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Periodic solutions of functional differential equations

Tetsuo FURUMOCHI

(Iwate University)

1. Introduction. The purpose of this paper is to discuss the existence problem of periodic solutions of functional differential equations. Particularly, we are concerned with a reduction of a given ω-periodic functional differential equation with the delay $r > \omega$ to an auxiliary ω-periodic functional differential equation with the delay ω. Moreover, we obtain a Razumikhin type theorem concerning the existence of ω-periodic solutions of functional differential equations by using a strongly convex Liapunov function.

Let R^+ and R denote the interval $0 \leqq t < \infty$ and $-\infty < t < \infty$, respectively. For a given r, $0 < r \leqq \infty$, C_r denotes the Banach space of continuous and bounded functions defined by

$$C_r = \{ \phi : [-r,\ 0] \to R^n,\ \text{continuous} \},\quad 0 < r < \infty,$$

or

$$C_\infty = \{ \phi : (-\infty,\ 0] \to R^n,\ \text{continuous and bounded} \},$$

with the uniform norm, $|\phi| = \sup\{ |\phi(\theta)| : -r < \theta \leqq 0 \}$, where $|\cdot|$ denotes the usual Euclidean norm in R^n. For a given continuous function $x(s)$, the symbol x_t will denote the element of C_r such that $x_t(\theta) = x(t+\theta)$, $-r < \theta \leqq 0$.

Let $f(t,\ \phi) : R \times C_r \to R^n$ be a completely continuous function which is ω-periodic in t, that is, $f(t+\omega, \phi) = f(t, \phi)$ for all $(t, \phi) \in R \times C_r$ and some positive constant ω.

Consider a functional differential equation

$$(1) \qquad \dot{x}(t) = f(t, x_t),$$

where $\dot{}$ denotes the right hand derivative.

2. Reductions of Equation (1). In this section, we shall consider reductions of Equation (1) with $r > \omega$ to auxiliary ω-periodic functional differential equations with the delay ω. Let $\sigma_t(\psi) : R \times C_\omega \to C_r$ be a mapping such that $\sigma_t(\psi)$ is ω-periodic in t, continuous on $R \times C_\omega$, takes bounded sets in $R \times C_\omega$ into bounded sets, $\sigma_t(\psi)(\theta)$ is ω-periodic in θ on $(-r, 0]$ if $\psi(0) = \psi(-\omega)$, and that $\sigma_t(\psi)(\theta)$ is ω-periodic in θ on $(-r, -\omega]$ if $r > 2\omega$. A simple example is :

$$(2) \quad \sigma(\psi)(\theta) = \begin{cases} \psi(\theta) \\ \psi(\theta+k\omega) - \dfrac{\theta+(k+1)\omega}{\omega}(\psi(0)-\psi(-\omega)), & -\min\{r, (k+1)\omega\} \leqq \theta < -k\omega, k \geqq 1. \end{cases}$$

For such a mapping $\sigma_t(\psi)$, let $g(t, \psi) : R \times C_\omega \to R^n$ be a function defined by

$$(3) \qquad g(t, \psi) = f(t, \sigma_t(\psi)).$$

Clearly $g(t, \psi)$ is completely continuous, and ω-periodic in t. For $g(t, \psi)$ defined by (3), consider an auxiliary equation

$$(4) \qquad \dot{x}(t) = g(t, x_t).$$

Then this equation always has a solution for the initial value problem, while Equation (1) may fail to have a solution for some initial value problem (see Seifert [3]). And we have the following theorem.

Theorem 1. An ω-periodic solution $x(t)$ ($t \in R$) of Equation (1) is a solution of Equation (4), and vice versa.

The proof of this theorem is clear from the definition of $g(t, \psi)$ and the properties of $\sigma_t(\psi)$.

3. Existence of periodic solutions of Equation (1). In this section, we shall show the existence of ω-periodic solutions of (1) via the existence of ω-periodic solutions of (4). Here we consider a strongly convex Liapunov function defined by strengthening the conditions for a convex Liapunov function in [2].

A function $V(t, x) : R \times R^n \to R$ is said to be a Liapunov function if $V(t, x)$ is continuous on $R \times R^n$ and satisfies $V(t, x) \geq a(|x|)$ for a continuous function $a(u)$ such that $a(u) \to \infty$ as $u \to \infty$. We shall say a Liapunov function $V(t, x)$ is convex if for each fixed $t \in R$ the set $\{ x \in R^n : V(t, x) \leq k\}$ is convex in R^n. Moreover, a convex Liapunov function $V(t, x)$ is said to be strongly convex if for each fixed $t \in R$ the set $\{ x \in R^n : V(t, x) = k\}$ is the boundary of $\{ x \in R^n : V(t, x) \leq k\}$. We define $V'_{(1)}(t, \phi)$ by

$$V'_{(1)}(t, \phi) = \limsup_{\tau \to 0+} \frac{1}{\tau} \{V(t+\tau, x(t+\tau, t, \phi)) - V(t, \phi(0))\}.$$

Clearly we have

$$(5)\quad V'_{(1)}(t, \phi) = \limsup_{\tau \to 0+} \frac{1}{\tau} \{V(t+\tau, \phi(0)+\tau f(t, \phi)) - V(t, \phi(0))\}$$

if $V(t, x)$ is locally Lipschitzian with respect to x.

Now consider the following scalar equation

$$(6)\qquad \dot{u} = h(t, u),$$

where $h(t, u) : R \times R \to R$ is continuous and locally Lipschitzian with respect to u.

The following theorem is given in [1], where the delay r is finite.

Theorem 2. Let $V : R \times R^n \to R^+$ be a continuous, ω-periodic, convex Liapunov function, and let $L : R \to R$ be a continuous, non-decreasing function such that

$$(7)\qquad L(u) > u \text{ for all } u > 0,$$

or

$$(8)\qquad L(u) = u \text{ for all } u > 0.$$

Suppose that there exists a continuous function $u(t)$ defined on $[t_0-r, t_0+\omega]$ for some t_0 such that $u(t)$ is a solution of (6) on $[t_0, t_0+\omega]$ which satisfies $u(t_0+\omega+\theta) \leqq u(t_0+\theta)$ for $-r \leqq \theta \leqq 0$, $u(t) \geqq V(t, 0)$ for $t_0 \leqq t \leqq t_0+\omega$, and $u(t+\theta) \leqq L(u(t))$ for $t_0 \leqq t \leqq t_0+\omega$, $-r \leqq \theta \leqq 0$, and that we have

$$(9)\qquad V'_{(1)}(t, \phi) \leqq h(t, V(t, \phi(0)))$$

for all functions $\phi \in C_r$ with the property that

(10) $V(t,\ \phi(0)) \geqq u(t)$, $V(t+\theta,\ \phi(\theta)) \leqq L(V(t,\ \phi(0)))$ for $-r \leqq \theta \leqq 0$.

Then Equation (1) has an ω-periodic solution.

Now we can obtain the following theorem concerning the existence of an ω-periodic solution of Equation (1) with $r = \infty$ by combining Theorem 1 and Theorem 2.

Theorem 3. Let $V : R \times R^n \to R^+$ be a continuous, ω-periodic in t, strongly convex Liapunov function which is locally Lipschitzian with respect to x, and let $L : R \to R$ be a nondecreasing continuous function which satisfies (7) or (8). Suppose that Equation (6) has a solution $u(t)$ on $[0,\ \omega]$ such that $u(0) \geqq u(\omega)$, $u(t) > V(t,\ 0)$ for $0 \leqq t \leqq \omega$, $\bar{u} \leqq L(\underline{u})$ where $\bar{u} = \max_{0 \leqq t \leqq \omega} u(t)$, $\underline{u} = \min_{0 \leqq t \leqq \omega} u(t)$, and that we have

(11) $$V'_{(1)}(t,\ \phi) \leqq h(t,\ V(t,\ \phi(0)))$$

for all functions $\phi \in C_\infty$ with the property that

(12) $\phi \in P_\infty$, $V(t, \phi(0)) \geqq u(t)$, $V(t+\theta, \phi(\theta)) \leqq L(V(t, \phi(0)))$ for $-\infty < \theta \leqq 0$,

where $P_\infty = \{\ \phi \in C_\infty : \phi(\theta)$ is ω-periodic on $(-\infty,\ -\omega]\}$. Then Equation (1) has an ω-periodic solution $x(t)$ such that $V(t,\ x(t)) \leqq u(t)$ for $0 \leqq t \leqq \omega$.

This theorem can be proved in the following manner. First, we define a function $\bar{\sigma}_t(\psi) : R \times C_\omega \to C_\infty$ which is similar to $\sigma_t(\psi)$ in Section 2 by using the strongly convex Liapunov function $V(t,\ x)$. For any $(t,\ \psi) \in R \times C_\omega$, let $v = \max_{-\omega \leqq \theta \leqq 0} V(t+\theta,\ \psi(\theta))$ and

$v^* = \max\{v, L(\underline{u})\}$. Define a set Σ^* by

$$\Sigma^* = \{(s, x) \in R \times R^n : V(s, x) \leqq v^*\}.$$

Now let $\sigma(\psi)$ be a function defined by (2). For $\theta \leqq 0$ such that $(t+\theta, \sigma(\psi)(\theta)) \in \Sigma^*$, define $\bar{\sigma}_t(\psi)(\theta)$ by $\bar{\sigma}_t(\psi)(\theta) = \sigma(\psi)(\theta)$. For $\theta < -\omega$ such that $(t+\theta, \sigma(\psi)(\theta)) \notin \Sigma^*$, define $\bar{\sigma}_t(\psi)(\theta)$ by $\bar{\sigma}_t(\psi)(\theta) = \lambda(\theta)\sigma(\psi)(\theta)$, where $\lambda(\theta)$ is a uniquely determined number such that $\lambda(\theta) \in (0, 1)$ and $V(t+\theta, \lambda(\theta)\sigma(\psi)(\theta)) = v^*$. Clearly the function $\bar{\sigma}_t(\psi)$ has the same properties with $\sigma_t(\psi)$ in Section 2. Moreover, we have $V(t+\theta, \bar{\sigma}_t(\psi)(\theta)) \leqq v^*$ for $\theta \leqq 0$ by the definition of $\bar{\sigma}_t(\psi)$. Using this $\bar{\sigma}_t(\psi)$, define an auxiliary function $g^*(t, \psi)$ by

(13) $$g^*(t, \psi) = f(t, \bar{\sigma}_t(\psi)),$$

and consider the following auxiliary equation

(14) $$\dot{x}(t) = g^*(t, x_t).$$

If we show that

(15) $$V'_{(14)}(t, \psi) \leqq h(t, V(t, \psi(0)))$$

under the condition

(16) $V(t, \psi(0)) \geqq u(t)$, $V(t+\theta, \psi(\theta)) \leqq L(V(t, \psi(0)))$ for $-\omega \leqq \theta \leqq 0$,

then we can apply Theorem 2 to Equation (14). Let (t, ψ) satisfy (16), and let $v = \max_{-\omega \leqq \theta \leqq 0} V(t+\theta, \psi(\theta))$ and $v^* = \max\{v, L(\underline{u})\}$. If we take $\phi = \bar{\sigma}_t(\psi)$ for (t, ψ), then we obtain

$$V(t+\theta, \phi(\theta)) \leqq v^* \leqq L(V(t, \phi(0))) \text{ for } -\infty < \theta \leqq 0.$$

Thus (16) implies (12), and consequently we have (11), which is not different from (15) by (5) and (13), since $V(t, x)$ is locally Lipschitzian with respect to x.

Next, $u(t)$ can be extended continuously on $[-\omega, \omega]$ by defining $u(t) = \max\{u(0), u(t+\omega)\}$ for $-\omega \leqq t \leqq 0$. Then all assumptions of Theorem 2 are satisfied with this $u(t)$ and $t_0 = 0$, and Equation (14) has an ω-periodic solution. Thus we can obtain the conclusion of this theorem by Theorem 1.

Finally, we present an application of Theorem 3. Consider a scalar ω-periodic functional differential equation

$$(17) \qquad \dot{x}(t) = -x(t) + 4x(t-\omega) - 4x(t-2\omega) + f(t, x_t),$$

where $f(t, \phi) : R \times C_\infty \to R$ is completely continuous, ω-periodic in t, and satisfies

$$(18) \qquad |f(t, \phi)| \leqq |\phi| \text{ if } t \in R, \ |\phi(0)| > K$$

for some positive constant K. If we take $L(u) = u$ and $V(x) = \frac{x^2}{2}$, then V is a strongly convex Liapunov function which is locally Lipschitzian, and we have

$$V'_{(17)}(t, \phi) = -\phi^2(0) + \phi(0)f(t, \phi) \leqq -\phi^2(0) + \phi^2(0)\frac{|\phi|}{|\phi(0)|}\frac{|f(t, \phi)|}{|\phi|} \leqq 0$$

under the condition (12) if the solution $u(t)$ of (6) with $h(t, u) \equiv 0$ satisfies $u(t) > \frac{K^2}{2}$, where K is the one in (18). Therefore, by Theorem 3, there exists an ω-periodic solution of (17).

Suppose that (17) is an equation with the finite delay r. In this case, we cannot apply Theorem 37.1 in [4] to (17) to conclude that (17) has an ω-periodic solution, since r is greater than the period ω. Moreover, $V'_{(17)}(t, \phi)$ cannot be compared with $h(t, V(t, \phi(0))) \equiv 0$ under the condition (10). Thus we have no idea how to apply Theorem2.

References

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