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On Caratheodory Conditions for Functional Differential Equations with Infinite Delays

by

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For functional differential equations of retarded type where the delay is fixed and finite, local existence results for initial value problems analogous to the Picard and Peano theorems for ordinary differential equations are well known; cf., for example, the book by J. Hale [1].

For initial value problems involving equations with infinite delays, the results of R. Driver [2] were perhaps the first to appear. More recently, existence theorems for such equations have appeared in papers by J. Hale and J. Kato [3], K. Schumacher [4], and F. Kappel and W. Schappacher [5]. In [2], [3], and [5] existence theorems of Peano-type, where solutions are continuously differentiable on their intervals of existence, are obtained for equations on quite general delay spaces. For such Peano-type existence theorems an important hypothesis that certain t-dependent composites of the function in the equation with translates of the state space functions be continuous seems to be crucial. For the state space CB consisting of continuous bounded functions on

$(-\infty, 0]$ with supremum norm, it is known that such composites are not in general continuous, even for very smooth functions on CB; cf. [6] for an example. The example in [6], however, has a solution of Caratheodory type; i.e., a solution which is absolutely continuous on its interval of definition and satisfies the equation almost everywhere there.

Recently fairly general existence theorems for solutions of Caratheodory type have appeared; cf. [4], [5]. Earlier, A. Halany and J. Yorke [7] also stated such an existence theorem. As would be expected, a crucial condition in these results seems to be that the composites mentioned earlier be measurable.

In fact, recently other results involving Caratheodory type solutions such as continuous dependence of solutions on their initial functions also indicate the importance of such a measurability hypotheses; cf. [8].

Consequently, a natural question would seem to be: how smooth must the function be to guarantee such measurability for such composites?

It is the purpose of this paper to show that if the state space CB is used, there exist continuous linear functions on CB for which such composites fail to be measurable.

We use the following fairly standard notation:

(1.1) R and R^n denote respectively the set of reals and real n-vectors; $|x|$ is a fixed norm for $x \in R^n$.

(1.2) $\{CB, \| \ \|\}$ is the Banach space of R^n-valued functions continuous and bounded on $(-\infty, 0]$; for $\phi \in CB$, $\|\phi\| = \sup \{|\phi(s)|, s \leqq 0\}$.

(1.3) If $x(t)$; $(-\infty, b) \to R^n$, $b \leqq \infty$, then for fixed $t \in (-\infty, b)$, x_t denotes the function $x(t + s)$, $s \leqq 0$.

The simple example $x(t) = \sin t^2$ shows that even though $x(t)$ is continuous and bounded on R, x_t: $R \to CB$, may not be continuous anywhere. There are also examples of Lipschitz continuous functions $f : R \to R$ for which there exist functions $x(t)$ continuous and bounded on R such that the composite $f(x_t)$ fails to be continuous in a non-degenerate interval of R; cf. an example in [6], as has been already mentioned. However, in the example in [6], this composite is measurable and dominated by an integrable function.

Theorem. There exists a R^2-valued function $x(t)$ bounded and continuous on R such that given any R^2-valued function $g(t)$ on R such that $|g(t)| \leqq 1$ for all t, there exists a continuous linear functional $f : CB \to R^2$ such that either $f(x_t) = g(t)$ or $f(x_t) = g(-t)$ for $t \in R$.

An obvious consequence of this theorem is the following:

Corollary. There exists a R^2-valued function $x(t)$ bounded and continuous on R such that given any R^2-valued bounded even function $g(t)$ on R, there exists a continuous linear functional $f : CB \to R^2$ such that $f(x_t) = g(t)$ for $t \in R$.

To prove the theorem we use the following result due to W. Rudin which is Theorem 1 in his paper [9]. Rudin's result was pointed out to the author by R. Sine who together with J. Peters, the author's colleague at Iowa State, indicated how it can be used to prove our theorem.

Theorem 1. Let G be an infinite metrizable locally compact group which is not compact. Let $L^\infty(G)$ denote the set of bounded complex Borel functions on G. Then there exists a $\phi \in L^\infty(G)$ continuous on G such that given any complex function g on G such that $|g(t)| \leqq 1$ for $t \in G$, there exists a continuous linear complex functional f on $L^\infty(G)$ such that $g(t) = f(\phi_t)$ for $t \in G$; here $\phi_t = \phi(ts)$, $s \in G$.

Moreover, f is multiplicative; i.e., for $\phi, \psi \in L^\infty(G)$ we have $f(\phi\psi) = f(\phi)f(\psi)$, where $\phi\psi$ is defined to be the pointwise product, and $f(\mu) = 1$ where $\mu \in L^\infty(G)$ is the function with constant value 1 on G.

Proof of Theorem. Since Theorem 1 deals with complex valued functions, we may regard CB as consisting of complex-valued functions, and make the identification with R^2-valued functions in the obvious way.

Let $G = R$ with group operation being addition and having the usual topology.

(i) For any $\phi^- \in CB$, define a unique $\phi \in L^\infty(R)$ by

$$\phi(\theta) = \phi^-(\theta),\ \theta \leqq 0, \text{ and } \phi(\theta) = 0,\ \theta > 0.$$

(ii) For any continuous linear complex functional f on $L^\infty(R)$ define f^- on CB by $f^-(\phi^-) = f(\phi)$, $\phi^- \in CB$. Thus f^- is easily seen to be a continuous linear functional on CB.

(iii) For any continuous $\phi \in L^\infty(R)$, define $\phi^- \in CB$ by $\phi^-(\theta) = \phi(\theta)$, $\theta \leqq 0$.

(iv) For any $\phi \in L^\infty(R)$, define $\phi_t^R \in L^\infty(R)$ by $\phi_t^R = \phi(t+s)$, $s \in R$.

(v) Define the functions μ, μ^+, μ^- on R by $\mu(\theta) = 1$, $\theta \in R$,

$$\mu^+(\theta) = 1,\ \theta \geqq 0,\ \mu^+(\theta) = 0,\ \theta < 0.$$

$$\mu^-(\theta) = 1,\ \theta \leqq 0,\ \mu^-(\theta) = 0,\ \theta > 0.$$

By Theorem 1 there exists a continuous $\phi \in L^\infty(R)$ such that given any complex function g on R such that $|g(t)| \leqq 1$, there exists a continuous linear complex functional f with properties as stated there such that $f(\phi_t^R) = g(t)$ on R. Henceforth in this proof, ϕ, g, and f are these fixed specific functions.

Since $f(\mu^-) = f(\mu^-\mu^-) = (f(\mu^-))^2$, it follows that $f(\mu^-)$

$= 0$ or $f(\mu^-) = 1$.

Case 1. $f(\mu^-) = 1$. For any continuous $\psi \in L^\infty(R)$, $f^-(\psi^-) = f(\psi\mu^-) = f(\psi)f(\mu^-)$; here $\psi^- \in CB$ is given by (iii), and f^- by (ii). Therefore $f^-(\phi_t) = f(\phi_t^R \mu^-) = f(\phi_t^R)f(\mu^-)$ $= f(\phi_t^R) = g(t)$, and our theorem follows in this case.

Case 2. $f(\mu^-) = 0$. Since $1 = f(\mu) = f(\mu^+ + \mu^-) = f(\mu^+)$ $+ f(\mu^-)$, we have $f(\mu^+) = 1$; note that $\mu = \mu^+ + \mu^-$ almost everywhere on R and in $L^\infty(R)$, we do not distinguish functions which differ only on sets of measure zero; cf. the remark in p.73 in [9].

We now repeat the same argument as in Case 1, replacing CB by CB^+, the space of continuous bounded functions on $t \geqq 0$ with supremum norm there, and conclude that there exists a continuous linear function f^+ on CB^+ such that $f^+(\phi_t{}^+)$ $= g(t)$; here $\phi_t{}^+ = \phi(t + s)$, $s \geqq 0$; i.e., $\phi_t{}^+ \in CB^+$.

Define $\hat{\phi}(t) = \phi(-t)$, $t \in R$; for fixed $t \in R$ we have

$$
\begin{aligned}
\hat{\phi}_t = \hat{\phi}(t + s) &= \phi(-t-s), \quad s \leqq 0 \\
&= \phi(-t+s), \quad s \geqq 0 \\
&= \phi_{-t}{}^+.
\end{aligned}
$$

Since $f^+(\phi_t{}^+) = g(t)$, we have $f^+(\phi_{-t}{}^+) = g(-t)$; i.e., $f^+(\hat{\phi}_t) = g(-t)$ on R. Now f^+ can be considered a continuous linear function on CB; we can define f^+ on CB by $f^+(\psi) = f^+(\hat{\psi})$ for $\psi \in CB$ where $\hat{\psi}(s) = \psi(-s)$, $s \geqq 0$; i.e. $\hat{\psi} \in CB^+$. This proves our theorem for this case too.

Remarks. Our proof uses the multiplicative property of the bounded linear function on $L^\infty(R)$ very strongly. On the other hand, the condition $f(\mu) = 1$ for the unit function μ is not crucial; it is easy to verify that $f(\mu) = 0$, which is the only other possibility due to the multiplicative property of f, implies $f(\phi) = 0$ for all $\phi \in L^\infty(R)$, so if $g(t)$ is not the identically zero function, the condition $f(\mu) = 1$ necessarily holds.

As is pointed out in [9], the fact that the group G in Theorem 1 is not compact allows us to assert that the $\phi \in L^\infty(G)$ is in fact continuous. This is clearly an important condition used in our proof. Rudin's theorem, however, guarantees the existence of an $\phi \in L^\infty(G)$ and an f as in Theorem 1 even for G compact. We also point out that in Rudin's theorem G need not be abelian.

The following questions suggest themselves:

(i) If S is a commutative semigroup with unit, does Theorem 1 told if G is replaced by S ? If so, our theorem would be an immediate consequence of it.

(ii) Does our corollary hold if the condition that g be even is omitted ? The author suspects the answer is in the affirmative but has not yet been able to prove it.

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