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Fixed Point Theorems in Nonlinear Analysis

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Let X be a given set and consider a mapping T of X into X. Then a point x such that $Tx = x$ is called a fixed point of T. Furthermore consider a mapping T of X into 2^X (the set of all subsets of X). Then a fixed point for T is a point x such that $x \in Tx$. A fixed point exists under suitable conditions of T and X. The theorems concerning fixed points are the so-called fixed point theorems and they are very useful in nonlinear analysis.

Let H be a real Hilbert space and let C be nonempty closed convex subset of H. A mapping $T: C \to C$ is called nonexpansive on C, or $T \in \text{Cont}(C)$ if $\|Tx - Ty\| \leq \|x - y\|$ for every $x, y \in C$. Let F(T) be the set of fixed points of T, that is, $F(T) = \{ z \in C : Tz = z \}$. Then, the set F(T) is obviously closed and convex. Let $S = \{ S(t) : t \geq 0 \}$ be a family of nonexpansive mappings of C into itself such that $S(0) = I$, $S(t+s) = S(t)S(s)$ for all $t, s \in [0,\infty)$ and $S(t)x$ is continuous in $t \in [0,\infty)$ for each $x \in C$. Then, S is called a nonexpansive semigroup on C. The fixed point set F(S) of S is defined by

$$F(S) = \{ x \in C : S(t)x = x \text{ for all } t \in [0,\infty) \}.$$

The first nonlinear ergodic theorem for nonexpansive mappings was established by Baillon [1]: Let $C \subset H$, $T \in \text{Cont}(C)$ and

$F(T) \neq \phi$. Then, Cesàro means

$$S_n(x) = \frac{1}{n} \sum_{k=0}^{n-1} T^k x$$

converge weakly as $n \to \infty$ to a fixed point of T for each $x \in C$. A corresponding result for nonexpansive semigroups on C was given by Baillon [2] and Baillon-Brézis [3]. Nonlinear ergodic theorems for general commutative semigroups of nonexpansive mappings were given by Brézis-Browder [6] and Hirano-Takahashi [13].

In this talk, we prove a nonlinear ergodic theorem for non-commutative semigroups of nonexpansive mappings in a Hilbert space. By the same method, we give a necessary and sufficient condition for a non-commutative semigroup to have a fixed point. This is a generalization of Pazy's results [15], [17]. Secondly, we give a necessary and sufficient condition under which a variational inequality [22] defined on unbounded sets in a Banach space has a solution. Using this, we solve the complementarity problem [14], [23] and a fixed point theorem. We also establish a necessary and sufficient condition under which the minimax equality on unbounded sets holds. Finally, using the Ky Fan-Browder fixed point theorem [7], [10], we obtain Fan's existence theorem [9] concerning systems of convex inequalities in topological vector spaces. Then we present a generalization of the Hahn-Banach theorem and a separation theorem on a linear space.

§1. Nonlinear ergodic theorem.

Let S be an abstract semigroup and m(S) the Banach space of all bounded real valued functions on S with the supremum norm. For each $s \in S$ and $f \in m(S)$, we define elements f_s and f^s in m(S) given by $f_s(t) = f(st)$ and $f^s(t) = f(ts)$ for all $t \in S$. An element $\mu \in m(S)^*$ (the dual space of m(S)) is called a mean on S if $\|\mu\| = \mu(1) = 1$. A mean μ is called left [right] invariant if $\mu(f_s) = \mu(f)$ [$\mu(f^s) = \mu(f)$] for all $f \in m(S)$ and $s \in S$. An invariant mean is a left and right invariant mean. A semigroup which has a left [right] invariant mean is called left [right] amenable. A semigroup which has an invariant mean is called amenable. Day [8] proved that a commutative semigroup is amenable. We also know that $\mu \in m(S)^*$ is a mean on S if and only if

$$\inf\{ f(s) : s \in S \} \leqslant \mu(f) \leqslant \sup\{ f(s) : s \in S \}$$

for every $f \in m(S)$.

Now we prove a nonlinear ergodic theorem for noncommutative semigroups of nonexpansive mappings in a Hilbert space. The proof employs the methods of [16], [20] and [21].

THEOREM 1. Let C be a nonempty closed convex subset of a real Hilbert space H and S be an amenable semigroup of nonexpansive mappings t of C into itself. Suppose that

$$F(S) = \cap \{F(t) : t \in S\} \neq \phi.$$

Then, there exists a nonexpansive retraction P of C onto F(S) such that $Pt = tP = P$ for every $t \in S$ and $Px \in \overline{co}\{tx : t \in S\}$ for every $x \in C$, where $\overline{co}\,A$ is the closure of convex hull of A.

PROOF. Let μ be an invariant mean on S and $x \in C$. Then since $F(S) \neq \phi$, $\{tx : t \in S\}$ is bounded and hence, for each y in H, the real-valued function $t \to \langle tx, y\rangle$ is in m(S). Denote by $\mu_t\langle tx, y\rangle$ the value of μ at this function. By linearity of μ and of the inner product, this is linear in y; moreover, since

$$|\mu_t\langle tx, y\rangle| \leq \|\mu\| \cdot \sup_t |\langle tx, y\rangle| \leq (\sup_t \|tx\|) \cdot \|y\| ,$$

it is continuous in y, so by the Riesz theorem, there exists an $x_0 \in H$ such that

$$\mu_t\langle tx, y\rangle = \langle x_0, y\rangle$$

for every $y \in H$. Setting $Px = x_0$, we have

$$Px \in \overline{co}\{tx : t \in S\} .$$

In fact, if $Px \notin \overline{co}\{tx : t \in S\}$, then by the separation theorem there exists a $y_0 \in H$ such that

$$\langle Px, y_0\rangle < \inf\{\langle z, y_0\rangle : z \in \overline{co}\{tx : t \in S\}\} .$$

So, we have

$$\begin{aligned}\inf_t \langle tx, y_0\rangle \leq \mu_t\langle tx, y_0\rangle &= \langle Px, y_0\rangle \\ &< \inf\{\langle z, y_0\rangle : z \in \overline{co}\{tx : t \in S\}\}\end{aligned}$$

$$\leq \inf_t \langle tx, y_0 \rangle$$

This is a contradiction. Let $s \in S$. Then we have

$$0 \leq \|tx - x_0\|^2 - \|stx - sx_0\|^2$$

$$\leq \|tx - sx_0\|^2 + 2\langle tx - sx_0, sx_0 - x_0\rangle$$

$$+ \|sx_0 - x_0\|^2 - \|stx - sx_0\|^2$$

and hence

$$0 \leq \mu_t(\|tx - sx_0\|^2 + 2\langle tx - sx_0, sx_0 - x_0\rangle$$

$$+ \|sx_0 - x_0\|^2 - \|stx - sx_0\|^2)$$

$$= \mu_t\|tx - sx_0\|^2 + 2\langle x_0 - sx_0, sx_0 - x_0\rangle$$

$$+ \|sx_0 - x_0\|^2 - \mu_t\|tx - sx_0\|^2$$

$$= 2\langle x_0 - sx_0, sx_0 - x_0\rangle + \|sx_0 - x_0\|^2$$

$$= -\|x_0 - sx_0\|^2 .$$

This implies $sx_0 = x_0$ for every $s \in S$ and hence we have $sPx = Px$ for every $s \in S$. From

$$\langle Psx, y\rangle = \mu_t\langle tsx, y\rangle = \mu_t\langle tx, y\rangle = \langle Px, y\rangle$$

and

$$\langle P^2x, y\rangle = \mu_t\langle tPx, y\rangle = \mu_t\langle Px, y\rangle = \langle Px, y\rangle ,$$

it follows that $Ps = P$ for every $s \in S$ and $P^2 = P$. At last, we prove that P is nonexpansive. In fact, we have

$$\|Px - Py\|^2 = \langle Px - Py, Px - Py\rangle = \mu_t\langle tx - ty, Px - Py\rangle$$

$$\leqslant (\sup_t \|tx - ty\|)\cdot\|Px - Py\|$$

$$\leqslant \|x - y\|\cdot\|Px - Py\|$$

for every $x, y \varepsilon C$.

As a direct consequence, we have

COROLLARY 1. Let C be a nonempty closed convex subset of a real Hilbert space H and S be a commutative semigroup of nonexpansive mappings t of C into itself. Suppose that $F(S) \neq \phi$. Then there exists a nonexpansive retraction P of C onto F(S) such that $Pt = tP = P$ for every $t \varepsilon S$ and $Px \varepsilon \overline{co}\{ tx : t \varepsilon S\}$ for every $x \varepsilon C$.

By the method of Theorem 1, we can prove the following

THEOREM 2. Let C be a nonempty closed convex subset of a real Hilbert space H and S be a left amenable semigroup of nonexpansive mappings t of C into itself. Then, $F(S) \neq \phi$ if and only if there exists an $x_0 \varepsilon C$ such that $\{ tx_0 : t \varepsilon S \}$ is bounded.

As direct consequences, we obtain Pazy's results [15] and [17].

COROLLARY 2. Let C be a nonempty closed convex subset of a real Hilbert space H and T be a nonexpansive mapping of C into itself. Then, $F(T) \neq \phi$ if and only if there exists an element

$x_0 \in C$ such that the sequence $\{ T^n x_0 : n = 1, 2, \ldots \}$ is bounded.

COROLLARY 3. Let C be a nonempty closed convex subset of a real Hilbert space H and $S = \{ S(t) : t \geqslant 0\}$ be a nonexpansive semigroup on C. Then, $F(S) \neq \phi$ if and only if there exists an element $x_0 \in C$ such that $\{ S(t)x_0 : t \geqslant 0 \}$ is bounded.

§2. Variational inequalities.

Let E be a real reflexive Banach space and C be a closed convex subset of E. A mapping $T: C \to E^*$ is said to be monotone if $(Tx-Ty, x-y) \geqslant 0$ for all $x, y \in C$, and hemicontinuous on C if for any $u, v \in C$, the mapping $t \to T(tv+(1-t)u)$ of $[0,1]$ to E^* is continuous when E^* is endowed with the weak* topology. Also T is said to be coercive on C if for some $u \in C$,

$$\lim_{\substack{\|x\| \to \infty \\ x \in C}} (Tx, x-u)/\|x\| = +\infty.$$

A mapping $F: C \to E$ said to be nonexpansive if for any $x, y \in C$, $\|Fx - Fy\| \leqslant \|x - y\|$. We note that if E is a real Hilbert space and $F: C \to E$ is nonexpansive, then $I-F$ is a monotone mapping of C into E. Let H, K be nonempty closed subsets of the Banach space E, then we denote by $\partial_H K$ the set of $z \in K$ such that $U(z) \cap (H-K) \neq \phi$ for every neighborhood $U(z)$ of z and by $i_H K$ the set of $z \in K$ such that $U(z) \cap (H-K) = \phi$ for some neighborhood $U(z)$ of z.

THEOREM 3. Let C be a nonempty closed convex subset of a reflexive Banach space E and T be a monotone and hemicontinuous mapping of C into E^*. Then the following conditions are equivalent.

(1) There exists $x_0 \varepsilon C$ such that $(Tx_0, y-x_0) \geqslant 0$ for all $y \varepsilon C$;

(2) there exists a bounded closed convex subset K of C such that for each $z \varepsilon \partial_C K$, there exists $y \varepsilon i_C K$ which satisfies $(Tz, y-z) \leqslant 0$.

PROOF. First we show that (1) implies (2). Let x_0 be an element of C such that $(Tx_0, y-x_0) \geqslant 0$ for all $y \varepsilon C$. Set $d = \|x_0-y_0\|$ where $y_0 \varepsilon C$ and $y_0 \neq x_0$, and $K = \{x \varepsilon C: \|x-x_0\| \leqslant d\}$. Then we have $x_0 \varepsilon i_C K$. Let $z \varepsilon \partial_C K$. By the monotonicity of T, it follows that $(Tz, z-x_0) \geqslant (Tx_0, z-x_0) \geqslant 0$. Therefore, we have $(Tz, x_0-z) \leqslant 0$. Next we show that (2) implies (1). Let K be a bounded closed convex subset of C which satisfies the condition (2). Since K is weakly compact convex, there exists $x_0 \varepsilon K$ such that $(Tx_0, x-x_0) \geqslant 0$ for all $x \varepsilon K$ (cf. [4],[5]). If $x_0 \varepsilon i_C K$, then for each $y \varepsilon C$ we can choose $\lambda > 0$ so small that $x = \lambda y+(1-\lambda)x_0$ lies in K. Then $(Tx_0, \lambda y+(1-\lambda)x_0-x_0) \geqslant 0$ and hence $\lambda(Tx_0, y-x_0) \geqslant 0$. Cancelling λ, we have $(Tx_0, y-x_0) \geqslant 0$. If $x_0 \varepsilon \partial_C K$, then, by the hypothesis, there exists $z_0 \varepsilon i_C K$ such that $(Tx_0, z_0-x_0) \leqslant 0$. Since $(Tx_0, x-x_0) \geqslant 0$ for all $x \varepsilon K$, we have $(Tx_0, x-z_0) \geqslant 0$ for all $x \varepsilon C$. Since $z_0 \varepsilon i_C K$, for each

$y \in C$, there exists $\lambda > 0$ such that $x = \lambda y+(1-\lambda)z_0$ lies in K. Then $\lambda(Tx_0, y-z_0) \geqslant 0$. Cancelling λ, we have $(Tx_0, y-z_0) \geqslant 0$. Then since $(Tx_0, z_0-x_0) \geqslant 0$, we obtain $(Tx_0, y-x_0) \geqslant 0$.

The following corollaries are direct consequences of Theorem 3.

CORORALLY 4. Let C be a nonempty closed convex subset of a reflexive Banach space E and T be a monotone hemicontinuous mapping of C into E*. If T is coercive on C, then there exists $x_0 \in C$ such that $(Tx_0, y-x_0) \geqslant 0$ for all $y \in C$.

PROOF. It is sufficient to show that the coercivity condition implies the condition (2) of Theorem 3. By the definition of coercivity, there exist $y \in C$ and positive numbers c, k such that $\|y\| < c$ and $(Tx, x-y) \geqslant k\|x\|$ for $x \in C$ with $\|x\| \geqslant c$. If we set $K = \{x \in C: \|x\| \leqslant c\}$, then it is obvious that K satisfies the condition (2) of Theorem 3.

Corollary 4 has a very interesting interpretation when C is a closed convex cone.

COROLLARY 5. Let C be a nonempty closed convex cone in a reflexive Banach space E and T be a monotone hemicontinuous mapping of C into E*. If T is coercive, then there exists an $x_0 \in C$ such that $-Tx_0 \in C^*$ and $(Tx_0, x_0) = 0$ where $C^* = \{u \in E^*: (u, x) \leqslant 0$ for all $x \in C\}$.

PROOF. By Corollary 4, there exists $x_0 \in C$ such that $(Tx_0, y-x_0) \geqslant 0$ for all $y \in C$. It follows from Lemma 3.1 of [14] that $-Tx_0 \in C^*$ and $(Tx_0, x_0) = 0$.

COROLLARY 6. Let C be a nonempty closed convex subset of a Hilbert space H such that $0 \in C$ and T be a nonexpansive mapping of C into H. If there exists a bounded closed convex set $K \subset C$ such that $0 \in i_C K$ and $\|Tz\| \leqslant \|z\|$ for all $z \in \partial_C K$, then there exists an $x_0 \in C$ such that

$$\|x_0 - Tx_0\| = \min\{\|y - Tx_0\| : y \in C\}.$$

Particularly, if T mapps C into itself, there exists $x_0 \in C$ such that $Tx_0 = x_0$.

PROOF. It is obvious that the mapping I-T of C into H is monotone and hemicontinuous. Since $\|Tz\| \leqslant \|z\|$ for all $z \in \partial_C K$, we have $(z-Tz, -z) \leqslant 0$ for all $z \in \partial_C K$. Since $0 \in i_C K$, K satisfies the condition (2) of Theorem 3. Therefore there exists $x_0 \in C$ such that $(x_0 - Tx_0, y - x_0) \geqslant 0$ for all $y \in C$. Hence we obtain $\|x_0 - Tx_0\| \leqslant \|y - Tx_0\|$ for all $y \in C$. Particularly, if T mapps C into itself, we have $\min\{\|y - Tx_0\| : y \in C\} = 0$ and hence $Tx_0 = x_0$.

§3. Minimax theorem.

Next we consider a minimax theorem and establish a necessary and sufficient condition under which the minimax equality on unbounded sets holds.

THEOREM 4. Let X, Y be reflexive Banach spaces, and let $A \subset X$, $B \subset Y$ be nonempty closed convex sets. If F is a function on $A \times B$ such that for each $y \in B$, $F(\cdot, y)$ is an upper semi-continuous concave function on A and for each $x \in A$, $F(x, \cdot)$

is a lower semicontinuous convex function on B, then the following conditions are equivalent.

(1) $$\max_{x\in A}\min_{y\in B} F(x,y) = \min_{y\in B}\max_{x\in A} F(x,y);$$

(2) there exist bounded closed convex sets $K \subset A$ and $L \subset B$ such that for each $(x,y) \in (\partial_A K \times L) \cup (K \times \partial_B L)$, there exists a $(u,v) \in i_A K \times i_B L$ which satisfies $F(u,y) \geqslant F(x,v)$.

PROOF. First we show that (1) implies (2). If (1) holds, then there exists $(x_0,y_0) \in A \times B$ such that $F(x_0,y) \geqslant F(x_0,y_0) \geqslant F(x,y_0)$ for all $(x,y) \in A \times B$. Let $K = \{x \in A: \|x_0-x\| \leqslant \|x_0-a\|\}$ and $L = \{y \in B: \|y_0-y\| \leqslant \|y_0-b\|\}$, where $a \in A$, $b \in B$, $x_0 \neq a$ and $y_0 \neq b$. Then we have $(x_0,y_0) \in i_A K \times i_B L$ and $F(x_0,y) \geqslant F(x_0,y_0) \geqslant F(x,y_0)$ for all $(x,y) \in (\partial_A K \times L) \cup (K \times \partial_B L)$. Next we show that (2) implies (1). Let K and L be bounded closed convex sets which satisfy the condition (2). Then, by Theorem 3.8 of [4], there exists $(x_0,y_0) \in K \times L$ such that $F(x,y_0) \leqslant F(x_0,y_0) \leqslant F(x_0,y)$ for all $(x,y) \in K \times L$. Let $(x_0,y_0) \in i_A K \times i_B L$. Then for each $x \in A$ we can choose $\lambda > 0$ so small that $\lambda x+(1-\lambda)x_0 \in K$. Since $F(\cdot,y)$ is concave, we have

$$F(x_0,y_0) \geqslant F(\lambda x+(1-\lambda)x_0,y_0) \geqslant \lambda F(x,y_0)+(1-\lambda)F(x_0,y_0)$$

and hence $F(x,y_0) \leqslant F(x_0,y_0)$. Also we obtain that $F(x_0,y_0) \leqslant F(x_0,y)$ for all $y \in B$. so, (1) holds. Let $(x_0,y_0) \in (\partial_A K \times L) \cup (K \times \partial_B L)$. Then by the condition (2) there exists $(u,v) \in i_A K \times i_B L$ such that $F(u,y_0) \geqslant F(x_0,v)$. Since $F(x,y_0) \leqslant F(x_0,y_0) \leqslant F(x_0,y)$ for all $(x,y) \in K \times L$, we have $F(u,y_0) = F(x_0,y_0) = F(x_0,v)$. For each $x \varepsilon A$, we take $\lambda > 0$ so small that $\lambda x+(1-\lambda)u \varepsilon K$. Then

$$F(x_0,y_0) \geqslant F(\lambda x+(1-\lambda)u,y_0) \geqslant \lambda F(x,y_0)+(1-\lambda)F(u,y_0)$$
$$= \lambda F(x,y_0)+(1-\lambda)F(x_0,y_0).$$

Hence we obtain that $F(x,y_0) \leqslant F(x_0,y_0)$. Also we obtain that $F(x_0,y_0) \leqslant F(x_0,y)$ for all $y \varepsilon B$. Theis completes the proof.

COROLLARY 7 (cf.[4]). Let X, Y, A, B and F satisfy the assumptions as in Theorem 4. If there exists $(x_0,y_0) \varepsilon A \times B$ such that

$$\lim_{\substack{\|x\|+\|y\|\to\infty \\ (x,y)\varepsilon A\times B}} \{F(x_0,y) - F(x,y_0\} = \infty,$$

then we have $\max_{x\varepsilon A} \min_{y\varepsilon B} F(x,y) = \min_{y\varepsilon B} \max_{x\varepsilon A} F(x,y)$.

PROOF. It is clear from the hypothesis that there exists $k > 0$ such that for every $(x,y) \varepsilon A \times B$ with $\|x\|+\|y\| \geqslant k$ we have $F(x_0,y) - F(x,y_0) > 0$. Let $K = \{x \varepsilon A: \|x_0-x\| \leqslant k\}$ and $L = \{y \varepsilon B: \|y_0-y\| \leqslant k\}$. Then for every $(x,y) \varepsilon (\partial_A K \times L) \cup (K \times \partial_B L)$, we obtain $F(x_0,y) > F(x,y_0)$. so, we obtain Corollary 7 from Theorem 4.

§4. Systems of convex inequalities.

Fan first proved the following lemma, and then Browder gave a different proof of it.

LEMMA 1(Ky Fan-Browder). Let X be a nonempty compact convex subset of a separated linear topological space and T be a multi-valued mapping on X such that for each $x \in X$, Tx is a nonempty convex subset of X and $T^{-1}y = \{x \in X : y \in Tx\}$ is open in X. Then there is an $x_0 \in X$ such that $x_0 \in Tx_0$.

Using this, we prove the following result obtained by Fan [9] which plays crucial roles to prove the main theorems.

LEMMA 2(Fan). Let X be a nonempty compact convex subset of a separated linear topological space and $\{f_\nu : \nu \in I\}$ be a family of lower semicontinuous convex functionals on X with values in $(-\infty, +\infty]$. If for any finite indices $\nu_1, \nu_2, \cdots, \nu_n$ and for any n nonnegative numbers $\lambda_1, \lambda_2, \cdots, \lambda_n$ with $\sum_{i=1}^{n} \lambda_i = 1$, there is a $y \in X$ such that

$$\sum_{i=1}^{n} \lambda_i f_{\nu_i}(y) \leqslant 0 ,$$

then there is an $x \in X$ such that

$$f_\nu(x) \leqslant 0 \quad \text{for every} \quad \nu \in I.$$

PROOF. Suppose that for each $x \in X$ there is a $\nu \in I$ such that $f_\nu(x) > 0$. Setting $G_\nu = \{x \in X : f_\nu(x) > 0\}$ for each $\nu \in I$, $\{G_\nu : \nu \in I\}$ is an open covering of X. Since X is compact, there is a finite subcovering $\{G_{\nu_1}, G_{\nu_2}, \cdots, G_{\nu_n}\}$ of $\{G_\nu : \nu \in I\}$. Let $g_1, g_2, \cdots, g_n$ be a partition of unity corresponding to $\{G_{\nu_1}, G_{\nu_2}, \cdots, G_{\nu_n}\}$, i.e., each g_i is a continuous mapping of X into $[0,1]$ which vanishes outside of G_{ν_i}, while

$$\sum_{i=1}^{n} g_i(x) = 1$$

for every $x \in X$. Then put

$$D(x,y) = \sum_{i=1}^{n} g_i(x) f_{\nu_i}(y), \quad (x,y) \in X \times X,$$

and

$$d(x) = D(x,x), \quad x \in X.$$

Since d is lower semicontinuous on X by [22, Lemma 3], d takes its minimum m. Hence we have

$$d(x) \geqslant m > 0, \quad x \in X.$$

Now we define a multi-valued mapping T on X by

$$Tx = \{y \in X : D(x,y) < m\}, \quad x \in X.$$

Then Tx is nonempty and convex by hypothesis and $T^{-1}y = \{x \in X : D(x,y) < m\}$ is open. Therefore there is an $x_0 \in X$ such that $d(x_0) < m$ by Lemma 1. This is a contradiction. This completes the proof.

A functional p defined on a linear space E into the real field R is said to be sublinear if $p(x+y) \leqslant p(x) + p(y)$ for all $x,y \in E$ and $p(\lambda x) = \lambda p(x)$ for all $\lambda \geqslant 0$ and all $x \in E$. If E is a linear space, we denote by E* the dual space of E which is the set of all linear functional from E into the real field. In our proof of Theorem 5, we shall need, not only Lemma 2, but also Lemma3 below, which is a special case of the Hahn-Banach theorem.

LEMMA 3. If p is sublinear on a linear space E and $x_0 \in E$, then there is an $f \in E^*$ such that $f(x) \leqslant p(x)$ for all $x \in E$ and $f(x_0) = p(x_0)$.

PROOF. Let F be the product space R^E, then F is a linear topological space. If we put

$$X_0 = \prod_{x \in E} [-p(-x), p(x)],$$

then X_0 is a compact convex subset of F. We consider a sequence $\{f_n\}$ in X_0 defined by

$$f_n(x) = p(x + nx_0) - p(nx_0), \quad x \in E.$$

Since X_0 is compact, there is a subnet $\{f_{n_\alpha}\}$ of $\{f_n\}$ which converges to $f_0 \in X_0$. It is easily seen that

$$-p(y-y) \leqslant f_0(x) - f_0(y) \leqslant p(x-y)$$

for all $x, y \in E$. If $\lambda \in R$, then there is α_0 such that $\lambda + n_\alpha > 0$ for all $\alpha \geqslant \alpha_0$. Hence

$$\begin{aligned} f_0(\lambda x_0) &= \lim_\alpha (p(\lambda x_0 + n_\alpha x_0) - p(n_\alpha x_0)) \\ &= \lim_\alpha ((\lambda + n_\alpha)p(x_0) - n_\alpha p(x_0)) \\ &= \lambda p(x_0). \end{aligned}$$

If we put

$$X_1 = \{f \in X_0 : -p(y-x) \leqslant f(x) - f(y) \leqslant p(x-y), \; x, y \in E \text{ and } f(\lambda x_0) = \lambda p(x_0), \; \lambda \in R\},$$

then X_1 is nonempty. It is easily seen that X_1 is compact and convex. We consider a commuting family $\{T_\mu : \mu \varepsilon R\}$ of continuous affine mappings of X_1 into itself defined by

$$(T_\mu f)x = f(x + \mu x_0) - f(\mu x_0), \quad f \varepsilon X_1, \quad x \varepsilon E.$$

By the Markov-Kakutani fixed point theorem, there is an $f_1 \varepsilon X_1$ such that

$$f_1(x + \mu x_0) = f_1(x) + f_1(\mu x_0),$$

for every $x \varepsilon E$ and $\mu \varepsilon R$. Hence if we put

$$X_2 = \{f \varepsilon X_1 : f(x + \mu x_0) = f(x) + f(\mu x_0) \text{ for every } x \varepsilon E \text{ and } \mu \varepsilon R\},$$

then X_2 is nonempty. Furthermore X_2 is compact and convex. We consider a commuting family $\{T_y : y \varepsilon E\}$ of continuous affine mappings of X_2 into itself defined by

$$(T_y f)x = f(x + y) - f(y), \quad f \varepsilon X_2, \quad x \varepsilon E.$$

By the Markov-Kakutani fixed point theorem again, there is an $f_2 \varepsilon X_2$ such that

$$f_2(x + y) = f_2(x) + f_2(y), \quad x, y \varepsilon E.$$

Hence if we put

$$X_3 = \{f \varepsilon X_1 : f(x + y) = f(x) + f(y), \quad x,y \varepsilon E\},$$

then X_3 is nonempty compact and convex. We consider a commuting family $\{S_\mu : \mu > 0\}$ of continuous affine mappings of X_3 into itself defined by

$$(S_\mu f)x = \frac{f(\mu x)}{\mu}, \quad f \in X_3, \quad x \in E.$$

By the Markov-Kakutani fixed point theorem, there is an $f_3 \in X_3$ such that

$$f_3(\mu x) = \mu f_3(x), \quad \mu > 0.$$

This implies that f_3 is linear, so the proof is complete.

THEOREM 5(Hirano-Komiya-Takahashi). Let p be a sublinear functional on a linear space E, let C be a nonempty convex subset of E, and let f be a concave functional on C such that $f(x) \leq p(x)$ for all $x \in C$, then there is an $f_0 \in E^*$ such that $f(x) \leq f_0(x)$ for all $x \in C$ and $f_0(y) \leq p(y)$ for all $y \in E$.

PROOF. Let F be the linear topological space R^E with the product topology and let X_0 be the compact convex subset

$$\prod_{x \in E} [-p(-x), p(x)]$$

of F. Let $B = \{g \in E^* : g(x) \leq p(x)$ for all $x \in E\}$, then B is nonempty by Lemma 3. Since X_0 is compact, B is compact convex. For each $x \in C$, we define a real valued functional G_x on B by

$$G_x(g) = f(x) - g(x), \quad g \in B.$$

By Lemma 3, for any $x \in C$, there is a $g \in E^*$ such that $G_x(g\ ) \leq 0$. If $x_1, x_2, \cdots, x_n \in C$ and $\lambda_1, \lambda_2, \cdots, \lambda_n \geq 0$ with $\Sigma \lambda_i = 1$, then

$$\sum_{i=1}^{n} \lambda_i G_{x_i}(g) = \sum_{i=1}^{n} \lambda_i (f(x_i) - g(x_i))$$
$$\leqslant f\left(\sum_{i=1}^{n} \lambda_i x_i\right) - g\left(\sum_{i=1}^{n} \lambda_i x_i\right)$$
$$\leqslant G_z(g)$$

for all $g \in B$, where $z = \Sigma\lambda_i x_i \in C$. Hence, by Lemma 2, there is an $f_0 \in B$ such that $G_x(f_0) \leqslant 0$ for all $x \in C$, that is, $f(x) \leqslant f_0(x)$ for all $x \in C$ and $f_0(y) \leqslant p(y)$ for all $y \in E$.

COROLLARY 8(The Hahn-Banach theorem). Let p be a sublinear functional on a linear space E, let L be a linear subspace of E, and let f be an element of L* such that $f(x) \leqslant p(x)$ for all $x \in L$, then there is an $f_0 \in E^*$ such that $f_0(x) = f(x)$ for all $x \in L$ and $f_0(y) \leqslant p(y)$ for all $y \in E$.

PROOF. By Theorem 5 there is an $f_0 \in E^*$ such that $f_0(x) \geqslant f(x)$ for all $x \in L$. Since L is a linear subspace of E*, we have $f_0(x) = f(x)$ for all $x \in L$.

Let p be a sublinear functional on E. For two nonempty subset A and B of E, we consider a number p(A,B) given by $\inf\{ p(x - y) : x \in A, y \in B \}$.

THEOREM 6(Hirano-Komiya-Takahashi). Let p be a sublinear functional on a linear space E. If C and D are nonempty convex subsets of E such that $p(C,D) > -\infty$, then there is an $f \in E^*$ such that

$$\inf\{ f(x) : x \in C \} = p(C,D) + \sup\{ f(y) : y \in D \}$$

and $f(x) \leqslant p(x)$ for all $x \in E$.

PROOF. We again consider the compact convex subset $B = \{ g \in E^* : g(x) \leqslant p(x) \text{ for all } x \in E \}$ of the linear topological space F. Let $p_0 = p(C,D)$. For each $x \in C$, we define a functional G_x on B with values in $(-\infty, +\infty]$ by

$$G_x(g) = \sup \{ g(y - x) : y \in D \} + p_0, \quad g \in B.$$

Then G_x is lower semicontinuous and convex. Also we have that if $x_1, x_2, \ldots, x_n \in C$ and $\lambda_1, \lambda_2, \ldots, \lambda_n \geqslant 0$ with $\Sigma\lambda_i = 1$, then

$$z = \sum_{i=1}^{n} \lambda_i x_i \in C \quad \text{and} \quad \sum_{i=1}^{n} \lambda_i G_{x_i} = G_z .$$

So, if we can show that for each $x \in C$, there is a $g \in B$ with $G_x(g) \leqslant 0$, then we obtain, by Lemma 2, that there is an $f \in B$ with $G_x(f) \leqslant 0$ for all $x \in C$. Hence we have

$$\sup\{ f(y - x) : y \in D \} + p_0 \leqslant 0$$

for all $x \in C$; that is,

$$\sup\{ f(y) : y \in D \} + p_0 \leqslant \inf\{ f(x) : x \in C \}.$$

Then

$$\begin{aligned} p_0 &\leqslant \inf\{ f(x) : x \in C \} - \sup\{ f(x) : y \in D \} \\ &\leqslant \inf\{ f(x - y) : x \in C, y \in D \} \\ &\leqslant \inf\{ p(x - y) : x \in C, y \in D \} \\ &= p_0 . \end{aligned}$$

Hence we have that $f(x) \leqslant p(x)$ for all $x \varepsilon E$ and

$$\inf\{ f(x) : x \varepsilon C \} = p(C,D) + \sup\{ f(y) : y \varepsilon D \} .$$

Now to complete the proof, we need only to show that for each $x \varepsilon C$ there is a $g \varepsilon B$ with $G_x(g) \leqslant 0$. Let $x \varepsilon C$. Then for each $y \varepsilon D$, we define a continuous affine fuctional H_y on B by

$$H_y(g) = g(y - x) + p_0, \qquad g \varepsilon B.$$

By Lemma 3, for each $y \varepsilon D$, there is a $g \varepsilon B$ such that $g(x - y) = p(x - y)$. Hence we have

$$\begin{aligned} H_y(g) &= -g(x - y) + p_0 \\ &= -p(x - y) + p_0 \\ &\leqslant 0 . \end{aligned}$$

Hence, by Lemma 2, there is a $g_0 \varepsilon B$ such that $H_y(g_0) \leqslant 0$ for all $y \varepsilon D$. Therefore we have

$$G_x(g_0) = \sup\{ H_y(g_0) : y \varepsilon D \} \leqslant 0.$$

Let N be a normed linear space and N' the dual space of N, that is, the set of all continuous linear functional from N into R. For two subsets A and B of N, the distance d(A,B) between A and B is given by $\inf\{ \|x - y\| : x \varepsilon A, y \varepsilon B \}$.

COROLLARY 9. If C and D are nonempty convex subsets of a normed linear space N such that $d(C,D) > 0$, then there is an $f \varepsilon N'$ such that $\|f\| = 1$ and

$$\inf\{ f(x) : x \in C \} = d(C,D) + \sup\{ f(y) : y \in D \} .$$

PROOF. By Theorem 6, there is an $f \in N'$ such that $f(x) \leqslant \|x\|$ for all $x \in N$ and

$$\inf\{ f(x) : x \in C \} = d(C,D) + \sup\{ f(y) : y \in D \}.$$

Then

$$\begin{aligned} d(C,D) &= \inf\{ f(x) : x \in C \} - \sup\{ f(y) : y \in D \} \\ &\leqslant \inf\{ f(x - y) : x \in C, y \in D \} \\ &\leqslant \inf\{ \|f\|\cdot\|x - y\| : x \in C, y \in D \} \\ &= \|f\| d(C,D) . \end{aligned}$$

Since $d(C,D) > 0$, we have $\|f\| \geqslant 1$ and hence $\|f\| = 1$.

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