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Kyoto University
Semigroup theory for functional differential
equations with infinite delay; a representation
of infinitesimal generators

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1. Semigroups and infinitesimal generators. Suppose $\mathcal{B}$
   $= \mathcal{B}(I, C^n)$ is a linear space of some functions $\phi$ mapping
an interval $I$ into $n$-dimensional complex linear space $C^n$,
where $I = [-r, 0]$, $0 \leq r < \infty$, or $I = (-\infty, 0]$. For a
$C^n$-valued given function $x$ and a parameter $t$ in $R$, the
function $x_t:I \to C^n$ is defined by $x_t(0) = x(t + 0)$ for $\theta$
in $I$ whenever $x(t + 0)$ is well defined. If $L: \mathcal{B} \to C^n$
is a given linear operator, we say that the relation

\[(1) \quad x'(t) = L(x_t)\]

is a linear functional differential equation—with finite
delay when $I = [-r, 0]$, or with infinite delay when $I$
$= (-\infty, 0]$. In this lecture, $L$ is always assumed to be con-
tinuous. Suppose for every $\phi$ in $\mathcal{B}$ Equation (1) has a
unique solution $x(t; \phi)$ for $t$ in $[0, \infty)$ with the ini-
tial condition $x_0 = \phi$. Then the solution operator $T(t): \mathcal{B} \to$
$\mathcal{B}$ is defined by the relation
\[ T(t)\phi = x_t(\phi), \quad \phi \text{ in } B, \quad t \geq 0. \]

In case \( B \) is the family of continuous functions on \([-r, 0]\) into \( C^N \), the family \( \{T(t): t \geq 0\} \) is a semigroup of class \( (C_0) \) of bounded linear operators on \( B \). Let \( A \) be the infinitesimal generator of \( T(t) \); that is, \( A\phi = \lim_{t \to 0^+} t^{-1}[T(t)\phi - \phi] \) whenever this limit exists. It is well known [2] that \( A \) is given by

\[
A\phi(\theta) = \begin{cases} 
L(\phi) \text{ for } \theta = 0 \\
\phi'(\theta) \text{ for } -r \leq \theta < 0
\end{cases}
\]

if and only if the function defined by the relation in the right hand side belongs to \( B \).

In the case when \( B = B(\langle -\infty, 0 \rangle, C^N) \), several models for \( B \) are proposed: for some special measure \( \mu \), \( B = L^p(\mu) \times C^N \) in which a norm is defined by \( |\phi| = [|\phi(0)|^p + \int_{-\infty}^0 |\phi(\theta)|^p \, d\mu(\theta)]^{1/p} \), \( 1 \leq p < \infty \); \( B = C^\omega \), the family of continuous functions \( \phi \) such that \( \phi(\theta) e^{Y\theta} \to \text{a limit as } \theta \to -\infty \), in which a norm is defined by \( |\phi| = \sup |\phi(\theta)e^{Y\theta}| \). In these cases the family \( T(t) \) is again a semigroup of class \( (C_0) \) of bounded linear operators on \( B \). Furthermore, the representation of \( A \) similar to Formula (2) is valid; the definition of \( \phi' \) is slightly changed according to the choice of \( B \) (cf [3, 4]).
In \([2, 5, 6]\), etc, Equation (1) is considered on some abstract phase space \(B\) which is defined to be a space satisfying some hypotheses. The systems of hypotheses are somewhat different from each other according to the problem under discussion. In all cases, however, \(T(t)\) becomes a semigroup of bounded linear operators on \(B\). Further results are known \([5, 6]\): an asymptotic estimate of the order of \(|T(t)|\) as \(t \to \infty\); informations about the spectrum of \(A\); and a construction of the fundamental matrix of Equation (1) with the variation-of-constants formula for the forced system of Equation (1), etc. However, it has been left unsolved to represent \(A\) in the manner analogous to Formula (2).

2. Formal approaches to the problem. To explain the reason why the representation of \(A\) is difficult to obtain, we go into the details of the hypotheses on \(B\) employed in \([5, 6]\).

\(\text{(H-0).}\) A seminorm \(|\cdot|\) is defined on \(B\): the quotient space \(\overline{B} = B/|\cdot|\) is a Banach space.

\(\text{(H-1).}\) If a function \(x:(-\infty, \sigma+\alpha) \to \mathbb{C}^n\), \(\alpha > 0\), is continuous on \([\sigma, \sigma+\alpha]\) and \(x_0\) is in \(B\), then \(x_t\) is in \(\overline{B}\) for every \(t\) in \([\sigma, \sigma+\alpha]\) and the map \(t \mapsto x_t\) is continuous.

\(\text{(H-2).}\) There exist positive continuous functions \(K(t)\) and \(M(t)\), where \(M(t)\) is submultiplicative, such that, for the function \(x\) arising in (H-2), \(|x_t| \leq K(t-\sigma) \times \sup\{|x(s)|: \sigma \leq s \leq t\} + M(t-\sigma)|x_\sigma|\) for \(\sigma \leq t < \sigma+\alpha\).
(H-3). \(|\phi(0)| \leq K|\phi|, \ \phi \text{ in } \mathcal{B}, \text{ for some constant } K.\)

From these hypotheses, it follows that the solution \(x(t;\phi)\) exists on \([0, \infty)\) uniquely: the solution operator \(T(t)\) is linear and continuous on \(\mathcal{B}\). Hypothesis (H-1) implies that the semigroup \(T(t)\) is of class \((C_0)\). However, notice that no measurability condition is assumed on \(\phi\) in \(\mathcal{B}\). We cannot, for example, discuss whether \(\phi\) in \(\mathcal{B}\) is absolutely continuous or not: the derivative \(\phi'\) has no meaning. To overcome this difficulty, we add more hypotheses on \(\mathcal{B}\), or else we interpret Formula (2) in a different manner than ever before. In this lecture, we proceed along the latter line.

To do this, let us introduce operators \(B\) and \(C_L\) defined formally by the relations

\[
B\phi(\theta) = \begin{cases} 
0 & \theta = 0 \\
\phi'(\theta) & \theta < 0
\end{cases}
C_L\phi(\theta) = \begin{cases} 
L(\phi) & \theta = 0 \\
0 & \theta < 0
\end{cases}
\]

To emphasize the operator \(L: \mathcal{B} \to \mathbb{R}^n\), \(T_L(t)\) denotes the solution semigroup of Equation (1) and \(A_L\) its infinitesimal generator. Then we can rewrite Formula (2) as

\[
(3) \quad A_L\phi = B\phi + C_L\phi.
\]

Observe that \(B = A_0\), the infinitesimal generator of the
solution semigroup \( T_0(t) \) of the trivial equation \( x'(t) = 0 \). Usually, we use a special symbol \( S(t) \) for \( T_0(t) \). The operator \( U_L(t) \) defined by \( U_L(t) = T_L(t) - S(t) \) is completely continuous; the decomposition \( T_L(t) = S(t) + U_L(t) \) was extremely useful to investigate the property of \( T_L(t) \), [3,5]. Hence Relation (3) is also expected to have some meaning. However, we soon notice that this formula contains a trivial contradiction; that is, the domains of \( A_L \) and \( B \) do not coincide with each other. Furthermore, we do not know whether \( C_L \) is well defined on the space \( B \) or not. Formula (3) has an ambiguity concerning the domain where it holds.

3. Representation of \( A \) in the dual space. Fortunately, the adjoint operators of \( A_L \) and \( B \) have the same domain. Stech [7] first discovered this interesting fact in the case where \( B \) is of the type \( L^p(\mu) \times C^n \); the author [5] proved the same result in the case where \( B \) is an abstract space satisfying a system of hypotheses similar to (H-0, ..., 4). Therefore, we hope that Formula (3) can be interpreted if the relation is transferred to the dual space. Let us introduce notations: \( X^* \) is a dual space of a Banach space \( X \), and \( T^* \) the dual operator of a linear operator \( T \) on \( X \) if it exists.

Before the demonstration of the final result, we again refer to the hypotheses on \( B \) which is sufficient to obtain the desired result. We leave Hypotheses (H-0) and (H-1) as
they are. The latter hypothesis implies that \( \mathcal{B} \) contains the space \( C \), the family of continuous functions on \( (-\infty, 0] \) with compact supports. Hypothesis (H-2) is replaced by (H-2)'. There exists a continuous function \( K(t) \) such that, if \( \phi \) in \( C \) has its support in \( [-t, 0] \), then

\[
|\phi| \leq K(t) \sup \{ |\phi(\theta)| : -t \leq \theta \leq 0 \}.
\]

Suppose \( \alpha : \mathcal{B} \to C \) is linear and continuous; that is, \( \alpha \) is a member of \( \mathcal{B}^* \). Then (H-2)' implies

\[
|\langle \alpha, \phi \rangle| \leq |\alpha| K(t) \sup \{ |\phi(\theta)| : -t \leq \theta \leq 0 \}
\]

for \( \phi \) arising in (H-2)'. This means that the restriction of \( \alpha \) on \( C \) is a Radon measure on \( (-\infty, 0] \). It is well known that, for such a measure \( \alpha \), there exists a function \( \eta(\alpha; \theta) = (\eta_1(\alpha; \theta), \ldots, \eta_n(\alpha; \theta)) \) locally of bounded variation for \( \theta \) on \( (-\infty, 0] \) such that

\[
\langle \alpha, \phi \rangle = \int_{-\infty}^{0} d_\theta \eta(\alpha; \theta) \phi(\theta) = \int_{-\infty}^{0} \sum_{i=1}^{n} d_\theta \eta_i(\alpha; \theta) \phi^i(\theta)
\]

for every \( \phi \) in \( C \). We can assume that \( \eta \) is normalized in the sense that \( \eta(\alpha; 0) = 0 \) and \( \eta(\alpha; \theta) \) is continuous to the left for \( \theta < 0 \). Then \( \eta(\alpha; \theta) \) is determined uniquely by \( \alpha \) in \( \mathcal{B}^* \). It is clear that the map \( \alpha \to \eta(\alpha; 0-) \), the left-hand limit of \( \eta \) at \( \theta = 0 \), is a linear operator on
$\mathcal{B}^*$ into $C^n$. We assume the following hypothesis.

(H-4). This operator $\alpha \to \eta(\alpha; 0-)$ is continuous.

Note that this condition holds [6] if we adds one more hypothesis to the system (H-0, 1, 2, 3).

Hypothesis (H-3) is also assumed and rewritten in the form

(H-3). The operator $D: \mathcal{B} \to C^n$ defined by $D(\phi) = \phi(0)$, $\phi \in \mathcal{B}$, is continuous.

Finally, we need the following.

(H-5). For every $t \geq 0$, $T_L(t)$ is well defined to be a continuous linear operator on $\mathcal{B}$.

Hypotheses (H-1, 5) imply that $T_L(t)$ is a semigroup of class $(C_0)$ of bounded linear operators on $\mathcal{B}$. It is known [2, 5] that (H-5) is derived from (H-1, 2, 3). In this lecture, we are not interested in this fact, but we devote ourself to the study of the representation of $A_L$. From this standpoint, we assume the above statement, while (H-2) is replaced by a weak Hypothesis (H-2)'.

Now observe that $(T_L(t) - S(t))\phi$ is a member of $C$ for every $\phi$ in $\mathcal{B}$. From Hypothesis (H-2)', for every $\alpha$ in $\mathcal{B}^*$ we can represent $<\alpha, (T_L(t) - S(t))\phi>$ in terms of Stirtjes integral. This implies the following Proposition.

**Proposition 1.** For every $\alpha$ in $\mathcal{B}^*$ and every $\phi$ in $\mathcal{B}$,

$$\lim_{t \to 0^+} t^{-1}[T_L^*(t) - S^*(t)]\alpha, \phi = \sum_{i=1}^{n} -\eta_i(\alpha; 0-) L_i^{1}(\phi).$$
Let $\delta^i_1(\phi)$ and $L^i_1(\phi)$ denote the $i$-th component of $D(\phi)$ and $L(\phi)$, respectively. Hypothesis (H-3) implies that $\delta^i_1$ is an element of $B^*$; clearly, $L^i_1$ is also in $B^*$. From the definition of infinitesimal generators and dual operators, we have the following proposition.

**Proposition 2.** Every $\delta^i_1$ belongs to the domain of $A^*_L$; and $A^*_L \delta^i_1 = L^i_1$ for $i = 1, \ldots, n$.

Define an operator $P : B^* \rightarrow B^*$ by the relation

$$P \alpha = -\eta(\alpha; 0^-)_D = \sum_{i=1}^{n} -\eta_i(\alpha; 0^-) \delta^i_1$$ for $\alpha$ in $B^*$.

It is easy to see that $PP = P$; while Hypothesis (H-4) implies that $P$ is continuous. Therefore $P$ is a continuous projection on $B^*$. Now we state the main theorem.

**Theorem 3.** Suppose Hypotheses (H-0,1,3,4,5) and (H-2)' hold and let $A^*_L$ be the infinitesimal generator of the solution semigroup $T_L(t)$ of Equation (1). Then the domain of $A^*_L$ is independent of the choice of the continuous linear operator $L : B \rightarrow C^n$. If the above projection $P$ is restricted on this common domain $D^*$, then the restriction, denoted by $P$ again, is also a projection on $D^*$. The all operators $A^*_L$, $B^*$ and $P$ are transformations of $D^*$ into $B^*$ and they are related with each other in the manner
\[ A_L^* = B^* + A_L^* F. \]

It is not difficult to prove this theorem if we use Proposition 1, 2 and the following result (cf. [1, p. 49]): the dual operator of the infinitesimal generator of a semi-group of class (C_0) is equal to the weak\(^*\) infinitesimal generator of the dual semigroup. Notice that, along this line, we can again prove the existence of the common domain \( D^* \).

**Corollary 4.** The domain \( D^* \) is decomposed into a direct sum \( D^* = N_L^* \oplus M_L^* \) as follows:

(i) \( A_L^* \alpha = B^* \alpha \) if and only if \( \alpha \) is in \( N_L^* \).

(ii) The restriction of \( A_L^* \) on \( M_L^* \) is an isomorphism of \( M_L^* \) onto the linear manifold generated by \( \{ L^1, ..., L^n \} \).

(iii) \( M_L^* \) is contained in \( PD^* \) and \( (I - P)D^* \) in \( N_L^* \).

On the other hand, the following conditions are equivalent:

(a) \( M_L^* = PD^* \), (b) \( N_L^* = (I - P)D^* \), (c) the family \( \{ L^1, ..., L^n \} \) are linearly independent.

**REFERENCES**


