so we would like to be satisfied if we get exact sequences involving \( \gamma^G_N(H) \) and \( \phi^N_H \) separately. Thus we consider only the case of a normal subgroup.

Let \( V \) and \( W \) be finite dimensional complex G-modules, and \( X \) be a pointed finite G-complex. The G-cofibration \( S^V_+ \to B^V_+ \to \Sigma^V \) induces the exact sequence

\[
\ldots \to \omega^d_{G}^{V-1}(S^V_+ \wedge X) \xrightarrow{\delta_V} \omega^d_G(X) \xrightarrow{\chi^V} \omega^d_{G}^{V}(X) \xrightarrow{\rho^V} \omega^d_{G}^{V}(S^V_+ \wedge X) \to \ldots,
\]

where \( \chi^V : \omega^d_G(X) \to \omega^d_{G}^{V}(X) \) is induced by the inclusion \( i_V : \Sigma^O \to \Sigma^V \). \( \chi^V = \chi^V_{+1} \in \omega^V_G \) is called the Euler class of \( V \) (after tom Dieck). When \( V^G \neq \{0\} \), then \( \chi^V \neq 0 \).

So we observe only the case that \( V \) contains no trivial G-modules. The commutative diagram of G-cofibrations

\[
\begin{array}{ccc}
S^{V \oplus W}_+ & \to & B^{V \oplus W}_+ \\
\downarrow & & \downarrow \\
S^{V \oplus W} / S^V_+ \times B^W & \to & B^{V \oplus W} / S^V_+ \times B^W \\
\| & & \| \\
\Sigma^V(S^W_+) & \to & \Sigma^V(B^W_+) \\
\| & & \| \\
\Sigma^V(i^W) & \to & \Sigma^V(i^W)
\end{array}
\]
induces the following commutative diagram of exact sequences:

\[
\cdots \to \omega_{G}^{d+W-1}(S_{+}^{V} \wedge X) \overset{\delta_{W}}\to \omega_{G}^{d}(X) \overset{\chi_{W}}\to \omega_{G}^{d+W}(X) \to \cdots
\]

(2.1)

Let \( G \supset N \), a normal subgroup. Put \( \text{Irr} \ (G) = \) the set of all isomorphism classes of complexes irreducible \( G \)-modules.

Decompose

\[
\text{Irr} \ (G) = A_{N} \sqcup B_{N}
\]

by "\( V \in A_{N} \iff \text{res}_{N}^{G} V \) is non-trivial" (\( \iff V^{N} = \{0\} \) as \( N \) is normal). Then \( V \in B_{N} \iff V^{N} = V \). Put

\[
\begin{align*}
\overline{A}_{N} &= \{ \text{finite sums of elements of } A_{N} \}, \\
\overline{B}_{N} &= \{ \text{finite sums of elements of } B_{N} \cup \{ \text{triv} \} \}.
\end{align*}
\]

Define

\[
\lambda_{G,N}^{\alpha}(X) = \text{colim}_{V \in \overline{A}_{N}} \{ \omega_{G}^{d+_V}(S_{+}^{V} \wedge X), \lambda_{V}^{d+_V}, V, W \in \overline{A}_{N} \}.
\]

Taking the colimit of the diagram (2.1) with respect to \( V \in \overline{A}_{N} \), we get the exact sequence

\[
(2.2) \quad \cdots \to \lambda_{G,N}^{\alpha}(X) \overset{\delta_{N}}\to \omega_{G}^{d}(X) \to \text{colim}_{V \in \overline{A}_{N}} \{ \omega_{G}^{d+_V}(X), \chi_{V} \} \to \cdots
\]

Now we get

**Proposition 2.3.** \( \text{colim}_{V \in \overline{A}_{N}} \{ \omega_{G}^{d+_V}(X), \chi_{V} \} \cong \omega_{G/N}^{\alpha}(X^{N}) \).

Thus we get the desired exact sequence

\[
(2.4) \quad \cdots \to \lambda_{G,N}^{\alpha}(X) \overset{\delta_{N}}\to \omega_{G}^{d}(X) \overset{\phi_{N}}\to \omega_{G/N}(X^{N}) \to \cdots
\]
which we call the fixed-point exact sequence.

3. Transfer-type morphisms.

Here we observe transfer-type morphisms for $\gamma_H$ and $\phi_H$. Let $G > H$. First we consider the transfer to $\gamma_H$. There exists a finite dimensional complex $G$-module $V$ and a $G$-embedding $i: G/H \subseteq V$. Let $\gamma(i)$ be the $G$-tubular neighborhood of $i(G/H)$. Then $\gamma(i) \approx_G G \times_H B^V$. Collapsing the outside of $\gamma(i)$ we get a $G$-map

$$\Sigma^V \to (G \times_H B^V)/(G \times_H S^V) \approx_G G_+ \wedge_H \Sigma^V,$$

which induces a $G$-map

$$\Sigma^V X \to (G_+ \wedge_H \Sigma^V X) \approx_G G_+ \wedge_H (\Sigma^V X)$$

for any pointed $G$-complex $X$. Now we get the transfer to $\gamma_H$:

$$\text{(3.1)} \quad \text{tr} = \text{tr}_H^G : \omega_H^d (X) \to \omega_G^d (X)$$

as the composition of the following:

$$\omega_H^d (X) \approx \omega_H^{\gamma_H(d+V)} (\Sigma^V X) \cong \omega_G^{d+V} (G_+ \wedge_H (\Sigma^V X))$$

$$\longrightarrow \omega_G^{d+V} (\Sigma^V X) \approx \omega_G^d (X).$$

We have

**Proposition 3.2.** $\text{tr}_H^G \circ \gamma_H = [G/H]$,

the multiplication with $[G/H] \in A(G)$.

$\text{tr}_H^G$ can be decomposed as the following composition. We may assume $i\{H\} \subseteq S^V$. Decompose $\text{res}_H^G V = W \oplus R$, $R \supseteq i\{H\}$. 

5.
Then \( W = \text{res}_H^G (V - R) \subseteq R(H) + Z \), \( T_{i_H \downarrow}^* S^V = W \) and \( i(G/H) \subseteq S^V \). As the Gysin homomorphism for this inclusion we get

\[
\delta_v : \omega^{d+V-1}_H (S^V_+ \wedge X) \rightarrow \omega^{d+V-1}_G (S^V_+ \wedge X).
\]

Also, as the connecting morphism for the G-cofibration \( S^V_+ \rightarrow B^V_+ \rightarrow S^V \) we get

\[
\delta_v : \omega^{d+V-1}_G (S^V_+ \wedge X) \rightarrow \omega^d_G (X).
\]

Then we get

\[ \text{Proposition 3.5.} \quad \text{tr}^G_V = \delta_v \circ j_V. \]

Let \( G \triangleright N \), a normal subgroup. We define the transfer-type morphism for \( \phi^d_N \). Remark that \( B_N \) corresponds bijectively with \( \text{Irr } G/N \). Thus we have the natural inclusion \( R(G/N) + Z \subseteq R(G) + Z \). Any \( G/N \)-complex is naturally a \( G \)-complex. Thus we get a natural homomorphism

\[ \delta = \delta^G_{G/N} : \omega^d_G (X) \rightarrow \omega^d_G (X) \]

for \( d \in R(G/N) + Z \subseteq R(G) + Z \) and \( G/N \)-complex \( X \).

\[ \text{Proposition 3.6.} \quad \phi^d_N \circ \delta^G_{G/N} = \text{id}. \]

In particular, the fixed-point exact sequence splits for \( \lambda \in R(G/N) + Z \), i.e.,

\[ \text{Corollary 3.7.} \quad \text{Let } G \triangleright N, \quad \lambda \in R(G/N) + Z \quad \text{and } X \text{ be } G/N \text{-complex. Then} \]

\[ \omega^d_G (X) \cong \lambda^d_{G,N} (X) \oplus \omega^d_{G/N} (X). \]
We obtain also

\textbf{Proposition 3.8.} Let $G = K \cdot L$, semi-direct product, such that $G \triangleright N$. Then
\[ \psi_K \circ \theta^G_{G/N} = \text{id}. \]
for deg $d \in \mathbb{R} + \mathbb{Z}$.

Let $G = K \cdot N$, semi-direct product, as above. There exist $V \in \mathcal{A}_N$ and a $G$-embedding $G/H \subseteq S^V$. Let
\[ \kappa : \omega^{d+1}_{G} (S^V \wedge X) \to \lambda^d_{G,N}(X). \]
be the canonical map. Let $X$ be a $K$-complex and $d \in \mathbb{R} + \mathbb{Z}$. By composing (3.3) and (3.9) we get a natural homomorphism
\[ k = \kappa \circ j_V : \omega^d_{K}(X) \to \lambda^d_{G,N}(X). \]

\textbf{Proposition 3.11.} Under this situation
\[ \psi_{K} \circ \delta_{N}^* k = [G/K]_K : \omega^d_{K}(X) \to \omega^d_{K}(X), \]
the multiplication with $[G/K]_K = i^* [G/K]$, where $i : K \subseteq G$.

4. Power operations.

Let $G > H$, a subgroup, and $V$ an $H$-module. Remark that
\[ \text{ind}^G_H V = \bigcup (G \times_H V \to G/H), \]
the module of sections of the bundle $G \times_H V \to G/H$. Let $X$ be a pointed $H$-complex. In a parallel way we define
\[ \text{ind}^G_H X = \bigcup (G \times_H X \to G/H), \]
7.
the module of sections of the bundle $G \times_H X \to G/H$. \( \text{ind}_H^G X \) is a $G$-complex which is topologically $X^{\wedge |G/H|}$. $G$-actions on $\text{ind}_H^G X$ preserves axises $\bigcup X \times \ldots \times X \times \{\text{pt}\} \times X \times \ldots \times X$. Thus, passing to quotients we get

$$\widetilde{\text{ind}}_H^G X = (\text{ind}_H^G X)/\{\text{axises}\},$$

which is topologically $X^{\wedge |G/H|}$.

Let $x \in \omega^d_H(X)$, represented by an $H$-map $f : \sum U \otimes V \to \sum U \otimes V$. Take

$$f^{\wedge |G/H|} : \widetilde{\text{ind}}_H^G (\sum U \otimes V) \to \widetilde{\text{ind}}_H^G (\sum U \otimes V),$$

which is a $G$-map because the corresponding map of bundles is a $G$-map. We see that $f^{\wedge |G/H|}$ represents

$$\bigotimes_{\text{ext}}^n(x) \in \omega^d_G (\widetilde{\text{ind}}_H^G X),$$

called the **external power** of $x$. $\bigotimes_{\text{ext}}^n$ is not linear in general. However, let $\rho_{G,H}$ be the permutation representation of $G/H$. Put

$$\rho_{G,H} = \widetilde{\rho}_{G,H} \otimes 1.$$  

Then

**Proposition 4.3.** $\chi_f \circ \rho_{\text{ext}} : \omega^d_H(X) \to \omega^d_H (\text{ind}_H^G X + f(\widetilde{\text{ind}}_H^G X))$ is a linear homomorphism, where $\widetilde{\rho} = \widetilde{\rho}_{G,H}$.

Let $V$ be a $G$-module. Then

$$\text{ind}_H^G \otimes \text{res}_H^G V \simeq \rho_{G,H} \otimes V.$$  

Similarly, if $X$ is a $G$-complex, then

$$G \times_H X \simeq_G G/H \times X,$$
which induces a $G$-homeomorphism $\text{Ind} H \cdot X \simeq (G \times H) \cdot X^{(G/H)}$. Proposition 4.5. Let $G \triangleleft N$, a semi-direct product such that $G \cap N = \{e\}$. Let $X$ be a pointed $\mathbb{K}$-complex (i.e., $G \cdot X$ is induced by conjugation with respect to $G$). Then a $G$-complex through $G \rightarrow G/N = \mathbb{K}$, $x \mapsto g_i$, is induced by conjugation for a pointed $G$-complex $X$. 

For a pointed $G$-complex $X$, then $\mathcal{Y}_N \cdot \mathcal{P}(x) = \{G_i \cdot x\}$ is a G-map. Thus we get the internal power operation $\Delta X : X \rightarrow X^{\mathcal{P}(G)}$. In particular, the diagonal map $\Delta X \cdot \omega_h : X \rightarrow X \cdot \omega_h(x)$ induces an action of diagonal ones and permutations of factors by an automorphism of $G \cdot h$.
Under the same situation as above, let $V'$ be a finite dimensional complex $K$-module and $V = V' \oplus R^k$, $k \geq 1$. Then $W = \text{ind}^G_K V - V$ contain a real $\hat{\theta}_{G,K}$ as a summand. Remark that $W^N = \{0\}$ and $\phi_N \circ \chi_W = \phi_N$. We get two splittings of the fixed-point exact sequences for $\delta = -\nu$ :

$$
0 \rightarrow \lambda_{G,N}^{-\nu}(x) \xrightarrow{\delta_N} \omega_G^{-\nu}(x) \xrightarrow{\phi_N} \omega_K^{-\nu}(x) \rightarrow 0.
$$

**Proposition 4.6.** Under this situation $\psi_K \circ \chi_W \circ \rho = 0$.

Then the difference $\theta_{G/N}^G - \chi_W \circ \rho$ gives a homomorphism

$$
(4.7)
\hat{\theta}_K : \omega_K^{-\nu}(x) \rightarrow \lambda_{G/N}^{-\nu}(x)
$$

such that $\delta_N \circ \hat{\theta}_K = \theta_{G/N}^G - \chi_W \circ \rho$, $\psi_K \circ \delta_N \circ \hat{\theta}_K = \text{id}$.

**Theorem 4.8.** Under the above situation

$$
\psi_K \circ \delta_N : \lambda_{G,N}^{-\nu}(x) \rightarrow \omega_K^{-\nu}(x)
$$

is a split epimorphism with the splitting $\hat{\theta}_K$.

5. The equivariant Kahn-Priddy Theorem.

In Theorem 4.8 we put $N = \mathbb{Z}/2$ and $G = K \times \mathbb{Z}/2$. Using Clifford C($W$)-module (where $W$ is a $K$-module) and equivariant S-duality we can prove the isomorphism

$$
(5.1) \quad \lambda_{G,Z/2}^{-\nu}(x) \cong \omega_V^K(x; \text{RP}_+^\infty)
$$

which is natural with respect to $x$, where $\text{RP}^{\infty}$ is the real projective space in $\omega$-regular representation of $K$, regarded as a $K$-complex.
A combination of Prop.3.11, Theorem 4.8 and (5.1) implies

Theorem 5.2. There holds an epimorphism

\[
\omega_V^G(\mathbb{R} \mathcal{P}^\infty)^{(2)} \rightarrow (\omega_{V'}^G)^{(2)}
\]

at 2-primary components, where \( V = V' \oplus \mathbb{R}^k, k \geq 1, \) and \( V' \)
is a finite dimensional complex \( G \)-module.

The above theorem is the equivariant version of the
Kahn-Priddy theorem for \( p = 2. \)