数理解析研究所講究録
428巻 1981年 60-69

Stefan problems with the unilateral boundary condition on the fixed boundary

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§0. Introduction.

In this note we consider the following one-dimensional two-phase Stefan problem with the unilateral boundary condition on the fixed boundary : Given the data, ϕ and ℓ, find two functions $s = s(t)$ and $u = u(x,t)$ such that the pair (s, u) satisfies

(S)

(0.1) $s(0) = \ell, \quad 0 < s(t) < 1 \quad (0 \leqq t \leqq T),$

(0.2) $u_{xx} - c_0 u_t = 0 \quad (0 < x < s(t),\ 0 < t \leqq T),$

(0.3) $u_{xx} - c_1 u_t = 0 \quad (s(t) < x < 1,\ 0 < t \leqq T),$

(0.4) (a) $u_x(0,t) \in \gamma_0(u(0,t)) \quad (0 < t \leqq T),$

(b) $-u_x(1,t) \in \gamma_1(u(1,t)) \quad (0 < t \leqq T),$

(0.5) (a) $u(x,0) = \phi(x) \quad (0 \leqq x \leqq \ell),$

(b) $u(x,0) = \phi(x) \quad (\ell \leqq x \leqq 1),$

(0.6) $u(s(t),t) = 0 \quad (0 \leqq t \leqq T),$

and the free boundary condition

(0.7) $b\,\dot{s}(t) = u_x(s(t)+0,t) - u_x(s(t)-0,t) \quad (0 < t \leqq T)$

The quantities c_0, c_1 and b are positive physical parameters of the problem. T is a positive constant to be discussed later, and the function ϕ and the value ℓ, $0 < \ell < 1$, are the initial data for (S).

Each γ_i, $i = 0, 1$, is a maximal monotone graph in $\mathbb{R}^2$ with $\gamma_i(H_i) \ni 0$, where H_i is a constant satisfying $(-1)^i H_i \geqq 0$. We put the assumptions of signs for H_0 and H_1 from the physical reasoning. (0.4) are the unilateral boundary conditions and (0.7) is the so-called Stefan's condition.

The system (0.1)-(0.7) is a simple model of a heat-conduction system consisting of two phases (e. g. liquid and solid) of the same substance which are in perfect thermal contact at an interface. $u(x,t)$ represents the temperature distribution in the system, and the curve $s(t)$ represents the position of the interface which varies with time t as solid melts or liquid freezes. The unilateral boundary conditions (0.4) model several physical situations, including the temperature control through the boundary [7, Ch. 1] and the heat flow subject to the nonlinear cooling by the radiation on the boundary [9, Ch. 7]. The boundary conditions at the interface ((0.6), (0.7)) reflect respectively the facts that the temperature at the interface must be equal to the melting temperature (taken to be zero) and that the rate of melting is propotional to the rate of absorption of the heat energy at the interface. In formulating (0.7), we have assumed, without loss of generality, that the thermal conductivety in both phases is 1.

The problems of this type with linear boundary conditions on the fixed boundary have been considered by many authors (Rubinstein [15], Kamenomostkaja [10], Friedman [8, 9], Brézis [1], Cannon-Primicerio [4, 5], Cannon-Henry-Kotlow [6], Nogi [13], etc.). On the other hand Bénilan [22] has treated this type's Stefan problem of n-dimensional case by using the theory of nonlinear contraction semigroups in Banach space L^1. He got a integral solution. However we do not know the differentiability of the Benilan's integral solution. One-phase problem of this type was recently studied by Yotsutani [20].

Our purpose is to study the global existence and uniqueness of the classical solution.

In §1 we state the main results. In §2 we explain the difference scheme.

§1. Statements of main results.

We suppose that $0 < \ell < 1$, and $\phi(x)$ satisfies the following condition (A) for simplicity.

(A) $\phi(x)$ is a Lipshitz continuous function on $[0,1]$ satisfying $\phi(0) \in D(\gamma_0)$, $\phi(1) \in D(\gamma_1)$ and $(\ell - x)\,\phi(x) \geqq 0 \quad (0 \leqq x \leqq 1)$.

DEFINETION. The pair (s, u) is a solution of (S) if $s \in C([0,T]) \cap C^\infty(]0,T])$, $u \in C(\overline{D}_T)$, $u_{|D_{i,T}} \in C^\infty(D_{i,T}) \quad (i = 0, 1)$, $\int_0^T\int_0^1 t\, u_{xx}{}^2\, dxdt < +\infty$, and (1)-(7) hold, where $D_T = D_{0,T} \cup D_{1,T}$, $D_{i,T} = \{(x,t)\ ;\ (-1)^i\, i < (-1)^i\, x \leqq (-1)^i\, s(t)\} \quad (i = 0, 1)$.

THEOREM 1. Let $\phi(x)$ satisfy (A). Then there exist a critical time T^* $(0 < T^* \leqq +\infty)$ and a pair of functions (s, u) defined on $[0,T^*[$ such that (s, u) is a solution of (S) on $[0,T]$ for any T $(0 < T < T^*)$.

Further if $T^* < +\infty$, then the following relations hold,

$$\lim_{t\to T^*} s(t) = 0 \ \text{ or } \ 1, \quad 0 < s(t) < 1 \quad (0 < t < T^*).$$

REMARK. In particular, if γ_i $(i = 0, 1)$ is a single valued maximal monotone, then $u_{x|\tilde{D}_{i,T}} \in C(\tilde{D}_{i,T})$ $(\tilde{D}_{i,T} = D_{i,T} \cup \{x=i\})$ for any T,

$$(-1)^i\, u_x(i,t) = \gamma_i(u(i,t)) \quad (0 < t < T^*).$$

THEOREM 2. Under the same assumption of Theorem 1, the critical time T^* and the pair (s, u) on $[0, T^*[$ are uniquely determined.

REMARK. We can loosen the assumption (A) a little.

We construct a solution by using a primitive implicit difference scheme with only a device of capturing a free boundary explicitly through step-by-step process in time (see §2). Uniqueness is based upon the maximum principle, its strong form, a parabolic version of Hoph's lemma and the comparison theorems for the unilateral problem associated with the heat equation. The proofs of our theorems is give in [21].

§2. Difference scheme

Let ℓ be a rational number with $0 < \ell < 1$.

We use a net of rectanglar meshes with uniform space width h an variable time step $\{k_n\}$ $(n = 1, 2. 3, \cdots)$. Time steps $\{k_n\}$ are assumed to be unknown and they are determined by the rule that h/k_n gives gradient of a desired free boundary at each time $t = t_n$, so that the free boundary crosses each mesh lines just at each correspon mesh points. Let us introduce discrete coordinates

$$x_j = jh \qquad (j = 0, 1, 2, \cdots\cdots, M),$$

$$t_n = \sum_{p=1}^{n} k_p \qquad (n = 1, 2, 3, \cdots\cdots), \qquad t_0 = 0,$$

where h varies in such a way that $\ell/h = J_0$ and $1/h = M$ are integers, and net functions s_n and u_j^n which correspond to $s(t_n)$ and $u(x_j, t_n)$ respectively. By our rule we can put

$$(2.1) \qquad s_n = J_n \cdot h \qquad (J_n : \text{interger}, \; n = 0,\, 1,\, 2,\, \cdots\cdots).$$

Further we introduce usual divided differences :

$$(2.2) \qquad u^n_{jx} = \frac{1}{h}(u^n_{j+1} - u^n_j), \qquad u^n_{j\bar{x}} = \frac{1}{h}(u^n_j - u^n_{j-1}),$$

$$(2.3) \qquad u^n_{jx\bar{x}} = \frac{1}{h^2}\,(u^n_{j+1} - 2u^n_j + u^n_{j-1}), \qquad u^n_{j\bar{t}} = \frac{1}{k_n}\,(u^n_j - u^{n-1}_j), \qquad \text{etc.}$$

In our scheme the heat equation are replaced by pure implicit difference equations,

$$(2.4) \qquad u^n_{jx\bar{x}} - c_0 u^n_{j\bar{t}} = 0 \qquad (1 \leqq j \leqq J_n - 1),$$

$$(2.5) \qquad u^n_{jx\bar{x}} - c_1 u^n_{j\bar{t}} = 0 \qquad (J_n + 1 \leqq j \leqq M - 1).$$

The boundary and initial conditions are put in the following forms,

$$(2.6) \qquad \begin{cases} \text{(a)} & u^n_{0x} \in \gamma_0(u^n_0), \\ \text{(b)} & -u^n_{M\bar{x}} \in \gamma_1(u^n_M), \end{cases}$$

$$(2.7) \qquad \begin{cases} \text{(a)} & u^0_j = \phi_j \equiv \phi(x_j) \qquad (0 \leqq j \leqq J_0 - 1), \\ \text{(b)} & u^0_j = \phi_j \equiv \phi(x_j) \qquad (J_0 + 1 \leqq j \leqq M), \end{cases}$$

$$(2.8) \qquad u^n_{J_n} = 0.$$

The Stefan's condition is replaced by an explicit formula

$$(2.9) \qquad \pm b\frac{h}{k_n} = u^{n-1}_{J_{n-1}x} - u^{n-1}_{J_{n-1}\bar{x}}$$

where + or − sign correspond to the cases of positive heat flow to the interface or negative one at $t = t_n$ respectively.

Our algorism is the following. β : a positive constant.

1° $u_j^0 = \phi_j$ $(0 \leqq j \leqq J_0 - 1)$, $u_{J_0}^0 = 0$, $u_j^0 = \phi_j$ $(J_0 + 1 \leqq j \leqq M)$,

$s_0 = J_0 \cdot h = \ell$.

For $n = 1, 2, 3, \cdots$ successively,

2.1° if $u_{J_{n-1}x}^{n-1} - u_{J_{n-1}\bar{x}}^{n-1} > \beta h^{\frac{1}{2}}$, then we take $J_n = J_{n-1} + 1$

and get k_n from

$$b\frac{h}{k_n} = u_{J_{n-1}x}^{n-1} - u_{J_{n-1}\bar{x}}^{n-1},$$

2.2° if $u_{J_{n-1}x}^{n-1} - u_{J_{n-1}\bar{x}}^{n-1} < -\beta h^{\frac{1}{2}}$, then we take $J_n = J_{n-1} - 1$

and get k_n from

$$-b\frac{h}{k_n} = u_{J_{n-1}x}^{n-1} - u_{J_{n-1}\bar{x}}^{n-1},$$

2.3° if $|u_{J_{n-1}x}^{n-1} - u_{J_{n-1}\bar{x}}^{n-1}| \leqq \beta h^{\frac{1}{2}}$, then we take

$J_n = J_{n-1}$ and $k_n = bh^{\frac{1}{2}} / \beta$,

3° solve the system of difference equations (2.4) and (2.5) for $\{u_j^n\}_j$ under the boundary conditions (2.6) and (2.8) with the initial condition $\{u_j^{n-1}\}_j$,

4° if $J_n < J_{n-1} > J_{n-2}$ or $J_n > J_{n-1} < J_{n-2}$, then J_n and

k_n are revised as $J_n = J_{n-1}$ and $k_n = bh^{\frac{1}{2}}/\beta$, and then return again to the step 3^0 (as $n = 1$, we conseder $J_{-1} = J_0$).

5^0 if $J_n = 1$ or $M - 1$, then stop computing.

REMARK 2.1. Step 3^0 is well-defined by Lemma 4.1 of Yotsutani [20].

Acknowledgement. The author would like to express his hearty thanks to Prof. H. Brézis for his valuable advices.

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