Stefan problems with the unilateral boundary condition on the fixed boundary

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§0. Introduction.

In this note we consider the following one-dimensional two-phase Stefan problem with the unilateral boundary condition on the fixed boundary: Given the data, $\phi$ and $\ell$, find two functions $s = s(t)$ and $u = u(x,t)$ such that the pair $(s,u)$ satisfies

\begin{align*}
(0.1) \quad s(0) &= \ell, \quad 0 < s(t) < 1 \quad (0 \leq t \leq T), \\
(0.2) \quad u_{xx} - c_0 u_t &= 0 \quad (0 < x < s(t), \, 0 < t \leq T), \\
(0.3) \quad u_{xx} - c_1 u_t &= 0 \quad (s(t) < x < 1, \, 0 < t \leq T), \\
(a) \quad u_x(0,t) &\in \gamma_0(u(0,t)) \quad (0 < t \leq T), \\
(b) \quad -u_x(1,t) &\in \gamma_1(u(1,t)) \quad (0 < t \leq T), \\
(0.4) \quad u(x,0) &= \phi(x) \quad (0 \leq x \leq \ell), \\
(0.5) \quad u(x,0) &= \phi(x) \quad (\ell \leq x \leq 1), \\
(0.6) \quad u(s(t),t) &= 0 \quad (0 \leq t \leq T),
\end{align*}

and the free boundary condition

\begin{equation}
(0.7) \quad b \cdot \mathbf{s}(t) = u_x(s(t)+0,t) - u_x(s(t)-0,t) \quad (0 < t \leq T)
\end{equation}

The quantities $c_0$, $c_1$ and $b$ are positive physical parameters of the problem. $T$ is a positive constant to be discussed later, and the function $\phi$ and the value $\ell$, $0 < \ell < 1$, are the initial data for $(S)$. 

Each \( \gamma_i, i = 0, 1, \) is a maximal monotone graph in \( \mathbb{R}^2 \) with \( \gamma_i(H_1) \ni 0, \) where \( H_1 \) is a constant satisfying \((-1)^i H_1 \geq 0\). We put the assumptions of signs for \( H_0 \) and \( H_1 \) from the physical reasoning. (0.4) are the unilateral boundary conditions and (0.7) is the so-called Stefan's condition.

The system (0.1)-(0.7) is a simple model of a heat-conduction system consisting of two phases (e.g., liquid and solid) of the same substance which are in perfect thermal contact at an interface. \( u(x,t) \) represents the temperature distribution in the system, and the curve \( s(t) \) represents the position of the interface which varies with time \( t \) as solid melts or liquid freezes. The unilateral boundary conditions (0.4) model several physical situations, including the temperature control through the boundary [7, Ch. 1] and the heat flow subject to the nonlinear cooling by the radiation on the boundary [9, Ch. 7]. The boundary conditions at the interface ((0.6), (0.7)) reflect respectively the facts that the temperature at the interface must be equal to the melting temperature (taken to be zero) and that the rate of melting is proportional to the rate of absorption of the heat energy at the interface. In formulating (0.7), we have assumed, without loss of generality, that the thermal conductivity in both phases is 1.

The problems of this type with linear boundary conditions on the fixed boundary have been considered by many authors (Rubinstein [15], Kamenomostkaja [10], Friedman [8, 9], Brézis [1], Cannon-Primicerio [4, 5], Cannon-Henry-Kotlow [6], Nogu [13], etc.). On the other hand Bénilan [22] has treated this type's Stefan problem of \( n \)-dimensional case by using the theory of nonlinear contraction semigroups in Banach space \( L^1 \). He got a integral solution. However we do not know the differentiability of the Benilan's integral solution. One-phase problem of this type was recently studied by Yotsutani [20].
Our purpose is to study the global existence and uniqueness of the classical solution.

In §1 we state the main results. In §2 we explain the difference scheme.

§1. Statements of main results.

We suppose that $0 < \xi < 1$, and $\phi(x)$ satisfies the following condition (A) for simplicity.

(A) \( \phi(x) \) is a Lipshitz continuous function on $[0,1]$ satisfying
\[
\phi(0) \in D(\gamma_0), \quad \phi(1) \in D(\gamma_1) \quad \text{and} \quad (\xi - x) \phi(x) \geq 0 \quad (0 \leq x \leq 1).
\]

DEFINITION. The pair $(s, u)$ is a solution of $(S)$ if
\[
s \in C([0,T]) \cap C^\infty([0,T]), \quad u \in C(\overline{D}_T), \quad u|_{D_i,T} \in C^\infty(D_i,T) \quad (i = 0, 1),
\]
\[
\int_0^T \frac{x}{x^2} \, dx \, dt < +\infty, \quad \text{and} \quad (1)-(7) \quad \text{hold, where} \quad D_T = D_{0,T} \cup D_{1,T},
\]
\[
D_i,T = \left\{(x,t) \mid (-1)^i x < (-1)^i x \leq (-1)^i s(t) \right\} \quad (i = 0, 1).
\]

THEOREM 1. Let $\phi(x)$ satisfy (A). Then there exist a critical time $T^*$ $(0 < T^* \leq +\infty)$ and a pair of functions $(s, u)$ defined on $[0,T^*[ ]$ such that $(s, u)$ is a solution of $(S)$ on $[0,T]$ for any $T$ $(0 < T < T^*)$.

Further if $T^* < +\infty$, then the following relations hold,

\[
\lim_{t \to T^*} s(t) = 0 \quad \text{or} \quad 1, \quad 0 < s(t) < 1 \quad (0 < t < T^*).
\]

REMARK. In particular, if $\gamma_i$ $(i = 0, 1)$ is a single valued maximal monotone, then $u_x|_{\overline{D}_i,T} \in C(\overline{D}_i,T)$ $(\overline{D}_i,T = D_i,T \cup \{x=i\})$ for any $T$,

\[
(-1)^i u_x(i,t) = \gamma_i(u(i,t)) \quad (0 < t < T^*).
\]
THEOREM 2. Under the same assumption of Theorem 1, the critical
time $T^*$ and the pair $(s, u)$ on $[0, T^*]$ are uniquely determined.

REMARK. We can loosen the assumption (A) a little.

We construct a solution by using a primitive implicit difference
scheme with only a device of capturing a free boundary explicitly
through step-by-step process in time (see §2). Uniqueness is based
upon the maximum principle, its strong form, a parabolic version of
Hopf's lemma and the comparison theorems for the unilateral problem
associated with the heat equation. The proofs of our theorems is given
in [21].

§2. Difference scheme

Let $\lambda$ be a rational number with $0 < \lambda < 1$.

We use a net of rectangular meshes with uniform space width $h$ and
variable time step $\{k_n\}$ $(n = 1, 2, 3, \cdots)$. Time steps $\{k_n\}$ are
assumed to be unknown and they are determined by the rule that $h/k_n$
gives gradient of a desired free boundary at each time $t = t_n$, so
that the free boundary crosses each mesh lines just at each correspon-
ding mesh points. Let us introduce discrete coordinates

$$x_j = jh \quad (j = 0, 1, 2, \cdots, M),$$
$$t_n = \sum_{p=1}^{n} k_p \quad (n = 1, 2, 3, \cdots), \quad t_0 = 0,$$

where $h$ varies in such a way that $\lambda/h = J_0$ and $1/h = M$ are
integers, and net functions $s_n$ and $u^n_j$ which correspond to $s(t_n)$
and $u(x_j, t_n)$ respectively. By our rule we can put
(2.1) \( s_n = J_n \cdot h \quad (J_n : \text{integer}, \ n = 0, 1, 2, \ldots) \).

Further we introduce usual divided differences:

(2.2) \( u_{j+1}^n = \frac{1}{h} (u_j^n - u_{j+1}^n), \quad u_j^n = \frac{1}{h} (u_j^n - u_{j-1}^n), \)

(2.3) \( u_{j+1}^n = \frac{1}{h^2} (u_j^n + u_{j+1}^n - 2u_j^n), \quad u_j^n = \frac{1}{k_n} (u_j^n - u_{j-1}^n), \) etc.

In our scheme the heat equation are replaced by pure implicit difference equations,

(2.4) \( u_{j+1}^n - c_0 u_j^n = 0 \quad (1 \leq j \leq J_n - 1), \)

(2.5) \( u_{j+1}^n - c_1 u_j^n = 0 \quad (J_n + 1 \leq j \leq M - 1). \)

The boundary and initial conditions are put in the following forms,

(a) \( u_0^n = \gamma_0 (u_0^n), \)

(b) \( -u_M^n = \gamma_1 (u_M^n), \)

(a) \( u_j^0 = \phi_j \equiv \phi(x_j) \quad (0 \leq j \leq J_0 - 1), \)

(b) \( u_j^0 = \phi_j \equiv \phi(x_j) \quad (J_0 + 1 \leq j \leq M), \)

(2.8) \( u_j^n = 0. \)

The Stefan's condition is replaced by an explicit formula

(2.9) \( b_h \Delta \frac{u_{j+1}^n}{k_n} = u_{j+1}^{n-1} - u_j^{n-1} \)

where + or - sign correspond to the cases of positive heat flow.
to the interface or negative one at \( t = t_n \) respectively.

Our algorithm is the following. \( \beta \): a positive constant.

1° \( u_j^0 = \phi_j \quad (0 \leq j \leq J_0 - 1), \quad u_{J_0}^0 = 0, \quad u_j^0 = \phi_j \quad (J_0 + 1 \leq j \leq M), \)

\[ s_0 = J_0 \cdot h = \ell. \]

For \( n = 1, 2, 3, \ldots \) successively,

2.1° if \( u_{J_{n-1}}^{n-1} - u_{J_{n-1}}^{n-1} > \frac{1}{\beta h^2} \), then we take \( J_n = J_{n-1} + 1 \)

and get \( k_n \) from

\[ b \frac{h}{k_n} = u_{J_{n-1}}^{n-1} - u_{J_{n-1}}^{n-1} \]

2.2° if \( u_{J_{n-1}}^{n-1} - u_{J_{n-1}}^{n-1} < -\frac{1}{\beta h^2} \), then we take \( J_n = J_{n-1} - 1 \)

and get \( k_n \) from

\[ -b \frac{h}{k_n} = u_{J_{n-1}}^{n-1} - u_{J_{n-1}}^{n-1} \]

2.3° if \( |u_{J_{n-1}}^{n-1} - u_{J_{n-1}}^{n-1}| \leq \frac{1}{\beta h^2} \), then we take

\[ J_n = J_{n-1} \quad \text{and} \quad k_n = bh^2 / \beta, \]

3° solve the system of difference equations (2.4) and (2.5)

for \( \{u_j^n\}_j \) under the boundary conditions (2.6) and (2.8)

with the initial condition \( \{u_j^{n-1}\}_j \),

4° if \( J_n < J_{n-1} > J_{n-2} \) or \( J_n > J_{n-1} < J_{n-2} \), then \( J_n \) and
k_n are revised as \( J_n = J_{n-1} \) and \( k_n = \frac{1}{Z} \sqrt{b/\beta} \), and then

return again to the step \( 3^0 \) (as \( n = 1 \), we consider \( J_{-1} = J_0 \)).

5° if \( J_n = 1 \) or \( M - 1 \), then stop computing.

**REMARK 2.1.** Step \( 3^0 \) is well-defined by Lemma 4.1 of Yotsutani [20].

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