<table>
<thead>
<tr>
<th>Title</th>
<th>On p-Nilpotent Groups with Extremal p-Blocks (Skew Polynomial Rings, Group Rings and Related Topics)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Author(s)</td>
<td>NINOMIYA, YASUSHI</td>
</tr>
<tr>
<td>Citation</td>
<td>数理解析研究所講究録 1981, 438: 71-75</td>
</tr>
<tr>
<td>Issue Date</td>
<td>1981-09</td>
</tr>
<tr>
<td>URL</td>
<td><a href="http://hdl.handle.net/2433/102776">http://hdl.handle.net/2433/102776</a></td>
</tr>
<tr>
<td>Type</td>
<td>Departmental Bulletin Paper</td>
</tr>
<tr>
<td>Textversion</td>
<td>publisher</td>
</tr>
</tbody>
</table>

Kyoto University
ON p-NILPOTENT GROUPS WITH EXTREMAL p-BLOCKS

Yasushi NINOMIYA

Let \( G \) be a finite group, and \( p \) a fixed prime number. It is well known that

(I) if \( G \) is \( p \)-closed, then every \( p \)-block of \( G \) has full defect, and

(II) if \( G \) has the \( p \)TI-property, then every \( p \)-block of \( G \) has either full defect or defect zero.

Here, "\( G \) is \( p \)-closed" means that a Sylow \( p \)-subgroup of \( G \) is normal, and "\( G \) has the \( p \)TI-property" means that the intersection of two distinct Sylow \( p \)-subgroups of \( G \) is the identity. It is interesting to consider each converse of (I) and (II). In general, neither the converse of (I) nor of (II) is true. In fact, the Mathieu groups \( M_{22} \) and \( M_{24} \) have only one 2-block, but these groups are neither 2-closed nor have the 2TI-property. Moreover, if \( H \) is a \( p \)-solvable group of \( p \)-length greater than 1, then \( G = H/\text{O}_p(H) \) has only one \( p \)-block, but \( G \) is neither \( p \)-closed nor has the \( p \)TI-property. In case \( p = 2 \), several authors studied this problem. In what follows, we call \( G \) a \( p \)FD-group if every \( p \)-block of \( G \) has full defect. On the other hand, we call \( G \) a \( p \)FZD-group if every \( p \)-block of \( G \) has either full defect or defect zero.

In [2], Harada proved the following
Theorem 1 ([2, Theorem 1]). Suppose that a Sylow 2-subgroup of \( G \) is abelian. Then \( G \) is 2-closed if and only if it is a 2FD-group.

In [3], Herzog has generalized the above theorem as follows:

Theorem 2 ([3, Theorems 1 and 2]). (1) A group \( G \) is 2-closed if and only if it is a 2FD-group and every intersection of two distinct Sylow 2-subgroups of \( G \) is centralized by a Sylow 2-subgroup of \( G \).

(2) A group \( G \) has the 2TI-property if and only if it is a 2FZD-group and every intersection of two distinct Sylow 2-subgroups of \( G \) is centralized by a Sylow 2-subgroup of \( G \).

Furthermore, in [1], Chillag and Herzog proved the following theorems.

Theorem 3 ([1, Theorems 1 and 2]). Suppose that \( G \) is a 2-nilpotent group and a Sylow 2-subgroup of \( G \) is a dihedral group, a generalized quaternion group or a quasidihedral group. Then there holds the following:

(1) \( G \) is 2-closed if and only if it is a 2FD-group.

(2) \( G \) has the 2TI-property if and only if it is a 2FZD-group.

Theorem 4 ([1, Theorem 4]). Suppose that a Sylow 2-subgroup
of $G$ is a quaternion group of order 8. Then there holds the following:

(1) $G$ is 2-closed if and only if it is a 2FD-group.

(2) $G$ has the 2TI-property if and only if it is a 2FZD-group.

Here, we report that each converse of (I) and (II) is true if $G$ is a $p$-nilpotent group. At first, the following proposition is an immediate consequence of [5, Theorem 4].

Proposition. Let $G$ be a $p$-nilpotent group with a normal $p$-complement $N$. Then $G$ is a pFD-group if and only if, for every $x \in N$, $C_G(x)$ contains a Sylow $p$-subgroup of $G$.

By making use of the proposition, we can easily obtain the following

Theorem 5. Let $G$ be a $p$-nilpotent group. Then $G$ is $p$-closed if and only if it is a pFD-group.

Let $H$ be a normal subgroup of $G$ such that $|G/H|$ is relatively prime to $p$. Then by [4, Proposition 4.2], we see that if $G$ is a pFD-group then $H$ is also a pFD-group. Hence, we get the following, which contains [2, Lemma 1].

Corollary 1. Let $G$ be a $p$-solvable group. Then $G$ is
p-closed if and only if it is a pFD-group and has p-length 1.

Furthermore, by making use of Theorem 5, we can prove the following

Theorem 6*). Let $G$ be a $p$-nilpotent group, and $P \cap Q$ an intersection of maximal order of two distinct Sylow $p$-subgroups of $G$. Then there exists a $p$-block of $G$ with defect group $P \cap Q$.

As a corollary to this theorem, we get the following

Theorem 7. Let $G$ be a $p$-nilpotent group. Then $G$ has the $p$TI-property if and only if it is a pFZD-group.

Let $G$ be a $p$-solvable group. If $G$ has the $p$TI-property, then $G/O_p(G)$ also has the $p$TI-property, and hence $G$ has p-length 1. Therefore, Theorem 7 together with [4, Proposition 4.2] implies the following

Corollary 2. Let $G$ be a $p$-solvable group. Then $G$ has the $p$TI-property if and only if it is a pFZD-group and has p-length 1.

*) This theorem was suggested by Dr. T. Okuyama.
References


Shinshu University