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Kyoto University
A Class of Recurrence Relations on Acyclic Digraphs of Poset Type

by

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We shall talk about a systematic study on a class of elementary combinatorial functions related to the number of pathes (chains) on an acyclic digraph (poset). Let D be an acyclic digraph. Then, for each arc (s,t) in D we say that s is adjacent to t, written s→t. In this talk, each vertex in the vertex-set V(D) of D is regarded as a path of length 0. Let R and R[x] be the real field and the polynomial ring of one variable, respectively. Then, for each a and b in R we define a map \( f^{(a,b)} : V(D) \to R[x] \) by

\[
f^{(a,b)}(s) = \begin{cases} 
a & \text{if } s \text{ is a sink} \\ \sum_{s \to t} f^{(a,b)}(t)x + b & \text{otherwise.} \end{cases}
\]

Let \( \mathcal{L}(D) \) denote the set \( \{ f^{(a,b)} \mid a \text{ and } b \in R \} \) of all such maps over R. Then, for each f and g in \( \mathcal{L}(D) \) and each a in R the sum + and the scalar multiple af are defined as follows:

1. \( (f + g)(s) = f(s) + g(s) \)
2. \( (af)(s) = a(f(s)) \),

where s is any vertex in V(D).

Theorem 1. Under the sum + and the scalar multiple, \( \mathcal{L}(D) \) is a linear space over R isomorphic to the 2-dimensional linear space \( R^2 \).
Therefore, \( \{ f(1,0), f(0,1) \} \) is a base of the linear space \( \mathcal{L}(D) \), and each \( f(a,b) \) in \( \mathcal{L}(D) \) is uniquely representable as follows:

\[
f(a,b) = af(1,0) + bf(0,1).
\]

**Remark 1.** For \( f(1,0) \), \( f(0,1) \) and \( f(1,1) \) in \( \mathcal{L}(D) \) and each \( s \) in \( V(D) \),

1. the coefficient of \( x^i \) in \( f(1,0)(s) \) is the number of paths of length \( i \) from \( s \) to sinks in \( D \),
2. the coefficient of \( x^i \) in \( f(0,1)(s) \) is the number of paths of length \( i \) from \( s \) to vertices but sinks in \( D \),
3. the coefficient of \( x^i \) in \( f(1,1)(s) \) is the number of paths of length \( i \) from \( s \) in \( D \).

For each \( f \) in \( \mathcal{L}(D) \), we define \( \tilde{f} \) by \( \tilde{f} = \sum_{s \in V(D)} f(s) \), that is, \( \tilde{f} \) is a map from \( \mathcal{L}(D) \) into \( \mathbb{R}[x] \).

**Theorem 2.** \( \mathcal{L}(D) \) is the linear subspace \( \langle \tilde{f}(1,0), \tilde{f}(0,1) \rangle \) of \( \mathbb{R}[x] \) and each \( \tilde{f}(a,b) \) in \( \mathcal{L}(D) \) is uniquely representable as follows:

\[
\tilde{f}(a,b) = a\tilde{f}(1,0) + b\tilde{f}(0,1).
\]

**Remark 2.** For \( \tilde{f}(1,0) \), \( \tilde{f}(0,1) \) and \( \tilde{f}(1,1) \) in \( \mathcal{L}(D) \), we have the following facts:

1. the coefficient of \( x^i \) in \( \tilde{f}(1,0) \) is the number of all paths of length \( i \) to sinks in \( D \),
2. the coefficient of \( x^i \) in \( \tilde{f}(0,1) \) is the number of all paths of length \( i \) to vertices but sinks in \( D \),
3. the coefficient of \( x^i \) in \( \tilde{f}(1,1) \) is the number of all paths of length \( i \) in \( D \).
Example 1. \( f^{(1,0)}, f^{(0,1)}, f^{(1,1)} \) and each \( \tilde{f} \) are illustrated.

(1) \( f^{(1,0)} \)

\[
\begin{align*}
3x & \quad 4x^2 + x \\
\circ & \quad \circ
\end{align*}
\]

\[
\tilde{f}^{(1,0)} = 7x^2 + 5x + 2
\]

(2) \( f^{(0,1)} \)

\[
\begin{align*}
2x + 1 & \quad 3x + 1 \\
\circ & \quad \circ
\end{align*}
\]

\[
\tilde{f}^{(0,1)} = 5x + 5
\]

(3) \( f^{(1,1)} \)

\[
\begin{align*}
3x^2 + 2x + 1 & \quad 4x^2 + 4x + 1 \\
\circ & \quad \circ
\end{align*}
\]

\[
\tilde{f}^{(1,1)} = 7x^2 + 10x + 7
\]

If a digraph \( D \) represents the incidence relation of a poset \( P \), then \( D \) is said to be of poset type \( P \) and identified with \( P \). If a digraph \( D \) represents the Hasse diagram \( H(P) \) of a poset \( P \), then \( D \) is said to be of Hasse diagram type \( H(P) \) and identified with \( H(P) \).

Example 2. A digraph of poset type \( P \) and a digraph of Hasse diagram type \( H(P) \) are illustrated.

The Hasse diagram \( H(P) \) of a poset \( P \)

A digraph of poset type \( P \)

A digraph of Hasse diagram type \( H(P) \)

Remark 3. Each \( f^{(a,b)} \) in \( L(P) \) is rewritten as follows:

\[
f^{(a,b)}(s) = \begin{cases} 
   a & \text{(s = a minimal element)} \\
   \left( \sum_{s > t} f^{(a,b)}(t) \right) x + b & \text{otherwise}.
\end{cases}
\]

Each \( f^{(a,b)} \) in \( L(H(P)) \) is rewritten as follows:

\[
f^{(a,b)}(s) = \begin{cases} 
   a & \text{(s = a minimal element)} \\
   \left( \sum_{s \leq t} f^{(a,b)}(t) \right) x + b & \text{otherwise},
\end{cases}
\]
where s \triangleright t denotes "s covers t".

A poset $P$ is said to be connected if the incidence relation of $P$ is represented by a connected digraph.

**Theorem 3.** Let $P$ be a connected poset. Let $f^{(1,0)}$ and $f^{(0,1)}$ be in $\mathcal{L}(P)$. Then, $f^{(1,0)}(s) = (f^{(0,1)}(s))x$ for all $s$ but minimal elements in $P$ if and only if $P$ has a unique minimal element.

**Theorem 4.** Let $P$ be a poset with a unique maximal element $1$, of which cardinality $\geq 2$. Then, for $\tilde{f}^{(a,b)}$ in $\mathcal{L}(P)$ and $f^{(a,b)}$ in $\mathcal{L}(P)$, the following identity holds:

$$
\tilde{f}^{(a,b)} = ((x + 1)f^{(a,b)}(1) - b)/x.
$$

**Example 3.** Theorem 3 and 4 are illustrated.

Each upper poly. is $f^{(1,0)}(s)$. $f^{(1,1)}(1) = 5x^2 + 5x + 1$.
Each lower poly. is $f^{(0,1)}(s)$. $\tilde{f}^{(1,1)} = ((x + 1)f^{(1,1)}(1) - 1)/x = 5x^2 + 10x + 6$.

**Remark 4.** Let $P$ be a poset. Then, in [1] the following notations are used. For any $s$ and $t$ in $P$ such that $s \geq t$,

$C(s,t;x)$: the command flow polynomial from $s$ to $t$.

Note that the coefficient of $x^i$ in $C(s,t;x)$ is the number of covering chains of length $i$ from $s$ to $t$. 

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C(s,t): the command flow number from s to t (= C(s,t;1)).
Note that C(s,t) is the number of covering chains from s to t.

C(s;x): the command flow polynomial from s.
Note that the coefficient of $x^i$ in C(s;x) is the number of covering chains of length i from s.

C(s): the command flow number from s (= C(s;1)).
Note that C(s) is the number of covering chains from s.

C(P;x): the command flow polynomial of P.
Note that the coefficient of $x^i$ in C(P;x) is the number of covering chains of length i in P.

C_P: the command flow number of P (= C(P;1)).
Note that C_P is the number of covering chains in P.

A: the adjacency matrix of H(P).

[t,s]: the closed interval.

Then, for $f^{(1,0)}$ in $\mathcal{L}(H([t,s]))$, $f^{(1,1)}$ in $\mathcal{L}(H(P))$, we have the following facts.

(1) $f^{(1,0)}(s) = C(s,t;x)$
(2) $f^{(1,0)}(s)|_{x=1} = C(s,t) = (E - A)^{-1}(t,s)$
(3) $f^{(1,0)}(s)|_{x=-1} = (E + A)^{-1}(t,s)$
(4) $f^{(1,1)}(s) = C(s;x)$
(5) $f^{(1,1)}(s)|_{x=1} = C(s) = \sum_{t \in P} (E - A)^{-1}(t,s)$
(6) $f^{(1,1)}(s)|_{x=-1} = \sum_{t \in P} (E + A)^{-1}(t,s)$
(7) $\tilde{f}^{(1,1)} = C(P;x)$
(8) $\tilde{f}^{(1,1)}|_{x=1} = C_P = \sum_{s \in P} \sum_{t \in P} (E - A)^{-1}(t,s)$
(9) $\tilde{f}^{(1,1)}|_{x=-1} = \sum_{s \in P} \sum_{t \in P} (E + A)^{-1}(t,s)$

Remark 5. Let P be a poset. Then, in [1] the following notations are used. For any s and t in P such that $s \preceq t$,
\( C^*(s,t;x) \): the total command flow polynomial from \( s \) to \( t \).

Note that the coefficient of \( x^i \) in \( C^*(s,t;x) \) is the number of chains of length \( i \) from \( s \) to \( t \).

\( C^*(s,t) \): the total command flow number from \( s \) to \( t \)
\( (= C^*(s,t;1)) \).

Note that \( C^*(s,t) \) is the number of chains from \( s \) to \( t \).

\( C^*(s;x) \): the total command flow polynomial from \( s \).

Note that the coefficient of \( x^i \) in \( C^*(s;x) \) is the number of chains of length \( i \) from \( s \).

\( C^*(s) \): the total command flow number from \( s \) (= \( C^*(s;1) \)).

Note that \( C^*(s) \) is the number of chains from \( s \).

\( C^*(P;x) \): the total command flow polynomial of \( P \).

Note that the coefficient of \( x^i \) in \( C^*(P;x) \) is the number of chains of length \( i \) in \( P \).

\( C^*_P \): the total command flow number of \( P \) (= \( C^*(P;1) \)).

Note that \( C^*_P \) is the number of chains in \( P \).

\( \mu \): the Möbius function of \( P \) (= \( \zeta^{-1} = (\delta + n)^{-1} \), \( \zeta \): the zeta function, \( n \): the incidence matrix, \( \delta \): the delta function).

\([t,s]\): the closed interval.

Then, for \( f^{(1,0)} \) in \( \mathcal{L}([t,s]) \) and \( f^{(1,1)} \) in \( \mathcal{L}(P) \), we have the following facts.

1. \( f^{(1,0)}(s) = C^*(s,t;x) \)
2. \( f^{(1,0)}(s)|_{x=1} = C^*(s,t) = (\delta - n)^{-1}(t,s) \)
3. \( f^{(1,0)}(s)|_{x=-1} = \mu(t,s) = (\delta + n)^{-1}(t,s) \)
4. \( f^{(1,1)}(s) = C^*(s;x) \)
5. \( f^{(1,1)}(s)|_{x=1} = C^*(s) = \sum_{t \in P} (\delta - n)^{-1}(t,s) \)
6. \( f^{(1,1)}(s)|_{x=-1} = \sum_{t \in P} \mu(t,s) = \sum_{t \in P} (\delta + n)^{-1}(t,s) \)
7. \( f^{(1,1)} = C^*(P;x) \)
\[ \hat{f}(1,1) |_{x=1} = C_P^* = \sum_{s \in P} \sum_{t \in B} (\delta - n)^{-1}(t,s) \]
\[ \hat{f}(1,1) |_{x=-1} = \sum_{s \in P} \sum_{t \in B} u(t,s) = \sum_{s \in P} \sum_{t \in B} (\delta + n)^{-1}(t,s) \]

The Command Flow Number Theory on Boolean Lattices

For a Boolean lattice \( B_n \) of \( n \) atoms, we use the following notations on the command flow polynomials.

\( C_B^{(1,0)}(n;x) \): the command flow polynomial from the top 1 to the bottom 0, i.e., \( C(1,0;x) \).

\( C_B^{(1,1)}(n;x) \): the command flow polynomial from the top 1, i.e., \( C(1;x) \).

\( \tilde{C}_B^{(1,1)}(n;x) \): the command flow polynomial of \( B_n \), i.e., \( C(B_n;x) \).

Then, we have the following formulas.

(1) \[ C_B^{(1,0)}(n;x) = \begin{cases} 1 & (n = 0) \\ nC_B^{(1,0)}(n-1;x) & (n \geq 1) \end{cases} \]

(2) \[ C_B^{(1,0)}(n;x) = n!x^n \]

(3) \[ C_B^{(1,0)}(n;1) = n! \text{ (the CF-number from 1 to 0)} \]

(4) \[ C_B^{(1,0)}(n;-1) = (-1)^n n! \]

(5) \[ C_B^{(1,1)}(n;x) = \begin{cases} 1 & (n = 0) \\ nC_B^{(1,1)}(n-1;x)x + 1 & (n \geq 1) \end{cases} \]

(6) \[ C_B^{(1,1)}(n;x) = \sum_{k=0}^{n} P_k x^k \]

(7) \[ C_B^{(1,1)}(n;1) = \left( \sum_{k=0}^{n} \frac{1}{k!} \right) n! = (e)_n n! \text{ (the CF-number from 1)}, \]

where \((e)_n\) denotes the first \( n+1 \) terms of Maclaurin expansion of the constant \( e \).

(8) \[ C_B^{(1,1)}(n;-1) = \left( \sum_{k=0}^{n} \frac{1}{(-1)^{n-k} k!} \right) n! = (-1)^n (e^{-1})_n n! \]

\[ = (-1)^n D(n), \]

where \((e^{-1})_n\) denotes the first \( n+1 \) terms of Maclaurin expansion of the constant \( e^{-1} \) and \( D(n) \) denotes "the well-known number of
permutations admitting no coincidences (derangements) of \( n \) objects.

(9) \( \tilde{C}_B^{(1,1)}(n;x) = \sum_{k=0}^{n} \binom{n}{k} C_B^{(1,1)}(k;x) \)

Let \( G(\tilde{C}_B^{(1,1)}(n;x);z) \) denote the exponential generating function of \( \tilde{C}_B^{(1,1)}(n;x) \).

(10) \( G(\tilde{C}_B^{(1,1)}(n;x);z) = \frac{e^{e^2z}}{1-xz} \)

(11) \( \tilde{C}_B^{(1,1)}(n;x) = \sum_{k=0}^{n} \binom{n}{k} \tilde{z}^{n-k} n! k^k = \left( \sum_{k=0}^{n} \frac{2^k}{k!} x^{n-k} \right) n! \)

(12) \( \tilde{C}_B^{(1,1)}(n;1) = \left( \sum_{k=0}^{n} \frac{2^n}{k!} \right) n! = (e^2)_n \cdot n! \) (the CF-number of \( B_n \))

(13) \( \tilde{C}_B^{(1,1)}(n;-1) = (-1)^n (e^{-2})_n \cdot n! \)

(14) \( e \cdot n! - C_B^{(1,1)}(n;1) = O\left( \frac{1}{n} \right) \)

(15) \( e^2 \cdot n! - \tilde{C}_B^{(1,1)}(n;1) = O\left( \frac{2^n}{n} \right) \)

We use the following notations on the total command flow polynomials.

\( C_B^{(1,0)}(n;x) \): the total command flow polynomial from 1 to 0, i.e., \( C^*(1,0;x) \).

\( C_B^{(1,1)}(n;x) \): the total command flow polynomial from 1, i.e., \( C^*(1,1;x) \).

\( \tilde{C}_B^{(1,1)}(n;x) \): the total command flow polynomial of \( B_n \), i.e., \( C^*(B_n;x) \).

Then, we have the following formulas.

(1) \( C_B^{(1,0)}(n;x) = \begin{cases} \frac{1}{n} & (n = 0) \\ \sum_{k=0}^{n-1} \binom{R}{k} C_B^{(1,0)}(k;x) & (n \geq 1) \end{cases} \)

(2) \( G(C_B^{(1,0)}(n;x);z) = \frac{1}{1 + x - xe^z} \),

where \( G(C_B^{(1,0)}(n;x);z) \) is the exponential generating function
of $C_B^{(1, 0)}(n; x)$.

(3) $C_B^{(1, 0)}(n; x) = \sum_{j=0}^{\infty} \frac{x^j}{(x+1)^{j+1}} j^n = \sum_{k=0}^{n} M(n, k, 0) x^k$,

$M(n, k, 0) = \sum_{i+j=k} (-1)^i \binom{k}{i} j^n$ (the number of surjections from $A$ ($|A| = n$) to $B$ ($|B| = k$)) = $k! S(n, k)$, where $S(n, k)$ is the Stirling number of second kind.

(4) $C_B^{(1, 0)}(n; 1) = \sum_{j=0}^{\infty} \frac{j^n}{2^{j+1}} = \sum_{k=0}^{n} M(n, k, 0)$

Remark 6. In p. 15 and 149 of [4], the following number is defined: for $N_n = \{1, 2, \ldots, n\}$,

$S_n$: the number of mappings $f$ from $N_n$ into itself such that if $f$ takes a value $i$ then it also takes each value $j$, $1 \leq j \leq i$, $S_0 = 1$. Also, in [5], the following number is dealt:

$P(n)$: the number of total preorders on a $n$-set.

Each recurrence relation for $S_n$ and $P(n)$ is equal to $(1) \big|_{x=1}$, and therefore we have the following equality:

$C_B^{(1, 0)}(n; 1) = S_n = P(n)$.

(5) $C_B^{(1, 0)}(n; -1) = \sum_{k=0}^{n} (-1)^k M(n, k, 0) = \sum_{k=0}^{n} (-1)^k k! S(n, k)$

$= (-1)^n$ (the Möbius function of $B_n$)

(6) $C_B^{(1, 1)}(n; x) = \begin{cases} 1 & (n=1) \\ \frac{1}{n-1} \sum_{k=0}^{n-1} (k) C_B^{(1, 1)}(k; x) x + 1 & (n \geq 1) \end{cases}$

(7) $G(C_B^{(1, 1)}(n; x); z) = \frac{e^z}{1 + x - xe^z}$

(when $x = 1$, $G(C_B^{(1, 1)}(n; 1); z) = \frac{e^z}{2 - e^z}$, by H. Enomoto), hereafter $G(f; z)$ denotes the exponential generating function of $f$. 9
(8) \( C_B^{(1,1)}(n;x) = \sum_{j=0}^{\infty} \frac{x^j}{(x+1)^{j+1}} (j+1)^n = \sum_{k=0}^{n} M(n,k,1)x^k, \)

\( M(n,k,1) = \sum_{i+j=k} (-1)^i \binom{k}{i} (j+1)^n \) (with M. Tsuchiya).

(9) \( C_B^{(1,1)}(n;1) = \sum_{j=0}^{\infty} \frac{(j+1)^n}{2^{j+1}} = \sum_{k=0}^{n} M(n,k,1) \)

Note that for \( n \neq 0 \), from Theorem 1 and 3 or comparing with (4),

\( C_B^{(1,1)}(n;1) = 2C_B^{(1,0)}(n;1). \)

(10) \( C^*_B(1,1)(n;x) = \sum_{k=0}^{n} \binom{n}{k} C_B^{(1,1)}(k;x) \)

(11) \( G(\tilde{C}_B^{(1,1)}(n;x);z) = \frac{e^{2z}}{1 + x - xe^z} \)

(12) \( \tilde{C}_B^{(1,1)}(n;x) = \sum_{j=0}^{\infty} \frac{x^j}{(x+1)^{j+1}} (j+2)^n = \sum_{k=0}^{n} M(n,k,2)x^k, \)

\( M(n,k,2) = \sum_{i+j=k} (-1)^i \binom{k}{i} (j+2)^n. \)

(13) \( \tilde{C}_B^{(1,1)}(n;1) = \sum_{j=0}^{\infty} \frac{(j+2)^n}{2^{j+1}} = \sum_{k=0}^{n} M(n,k,2), \)

Note that from Theorem 4 or comparing with (9),

\( \tilde{C}_B^{(1,1)}(n;1) = 2C_B^{(1,1)}(n;1) - 1. \)

We now stand on a stage of introducing the following polynomial with respect to \( t \):

\( M(n,k,t) = \sum_{i+j=k} (-1)^i \binom{k}{i} (j+t)^n. \)

This polynomial has the following property:

\[ M(n,k,t-l) + M(n,k+1,t-l) \quad (0 \leq k \leq n-l) \]
\[ M(n,k,t) = \begin{cases} n! & (k=n) \\ 0 & (k \geq n+1). \end{cases} \]

From \( M(n,n,t) = n! \), we obtain the following formula:

\[ \sum_{k=0}^{n} (-1)^k \binom{n}{k} k^i = \begin{cases} 0 & (0 \leq i < n) \\ (-1)^n n! & (i = n) \end{cases} \]

(\( = C_B^{(1,0)}(n;-1) \))
Also, by putting \( C^*(n, t; x) = \sum_{k=0}^{n} M(n, k, t)x^k \), we obtain the following formulas:

(i) \( C^*(n, t; x) = \sum_{k=0}^{n} \binom{n}{k} C^*(n, t-1; x) \)

(ii) \( C^*(n, t; x) = (C^*(n, t-1; x)(1 + x) - (t-1)^n) / x \)

(iii) \( G(C^*(n, t; x); z) = \frac{e^{tz}}{1 + x - xe^z} \).

(14) \( \lim_{n \to \infty} \binom{C^*_B(1, 0)}{n; 1} / \frac{1}{2} \left( \frac{1}{\log 2} \right)^{n+1} n! = 1 \) (with T. Ohya)

\( \lim_{n \to \infty} \frac{1}{2} \left( \frac{1}{\log 2} \right)^{n+1} n! = 1 \) in Lovász [4])

\( \lim_{n \to \infty} \frac{1}{2} \left( \frac{1}{\log 2} \right)^{n+1} n! = 1 \) in Barthelemy [5])

(15) \( \limsup_{n \to \infty} \binom{C^*_B(1, 0)}{n; 1} - \frac{1}{2} \left( \frac{1}{\log 2} \right)^{n+1} n! = \infty \)

\( \liminf_{n \to \infty} \binom{C^*_B(1, 0)}{n; 1} - \frac{1}{2} \left( \frac{1}{\log 2} \right)^{n+1} n! = -\infty \) (with T. Hilano)

(16) \( \binom{C^*_B(1, 0)}{n; 1} = n! \left( \frac{1}{2} \left( \frac{1}{\log 2} \right)^{n+1} + \sum_{k=1}^{\infty} \Re(z^{n+1}) \right) \)

\( \Re(z^{n+1}) = \frac{1}{(\sqrt{\log 2})^2 + (2\pi n)^2} \cos(n+1)\theta_k \)

\( \tan\theta_k = (2\pi k) / \log 2 \)

\( P(n) = \frac{n!}{2(\log 2)^{n+1}} + \sum_{p=0}^{\infty} \frac{B_{n+p+1}}{(n+p+1)!} (\log 2)^p \)

\( B_{n+p+1} \) is the Bernoulli number, in Barthelemy [5])

The following lemma is useful in obtaining a generating function.

**Lemma.** Let \( F(z) \) denote the exponential generating function \( \sum_{n=0}^{\infty} f(n)(z^n / n!) \) of \( f(n) \) and \( G(z) \) denote the exponential generating function \( \sum_{n=0}^{\infty} g(n)(z^n / n!) \) of \( g(n) = \sum_{k=0}^{n} \binom{n}{k} f(k) \).
Then, the following identity holds.
\[ G(z) = e^{zF(z)}. \]

Remark 7. Generally speaking, the so-called computational complexity of the well-known method with matrix operations for computing the number of paths in a given acyclic digraph is \( O(n^{2+\alpha}) \) for \( n = \) the number of vertices. But, the complexity of our method is \( O(\ell) \) for \( \ell = \) the number of arcs. Note that \( \ell \leq n^2 \).

The author thinks that "the command flow complexity of a social system" with an order relation is evaluated by the command flow numbers on the system.

Finally, we restate the following open problem.

**Open Problem.** Decide whether or not there exists \( n \geq 17 \) such that \[ |\binom{(1,0)}{n;1} - \frac{1}{2} \left( \frac{1}{\log 2} \right)^{n+1} \cdot n! | < \frac{1}{2}. \]

References