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Equivariant homotopy groups, Power operations and the equivariant Kahn-Priddy Theorem

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1. <u>Introduction</u>. Let G be a finite group and V a finite dimensional real G-module with an invariant metric. S^V amd B^V will denote the unit sphere and ball in V, and $\Sigma^V = B^V/S^V$. Let X be a finite G-complex with base point $(\in X^G)$. The stable G-cohomotopy group $\omega_G^{\alpha}(X)$ is usually defined for $\alpha \in RO(G)$ by the formula

$$\omega_G^{\alpha}(X) = \operatorname*{colim}_{U} [\Sigma^{U \oplus W} X, \Sigma^{U \oplus V}]^G,$$

where $\alpha = V - W$ and U runs through finite dimensional G-modules. But we are interested here in multiplicative structure with respect to smath products, so we restrict ourselves to $(R(G)+\underline{Z})$-graded theory [redacted], i.e., $\omega_G^{\alpha}(X)$ is defined only for $\alpha \in R(G)+\underline{Z}$ by the same formula as above restricting all G-modules U, V and W to complex ones up to oriented real trivial summands. Then the multiplication is always commutative in graded sense.

In equivariant homotopy theory there are two types of natural homomorphisms for $H < G$:

$$\psi_H = \psi_H^G : \omega_G^{\alpha}(X) \longrightarrow \omega_H^{\psi_H \alpha}(X)$$

called the <u>forgetful</u> (or <u>restriction</u>) <u>morphism</u>, and

$$\phi_H = \phi_H^G : \omega_G^{\alpha}(X) \longrightarrow \omega_{W(H)}^{\phi_H \alpha}(X^H)$$

called the <u>fixed-point morphism</u>, where, for $\alpha = V-W \in R(G)+Z$,

$$\psi_H \alpha = \mathrm{res}_H^G V - \mathrm{res}_H^G W \in R(H)+Z,$$
$$\phi_H \alpha = V^H - W^H \in R(W(H))+Z, \quad W(H) = N(H)/H.$$

These are multiplicative as is easily seen.

In case $G = Z/2$, there are exact sequences involving the forgetful and fixed-point morphisms (after Landweber), which played important roles in our previous work to compute $\pi_{p,q}^S$, $p+q \leqq 13$ (jointly with K. Iriye). We observed also that the combination of these two exact sequences and squaring operation gives the Kahn-Priddy theorem for $Z/2$.

Here we observe how these machines for $G = Z/2$ can be generalized for more general groups, which implies the equivariant Kahn-Priddy theorem at least for $p = 2$.

2. <u>The forgetful and fixed-point exact sequences</u>.

Let $G > H$, and X be a finite H-complex. There holds the canonical isomorphism

$$c : \omega_G^{\alpha}(G_+ \wedge_H X) \approx \omega_H^{\psi_H \alpha}(X).$$

When X is a G-complex, $G_+ \wedge_H X \approx_G (G/H)_+ \wedge X$ and the map $G/H \rightarrow pt$ induces the forgetful morphism

$$\psi_H : \omega_G^{\alpha}(X) \longrightarrow \omega_H^{\psi_H \alpha}(X).$$

Thus the G-cofibration $(G/H)_+ \to C(G/H)_+ \to \Sigma(G/H)$ (where C and Σ denote unreduced cone and suspension) induces an exact sequence involving the forgetful morphism, which may be called the <u>forgetful exact sequence</u> (even though $\omega_G^{\alpha}(\Sigma(G/H) \wedge X)$ might be generally not simple to discuss).

As to the fixed-point morphism

$$\phi_H : \omega_G^{\alpha}(X) \longrightarrow \omega_{W(H)}^{\phi_H \alpha}(X^H),$$

first we remark that it can be decomposed as

$$\phi_H = \phi_H^{N(H)} \circ \psi_{N(H)}^{G} ,$$

so we would like to be satisfied if we get exact sequences involving $\psi_{N(H)}^{G}$ and $\phi_H^{N(H)}$ separately. Thus we consider only the case of a normal subgroup.

Let V and W be finite dimensional complex G-modules, and X be a pointed finite G-complex. The G-cofibration $S_+^V \to B_+^V \to \Sigma^V$ induces the exact sequence

$$\cdots \to \omega_G^{\alpha+V-1}(S_+^V \wedge X) \xrightarrow{\delta_V} \omega_G^{\alpha}(X) \xrightarrow{\chi_V} \omega_G^{\alpha+V}(X) \xrightarrow{\beta_V} \omega_G^{\alpha+V}(S_+^V \wedge X) \to \cdots,$$

where $\chi_V : \omega_G^{\alpha}(X) \to \omega_G^{\alpha+V}(X)$ is induced by the inclusion $i_V : \Sigma^0 \subset \Sigma^V$. $\chi_V = \chi_V^{*}1 \in \omega_G^V$ is called the <u>Euler class</u> of V (after tom Dieck). When $V^G \neq \{0\}$, then $\chi_V = 0$. So we observe only the case that V contains no trivial G-modules. The commutative diagram of G-cofibrations

$$\begin{tikzcd}
S_+^{V\oplus W} \arrow[r] \arrow[d] & B_+^{V\oplus W} \arrow[r] \arrow[d] & \Sigma^{V\oplus W} \arrow[d, equal] \\
S^{V\oplus W}/S^V \times B^W \arrow[r] \arrow[d, phantom, "\wr"] & B^{V\oplus W}/S^V \times B^W \arrow[r] \arrow[d, phantom, "\wr"] & \Sigma^{V\oplus W} \\
\Sigma^V(S_+^W) \arrow[r] & \Sigma^V(B_+^W) \arrow[ur, "\Sigma^V(i_W)"'] &
\end{tikzcd}$$

induces the following commutative diagram of exact sequences:

$$
\begin{tikzcd}
\cdots \arrow[r] & \omega_G^{\alpha+W-1}(S_+^W \wedge X) \arrow[r, "\delta_W"] \arrow[d, "\xi_{W,W\oplus V}"] & \omega_G^{\alpha}(X) \arrow[r, "\chi_W"] \arrow[d, equal] & \omega_G^{\alpha+W}(X) \arrow[r] \arrow[d, "\chi_V"] & \cdots \\
\cdots \arrow[r] & \omega_G^{\alpha+V+W-1}(S_+^{V\oplus W} \wedge X) \arrow[r, "\delta_{V+W}"] & \omega_G^{\alpha}(X) \arrow[r, "\chi_{V+W}"] & \omega_G^{\alpha+V+W}(X) \arrow[r] & \cdots .
\end{tikzcd}
\tag{2.1}
$$

Let $G \triangleright N$, a normal subgroup. Put Irr (G) = the set of all isomorphism classes of comples irreducible G=modules. Decompose

$$\mathrm{Irr}\ (G) = A_N \amalg B_N$$

by "$V \in A_N \Leftrightarrow \mathrm{res}_N^G V$ is non-trivial" ($\Leftrightarrow$ $V^N = \{0\}$ as N is normal). Then $V \in B_N \Leftrightarrow V^N = V$. Put

$$\bar{A}_N = \{\text{finite sums of elements of } A_N\},$$

$$\bar{B}_N = \{\text{finite sums of elements of } B_N \cup \{R_{triv}\}\}.$$

Define

$$\lambda_{G,N}^{\alpha}(X) = \operatorname*{colim}_{V \in \bar{A}_N} \{\omega_G^{\alpha+V-1}(S_+^V \wedge X),\ \xi_{V,V+W},\ V,\ W \in \bar{A}_N\}.$$

Taking the colimit of the diagram (2.1) with respect to $V \in \bar{A}_N$ we get the exact sequence

$$\cdots \to \lambda_{G,N}^{\alpha}(X) \xrightarrow{\delta_N} \omega_G^{\alpha}(X) \to \operatorname*{colim}_{V \in \bar{A}_N} \{\omega_G^{\alpha+V}(X),\ \chi_V\} \to \cdots . \tag{2.2}$$

Now we get

<u>Proposition</u> 2.3. $\operatorname*{colim}_{V \in \bar{A}_N} \{\omega_G^{\alpha+V}(X),\ \chi_W\} \approx \omega_{G/N}^{\phi_N \alpha}(X^N).$

Thus we get the desired exact sequence

$$\cdots \to \lambda_{G,N}^{\alpha}(X) \xrightarrow{\delta_N} \omega_G^{\alpha}(X) \xrightarrow{\phi_N} \omega_{G/N}^{\phi_N\alpha}(X^N) \to \cdots , \tag{2.4}$$

which we call the <u>fixed-point exact sequence</u>.

3. <u>Transfer-type morphisms</u>.

Here we observe transfer-type morphisms for ψ_H and ϕ_H. Let $G > H$. First we consider the transfer to ψ_H. There exists a finite dimensional complex G-module V and a G-embedding $i : G/H \subset V$. Let $\nu(i)$ be the G-tubular neighborhood of $i(G/H)$. Then $\nu(i) \approx_G G \times_H B^V$. Collapsing the outside of $\nu(i)$ we get a G-map

$$\Sigma^V \to (G \times_H B^V)/(G \times_H S^V) \approx_G G_+ \wedge_H \Sigma^V,$$

which induces a G-map

$$\Sigma^V X \to (G_+ \wedge_H \Sigma^V) \wedge X \approx_G G_+ \wedge_H (\Sigma^V \wedge X))$$

for any pointed G-complex X. Now we get the <u>transfer</u> to ψ_H :

$$(3.1) \qquad tr = tr_H^G : \omega_H^{\psi_H \alpha}(X) \to \omega_G^{\alpha}(X)$$

as the composition of the following:

$$\omega_H^{\psi_H \alpha}(X) \approx \omega_H^{\psi_H(\alpha+V)}(\Sigma^V X) \overset{c}{\approx} \omega_G^{\alpha+V}(G_+ \wedge_H (\Sigma^V X))$$

$$\to \omega_G^{\alpha+V}(\Sigma^V X) \approx \omega_G^{\alpha}(X).$$

We have

<u>Proposition</u> 3.2. $tr_H^G \circ \psi_H^G = [G/H]$, <u>the multiplication with</u> $[G/H] \in A(G)$.

tr_H^G can be decomposed as the following composition. We may assume $i\{H\} \in S^V$. Decompose $res_H^G V = W \oplus \underline{R}$, $\underline{R} \ni i\{H\}$.

Then $W = \mathrm{res}^G_H\,(V - \underline{R}) \in R(H) + Z$, $T_{i\{H\}}S^V = W$ and $i(G/H) \subset S^V$. As the Gysin homomorphism for this inclusion we get

$$(3.3) \qquad j_V : \omega_H^{\psi_H \alpha}(X) \longrightarrow \omega_G^{\alpha+V-1}(S^V_+ \wedge X).$$

Also, as the connecting morphism for the G-cofibration $S^V_+ \to B^V_+ \to \Sigma^V$ we get

$$(3.4) \qquad \delta_V : \omega_G^{\alpha+V-1}(S^V_+ \wedge X) \longrightarrow \omega_G^{\alpha}(X).$$

Then we get

<u>Proposition</u> 3.5. $\quad \mathrm{tr}^G_V = \delta_V \circ j_V$.

Let $G \trianglerighteq N$, a normal subgroup. We define the transfer-type morphism for ϕ_N. Remark that B_N corresponds bijectively with Irr G/N. Thus we have the natural inclusion $R(G/N) + Z \subset R(G) + Z$. Any G/N-complex is naturally a G-complex. Thus we get a natural homomorphism

$$\theta = \theta^G_{G/N} : \omega^{\alpha}_{G/N}(X) \longrightarrow \omega^{\alpha}_G(X)$$

for $\alpha \in R(G/N) + Z \subset R(G) + Z$ and G/N-complex X.

<u>Proposition</u> 3.6. $\quad \phi_N \circ \theta^G_{G/N} = \mathrm{id}.$

In particular, the fixed-point exact sequence splits for $\alpha \in R(G/N) + Z$, i.e.,

<u>Corollary</u> 3.7. <u>Let</u> $G \triangleright N$, $\alpha \in R(G/N) + Z$ <u>and</u> X <u>be a</u> G/N-<u>complex</u>. <u>Then</u>

$$\omega^{\alpha}_G(X) \approx \lambda^{\alpha}_{G,N}(X) \oplus \omega^{\alpha}_{G/N}(X).$$

We obtain also

Proposition 3.8. Let $G = K\cdot L$, semi-direct product, such that $G \triangleright N$. Then

$$\psi_K \circ \theta^G_{G/N} = \mathrm{id}.$$

for deg $\alpha \in R(K) + \underline{Z}$.

Let $G = K\cdot N$, semi-direct product, as above. There exist $V \in \bar{A}_N$ and a G-embedding $G/H \subset S^V$. Let

$$(3.9) \qquad \varkappa : \omega_G^{\alpha+V-1}(S_+^V \wedge X) \longrightarrow \lambda^{\alpha}_{G,N}(X)$$

be the canonical map. Let X be a K-complex and $\alpha \in R(K) + Z$. By composing (3.3) and (3.9) we get a natural homomorphism

$$(3.10) \qquad k = \varkappa \circ j_V : \omega_K^{\alpha}(X) \longrightarrow \lambda^{\alpha}_{G,N}(X).$$

Proposition 3.11. Under this situation

$$\psi_K \circ \delta_N \circ k = [G/K]_K : \omega_K^{\alpha}(X) \longrightarrow \omega_K^{\alpha}(X),$$

the multiplication with $[G/K]_K = i^*[G/K]$, where $i : K \subset G$.

4. Power operations.

Let $G > H$, a subgroup, and V an H-module. Remark that

$$\mathrm{ind}_H^G V = \Gamma(G \times_H V \longrightarrow G/H),$$

the module of sections of the bundle $G \times_H V \to G/H$. Let X be a pointed H-complex. In a parallel way we define

$$\mathrm{ind}_H^G X = \Gamma(G \times_H X \longrightarrow G/H),$$

the module of sections of the bundle $G \times_H X \to G/H$. $\mathrm{ind}_H^G X$ is a G-complex which is topologically $X^{\times |G/H|}$. G-actions on $\mathrm{ind}_H^G X$ preserves axises $\bigcup X \times \ldots \times X \times \{pt\} \times X \times \ldots \times X$. Thus, passing to quotients we get

$$\widetilde{\mathrm{ind}}_H^G X = (\mathrm{ind}_H^G X)/\{\text{axises}\},$$

which is topologically $X^{\wedge |G/H|}$.

Let $x \in \omega_H^{\alpha}(X)$, represented by an H-map $f : \Sigma^{U\oplus W} X \to \Sigma^{U\oplus V}$. Take

$$f^{\wedge |G/H|} : \widetilde{\mathrm{ind}}_H^G(\Sigma^{U\oplus W} X) \to \widetilde{\mathrm{ind}}_H^G(\Sigma^{U\oplus V}),$$

which is a G-map because the corresponding map of bundles is a G-map. We see that $f^{\wedge |G/H|}$ represents

(4.1) $$\mathcal{P}_{ext}(x) \in \omega_G^{\mathrm{ind}_H^G \alpha}(\widetilde{\mathrm{ind}}_H^G X),$$

called the <u>external power</u> of x. $\mathcal{P}_{ext}$ is not linear in general. However, let $\rho_{G,H}$ be the permutation representation of G/H. Put

(4.2) $$\rho_{G,H} = \tilde{\rho}_{G,H} \oplus 1.$$

Then

<u>Proposition</u> 4.3. $\chi_{\tilde{\rho}} \circ \mathcal{P}_{ext} : \omega_H^{\alpha}(X) \to \omega_G^{\mathrm{ind}_H^G \alpha + \tilde{\rho}}(\widetilde{\mathrm{ind}}_H^G X)$ <u>is a linear homomorphism</u>, <u>where</u> $\tilde{\rho} = \tilde{\rho}_{G,H}$.

Let V be a G-module. Then

$$\mathrm{ind}_H^G \circ \mathrm{res}_H^G V \approx \rho_{G,H} \otimes V.$$

Similarly, if X is a G-complex, then

$$G \times_H X \approx_G G/H \times X,$$

which induces a G-homeomorphism

$$\widetilde{\mathrm{ind}}_H^G\, X \approx_G X^{\wedge|G/H|}$$

where G acts on the right hand side by the simultaneous actions of diagonal ones and permutations of factors by left actions on G/H. In particular, the diagonal map

$$\Delta_X : X \to X^{\wedge|G/H|}$$

is a G-map. Thus we get the <u>internal power</u> operation

$$\mathcal{P} = \Delta_X^* \circ \mathcal{P}_{ext} \;:\; \omega_H^{\alpha}(X) \to \omega_G^{\mathrm{ind}_H^G \alpha}(X)$$

for a pointed G-complex X.

<u>Proposition</u> 4.4. <u>Let</u> $G \trianglerighteq N$, <u>normal</u>, <u>and</u> $x \in \omega_N^{\alpha}(X)$ <u>for a pointed</u> G-<u>complex</u> X. <u>Then</u>

$$\psi_N \circ \mathcal{P}(x) = \prod_{i=1}^{|G/N|} x^{g_i},$$

<u>where</u> $G = \coprod g_i N$ <u>and</u> $x \mapsto x^{g_i}$ <u>is induced by conjugation with respect to</u> g_i.

<u>Proposition</u> 4.5. <u>Let</u> $G \cdot N$, <u>a semi-direct product such that</u> $G \trianglerighteq N$. <u>Let</u> X <u>be a pointed</u> K-<u>complex</u> (<u>regarded as a</u> G-<u>complex through</u> $G \to G/N = K$). <u>Let</u> $\alpha \in R(K) + Z$. <u>Then the following diagram is commutative</u>:

$$\begin{tikzcd}
\omega_K^{\alpha}(X) \arrow[r, "\mathcal{P}"] \arrow[dr, "\mathrm{id}"', "\approx"] & \omega_G^{\mathrm{ind}_K^G \alpha}(X) \arrow[d, "\phi_N"] \\
& \omega_{G/N}^{\alpha}(X).
\end{tikzcd}$$

Under the same situation as above, let V' be a finite dimensional complex K-module and $V = V' \oplus R^k$, $k \geqq 1$. Then $W = \mathrm{ind}_K^G V - V$ contain a real $\tilde{\rho}_{G,K}$ as a summand. Remark that $W^N = \{0\}$ and $\phi_N \circ \chi_W = \phi_N$. We get two splittings of the fixed-point exact sequences for $\alpha = -V$:

$$\begin{tikzcd} 0 \arrow[r] & \lambda^{-V}_{G,N}(X) \arrow[r, "\delta_N"] & \omega^{-V}_G(X) \arrow[r, "\phi_N"] & \omega^{-V}_K(X) \arrow[l, bend left, "\theta^G_{G/N}"] \arrow[l, bend left=70, "\chi_W \circ \rho"] \arrow[r] & 0 . \end{tikzcd}$$

Proposition 4.6. Under this situation $\psi_K \circ \chi_W \circ \rho = 0$.

Then the difference $\theta^G_{G/N} - \chi_W \circ \rho$ gives a homomorphism

(4.7) $$\tilde{\theta}_K : \omega_K^{-V}(X) \longrightarrow \lambda^{-V}_{G/N}(X)$$

such that $\delta_N \circ \tilde{\theta}_K = \theta^G_{G/N} - \chi_W \circ \rho$, $\psi_K \circ \delta_N \circ \tilde{\theta}_K = \mathrm{id}$.

Theorem 4.8. Under the above situation

$$\psi_K \circ \delta_N : \lambda^{-V}_{G,N}(X) \longrightarrow \omega_K^{-V}(X)$$

is a split epimorphism with the splitting $\tilde{\theta}_K$.

5. The equivariant Kahn-Priddy Theorem.

In Theorem 4.8 we put $N = Z/2$ and $G = K \times Z/2$. Using Clifforg $C(W)$-module (where W is a K-module) and equivariant S-duality we can prove the isomorphism

(5.1) $$\lambda^{-V}_{G,Z/2}(X) \approx \omega^K_V(X;\ RP_+^{\infty\rho})$$

which is natural with respect to X, where $RP^{\omega\rho}$ is the real projective space in ω·regular representation of K, regarded as a K-complex.

A combination of Prop.3.11, Theorem 4.8 and (5. 1) inplies

Theorem 5.2. There holds an epimorphism

$$\omega_V^G(RP^{\infty\rho})_{(2)} \longrightarrow (\omega_V^G)_{(2)}$$

at 2-primary components, where $V = V' \oplus R^k$, $k \geqq 1$, and V' is a finite dimensional complex G-module.

The above theorem is the equivariant version of the Kahn-Priddy theorem for $p = 2$.