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Theory of Hypermatroids

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Summary

Hypermatroid is a generalized notion of matroid and network flow. However, it is not a mere generalization but it gives us a deep insight into matroids and network flows and also provides us with new significant problems which are overlooked. This is an abstract of the series of the author's papers [1] $\sim$ [7].

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44

If E is a finite set, we let $\mathfrak{S}_0(E)$ denote the linear space of real-valued *modular functions* on 2^E which corresponds closely to the vector space R^E except on ϕ. Let μ^ϕ denote the *constant function* in $\mathfrak{S}_0(E)$ such that $\mu^\phi(X) = 1$ for any $X \subseteq E$ and let μ^i denote the *unit function* such that $\mu^i(X) = 1$ if $i \in X$ and 0 otherwise. For any $\xi \in \mathfrak{S}_0(E)$, we define

$$\mathrm{car}^{\pm}\xi = \{i \mid \xi(\{i\}) \gtrless \xi(\phi)\}.$$

A *hedron* $\mathfrak{B}$ in $\mathfrak{S}_0(E)$ is a compact non-empty subset of $\mathfrak{S}_0(E)$ satisfying the following

Exchange axiom for bases [1]. If $\xi, \eta \in \mathfrak{B}$ and $\xi \neq \eta$, then $\xi(\phi) = \eta(\phi)$ and $\exists i \in \mathrm{car}^-(\xi - \eta)$, $\exists j \in \mathrm{car}^+(\xi - \eta)$, $\exists \hat{c} > 0$;

$$(0 \leq)\forall c \leq \hat{c}\colon\ \xi + c(\mu^i - \mu^j),\ \eta + c(\mu^j - \mu^i) \in \mathfrak{B}.$$

We say the modular function $\xi \in \mathfrak{B}$ is a *base* of $\mathfrak{B}$. Since the Exchange axiom for bases has a self-dual form, it is obvious that if $\mathfrak{B}$ is a hedron in $\mathfrak{S}_0(E)$ then $-\mathfrak{B} \equiv \{-\xi \mid \xi \in \mathfrak{B}\}$ is also a hedron in $\mathfrak{S}_0(E)$, called the *inverse* of $\mathfrak{B}$.

The *hyperspace* (or *hedron space*) $\mathfrak{H}(E)$ is the linear space of all hedra in $\mathfrak{S}_0(E)$ such that the *sum* is defined by

$$\mathfrak{A} + \mathfrak{B} \equiv \{\xi + \eta \mid \xi \in \mathfrak{A},\ \eta \in \mathfrak{B}\}$$

and *scalar multiple* is defined by

$$c\mathfrak{B} \equiv \{c\eta \mid \eta \in \mathfrak{B}\} \quad (c \in R).$$

Obviously, $\mathfrak{S}_0(E)$ is a subspace of $\mathfrak{H}(E)$. Thus a hyperspace can be regarded as a natural generalization of a vector space.

Let $\mathfrak{S}_{\pm}(E)$ denote the convex cone of real-valued *super*[*sub*]-*modular functions* on 2^E. The *deficiency* [*rank*] *function* $\sigma_{\pm}$ of a hedron $\mathfrak{B}$ is defined by

$$\sigma_{\pm}(X) \equiv {\min \atop \max} \{\xi(X) \mid \xi \in \mathfrak{B}\}.$$

Then we have the following

Theorem 1. The deficiency [rank] function $\sigma_{\pm}$ of a hedron $\mathfrak{B}$ in $\mathfrak{H}(E)$ is a super[sub]modular function, that is $\sigma_{\pm} \in \mathfrak{S}_{\pm}(E)$.

The remarkable property of Theorem 1 is that its converse is also true.

Theorem 2. For any super[sub]modular function $\sigma_{\pm} \in \mathfrak{S}_{\pm}(E)$, the convex polyhedron $\mathfrak{B}_{\pm}$ in $\mathfrak{S}_0(E)$ defined by

$$\mathfrak{B}_{\pm} \equiv \{\xi \mid \xi \gtreqless \sigma_{\pm},\ \xi(E) = \sigma_{\pm}(E),\ \xi(\phi) = \sigma_{\pm}(\phi)\}$$

is a hedron in $\mathfrak{H}(E)$.

We have then established a one-to-one correspondence between the hedra in $\mathfrak{H}(E)$ and the super[sub]modular functions in $\mathfrak{S}_{\pm}(E)$.

The concept of *hypermatroids* may be defined in several different but equivalent ways. That is, a hypermatroid $\mathfrak{M}$ is a pair (E, $\mathfrak{B}$), (E, σ_+), (E, σ_-), etc., where E is a ground set, $\mathfrak{B}$ is a hedron in $\mathfrak{H}(E)$, and $\sigma_\pm(\in \mathfrak{S}_\pm(E))$ is the deficiency [rank] function of $\mathfrak{B}$.

The $*$-*dual* of $\sigma_\pm \in \mathfrak{S}_\pm(E)$ is defined by

$$\sigma^*_\pm(X) \equiv -\sigma_\pm(\bar{X}) + \sigma_\pm(E) + \sigma_\pm(\phi).$$

Theorem 3. The $*$-dual of the deficiency [rank] function of a hedron $\mathfrak{B}$ in $\mathfrak{H}(E)$ is the rank [deficiency] function of $\mathfrak{B}$, that is $\sigma^*_\pm = \sigma_\mp$.

Hereafter, for notational simplicity let ρ be a submodular function on 2^E. Define a *least upper vector* $\grave{\rho}$ and a *greatest lower vector* $\acute{\rho}$ of ρ by

$$\grave{\rho}(x) \equiv \rho(\{x\}) - \rho(\phi) \quad (^\forall x \in E),$$
$$\acute{\rho}(x) \equiv \rho(E) - \rho(E - \{x\}) \quad (^\forall x \in E),$$

respectively. And define a *least upper modular function* $\hat{\rho}$ and a *greatest lower modular function* $\check{\rho}$ of ρ by

$$\hat{\rho}(X) \equiv \sum \grave{\rho}(x) \quad (x \in X),$$
$$\check{\rho}(X) \equiv \sum \acute{\rho}(x) \quad (x \in X),$$

respectively. We say $\rho^\circ \equiv \hat{\rho} - \check{\rho}$ ($\in \mathfrak{S}_0(E)$) is the *oscillation* of ρ. The $\flat$-*dual* of ρ is defined by $\rho^\flat \equiv \rho^\circ - \rho$ and the $\natural$-*dual* of ρ is defined by $\rho^\natural \equiv (\rho^\flat)^*$. Then we call the hypermatroid $\mathfrak{M}^\natural =$

$(E, \rho^{\natural})$ defined by a rank function the *dual* of $\mathfrak{M} = (E, \rho)$. The hedron $\mathfrak{B}^{\natural}$ of $\mathfrak{M}^{\natural}$ is given by

$$\mathfrak{B}^{\natural} \equiv \{\rho^{\circ} - \xi \mid \xi \in \mathfrak{B}\},$$

and the deficiency function of $\mathfrak{M}^{\natural}$ is given by $\rho^{\flat}$. Obviously we have

$$(\mathfrak{M}^{\natural})^{\natural} = \mathfrak{M}.$$

Note that the duality of hypermatroids is slightly different from the duality of matroids.

Theorem 4. Any submodular function ρ can be decomposed into three parts as follows:

$$\rho = \rho(\phi)\mu^{\phi} + \check{\rho} + \tilde{\rho}. \tag{1}$$

We call $\tilde{\rho}$ in (1) the *proper* submodular function of ρ. Let $\tilde{\mathfrak{S}}_{-}(E)$ denote the set of all proper submodular functions in $\mathfrak{S}_{-}(E)$.

A hypermatroid $\mathfrak{M} = (E, \rho)$ is called *integral* if ρ is integer-valued. A *polymatroid* [8] $\mathfrak{M} = (E, \rho)$ is a hypermatroid satisfying $\rho(\phi) = 0$ and $\acute{\rho}(x) \geq 0$ ($^{\forall}x \in E$). A *matroid* is an integral polymatroid satisfying $\grave{\rho}(x) \leq 1$ ($^{\forall}x \in E$).

A *quasimatroid* [7] is an integral polymatroid such that its rank function is the direct sum of proper submodular functions satisfying

$$\left.\begin{array}{l}\rho(X\cup\{y\})+\rho(X\cup\{z\})-\rho(X)-\rho(X\cup\{y,z\})=0,\ 1\\ (y,\ z\notin X,\ y\neq z),\end{array}\right\}(2)$$

unit functions and the constant function. Obviously a matroid is a quasimatroid.

The following theorem solves the open question by Edmonds [8].

Extreme rays theorem [2]. The extreme rays of $\tilde{\mathfrak{S}}_{-}(E)$ are the proper submodular functions which satisfy the above condition (2) and have minimal sets of intervals $[X, X\cup\{y,z\}]$ such that the left hand side of (2) is equal to 1.

Let $N=(V, A; c)$ be the capacitated network, where V is a vertex set, A is a directed arc set and c is a capacity vector in R_+^A. Define the *cut function* $\gamma: 2^V\to R_+$ by

$$\gamma(X)\equiv c(X,\bar{X})\equiv\sum c(a)\quad(\partial^+a\in X,\ \partial^-a\in\bar{X},\ a\in A).$$

For any *flow* $f\in R_+^A$ satisfying $f\leq c$, define the *boundary function* $\partial f: 2^V\to R$ by

$$\partial f(X)\equiv\sum\{f(\delta^+v)-f(\delta^-v)\}\quad(v\in X).$$

Then γ is submodular and we have $\partial f(\phi)=\gamma(\phi)=0$, $\partial f(V)=\gamma(V)=0$ and $\partial f\leq\gamma$. Therefore, $\mathfrak{N}=(V,\gamma)$ is a hypermatroid defined by a rank function and every boundary function ∂f for a flow f is a base of $\mathfrak{N}$ [5],[6]. Thus we have known that a capacitated network N = (V, A; c) is a typical example of a hypermatroid.

References

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